Stability of the eigenvalue condition for the fine-structure constant α and short-distance behavior in strong interaction. I

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The short-distance behavior of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar coupling is studied without closed fermion loops. It is then shown that in the single-closed-fermion-loop contribution to the renormalized photon self-energy part the Adler-Baker-Johnson eigenvalue condition for the fine-structure constant α remains unaltered. The effective strong coupling vanishes at very high energies and is simply expressed in terms of well-known parameters. An alternative type of solution to the above problem is also discussed (in an appendix) which, however, is not physically and technically very attractive for various mentioned reasons.

I. INTRODUCTION

The short-distance behavior of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar (ps-ps) coupling is studied without closed fermion loops. The scaling equations for the proton propagator and the strong vertex are obtained and solved at short distances. It is then shown that in the *single*-closed-fermion-loop contribution $\pi_c^{[1]}$ to the renormalized photon self-energy part the Adler-Baker-Johnson eigenvalue condition[1,2] for the (renormalized) fine-structure constant[1] α, $F^{[1]}(\alpha)=0$, remains *unaltered*. This means that a possible zero of $F^{[1]}(x)$ does not "move" in the presence of the strong coupling. By definition, the object $\pi_c^{[1]}$ contains *one* over-all closed fermion loop. Accordingly, $\pi_c^{[1]}$ does not contain pion-pion, pion-photon, photon-photon scattering graphs, nor does it contain pion and photon self-energy parts. As a by-product of the work it is shown that the *effective* strong coupling vanishes at very high energies. This effective strong coupling is expressed in terms of well-known parameters.

The technique used here is in the spirit of Refs. 1 and 3 which make consistent use of the Callan-Symanzik scaling equations.[4] The method we assume to study these scaling equations is to sum up first in the electromagnetic coupling (to all orders), study the property of the scaling equations, and then finally sum up over the strong coupling as well. (This permits us to treat the derivative with respect to the strong coupling, appearing in the Callan-Symanzik scaling equations, in an elementary fashion.) Another type of solution to the above problem is clearly discussed in Appendix B, which is neither physically nor technically very attractive for various mentioned reasons. The single-fermion-loop contribution to the pion self-energy part and the implication of the present results on the finiteness problem of the full theory will be discussed in paper II.

The paper is organized as follows: In Sec. II we give a quick derivation of Schwinger-Dyson integral equations without closed fermion loops and discuss their renormalizability. In Sec. III the various gauge functions which make the respective renormalization constants in a perturbative expansion finite are calculated. The Callan-Symanzik scaling equations for the proton propagator and various vertex functions are studied in Sec. IV. The stability of the eigenvalue condition for α is demonstrated in Sec. V. Section VI deals with a brief conclusion. In Appendix A the gauge transformation of the multiplicative renormalization constants is discussed. The alternative type of solution mentioned above is discussed in Appendix B. Our metric is $g_{\mu\nu}=\mathrm{diag}[-1,1,1,1]$, $\{\gamma_\mu,\gamma_\nu\}=-2g_{\mu\nu}$, with γ_0, $i\gamma_k$, $i\gamma_5$ being Hermitian and $\{\gamma_\mu,\gamma_5\}=0$, $\gamma_5^2=-\underline{1}$. As a preliminary study of the present dynamics, we have also carried out an analysis (essentially oriented towards the ghost problem) in the spirit of the leading logarithms of perturbation theory (unpublished) which was a useful guide in leading to the present work. In this connection see also Ref. 5.

II. BASIC INTEGRAL EQUATIONS AND RENORMALIZATION

To derive the integral equations for the vertices, for example, and for other Green's functions as well, one may add to the initial (interaction) Lagrangian density $g_0\bar{\Psi}\gamma_5\Psi\phi+e_0\bar{\Psi}\gamma_\mu\Psi A^\mu$, where g_0, e_0 denote the respective unrenormalized strong and electromagnetic coupling constants, source terms

$$\delta\mathcal{L}(x)=\overline{\Psi}(x)\eta(x)+\overline{\eta}(x)\Psi(x)+J_5(x)\phi(x)+J_\mu(x)A^\mu(x), \tag{1}$$

where η, $\overline{\eta}$, J_5 and J_μ are classical sources with the usual properties.[6] We will be interested in the case when these sources are all set equal to zero. The Schwinger equation for the proton propagator $S(x,y)$,

$$-iS(x,y)=(-i)\frac{\delta}{\delta\overline{\eta}(x)}(-i)\frac{\delta}{\delta\eta(y)}\langle 0\,\text{out}|0\,\text{in}\rangle\Big/\langle 0\,\text{out}|0\,\text{in}\rangle, \tag{2}$$

is easily obtained by the well-known procedure by the use of Schwinger's action (dynamical, variational, etc.) principle[6] to be (when η and $\overline{\eta}$ are set equal to zero)

$$\left\{\frac{\gamma\partial}{i}+m_0-\hat{\gamma}_\xi\left[(-i)\frac{\delta}{\delta J_\xi(x)}+\langle A^\xi(x)\rangle\right]\right\}S(x,y)=\delta(x-y), \tag{3}$$

where we have introduced a convenient "five-dimensional" notation $\hat{\gamma}_\xi\equiv(e_0\gamma_\mu, g_0\gamma_5)$, $A^\xi\equiv(A^\mu,\phi)$; m_0 denotes the unrenormalized mass of the proton and

$$\langle A^\xi(x)\rangle\equiv\frac{\langle 0\,\text{out}|A^\xi(x)|0\,\text{in}\rangle}{\langle 0\,\text{out}|0\,\text{in}\rangle}=(-i)\frac{\delta}{\delta J_\xi(x)}\ln(\langle 0\,\text{out}|0\,\text{in}\rangle). \tag{4}$$

The object $\langle 0\,\text{out}|0\,\text{in}\rangle$ denotes the vacuum-to-vacuum transition amplitude in the presence of the external sources. In a matrixlike notation for the inverse proton propagator we have

$$S^{-1}=\left(\frac{\gamma\partial}{i}+m_0-\hat{\gamma}_\xi\langle A^\xi\rangle\right)+\Sigma, \tag{5}$$

with

$$\Sigma\equiv-i\hat{\gamma}_\xi D^{\xi\xi'}S\frac{\delta}{\delta\langle A^{\xi'}\rangle}S^{-1}, \tag{6}$$

and

$$-iD^{\xi\xi'}=(-i)(\delta/\delta J_\xi)(-i)(\delta/\delta J_{\xi'})\ln(\langle 0\,\text{out}|0\,\text{in}\rangle). \tag{7}$$

By a straightforward iteration[7] in the parentheses on the right-hand side of Eq. (5), we easily generate the set of diagrams contributing to Σ, for example, as shown in Fig. 1 in the absence of external sources. All the lines in this figure denote exact proton (S), pion ($D_\pi\equiv D_{55}$), and photon ($D_\gamma\equiv D_{\mu\nu}$) propagators.

The strong and electromagnetic vertices are in turn defined by

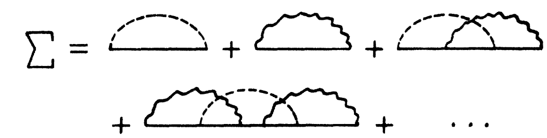

FIG. 1. Diagrammatic expansion of the proton self-energy part in terms of exact propagators. The dashed, solid, and wavy lines denote pions, protons, and photons, respectively. The external sources have all been set equal to zero.

$$\Gamma_5(x,x';\xi)\equiv-\frac{\delta}{g_0\delta\langle\phi(\xi)\rangle}S^{-1}(x,x'), \tag{8}$$

$$\Gamma_\mu(x,x';\xi)\equiv-\frac{\delta}{e_0\delta\langle A^\mu(\xi)\rangle}S^{-1}(x,x'). \tag{9}$$

Using the fact that Σ is a functional of S and $D^{\xi\xi'}$ we easily obtain (in symbolic notation)

$$-\frac{\delta}{g_0\delta\langle\phi\rangle}\Sigma=\int(-\delta\Sigma/\delta S)S(-\delta S^{-1}/g_0\delta\langle\phi\rangle)S+\int(-\delta\Sigma/\delta D)D(-\delta D^{-1}/g_0\delta\langle\phi\rangle)D, \tag{10}$$

where the notation $D\equiv D^{\xi\xi'}$ has been used for convenience. Clearly, the second term can be represented by two pieces joined exactly by two D lines as shown in Fig. 2 and hence gives rise to a closed fermion-loop contribution which by definition of the present work we omit. Repeating the same analysis for Γ_μ we easily see that in the *absence of closed fermion loops* the strong and the electromagnetic vertices satisfy the *same* integral equation (this is *not* true otherwise). In longhand we have (in the momentum representation)

$$\Gamma_\xi(p_+,p_-)=\gamma_\xi+\int\frac{(dp')}{(2\pi)^4}K(p_+,p_-;p'_+,p'_-)\times S(p'_+)\Gamma_\xi(p'_+,p'_-)S(p'_-), \tag{11}$$

where $\gamma_\xi\equiv(\gamma_5,\gamma_\mu)$, etc.; $p_\pm=p\pm k/2$, $p'_\pm=p'\pm k/2$, and $K\equiv-\delta\Sigma/\delta S$. All the sources are now set equal to zero. The diagrams contributing to K are easily obtained from the dependence of Σ on S (see, for example, Fig. 1) in Eq. (6). By the definition of

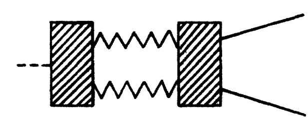

FIG. 2. A contribution to the strong vertex Γ_5 integral equation obtained by the functional differentiation of a D propagator [defined in Eq. (7)] with respect to $\langle A\rangle$. The zig-zag line (a D line) denotes, quite generally, a pion or a photon line with the external sources set equal to zero. This diagram clearly shows a closed fermion loop contribution to Γ_5.

the present work, closed fermion loops are to be omitted from K and from the over-all integral expression in (11). The kernel K does *not* also contain an intermediate state with *only* two proton lines cut vertically. We should emphasize that Eq. (11) is correct only in the absence of closed fermion loops. We could have also added to $\delta\mathcal{L}$ in Eq. (1), for example, a term

$$J_s(x)\phi_s(x)+\lambda\bar{\Psi}(x)\Psi(x)\phi_s(x)+\mathcal{L}_0[\phi_s],$$

where ϕ_s is a scalar field with the free Lagrangian density $\mathcal{L}_0$ (this may be also done in many other different ways) to obtain in the limit $J_s\to 0$ and $\lambda\to 0$ an integral equation for the scalar vertex defined by

$$\Gamma_s(x,x';\xi)\equiv\lim_{\lambda\to 0}\left[-\frac{\delta}{\lambda\delta\langle\phi_s(\xi)\rangle}S^{-1}(x,x')\right]. \quad (12)$$

Clearly, the integral equation for $\Gamma_s(p,p')$ is again identical to the one in Eq. (11) (in the absence of closed fermion loops). Therefore the index ξ in Eq. (11) now corresponds to $(s, 5, \mu)$, etc.

We introduce vertex-function renormalization constants Z_5, Z_1, Z_s and rewrite the renormalized version of Eq. (11) as, for example,

$$\tilde{\Gamma}_5=Z_5\gamma_5+\int\tilde{K}\tilde{S}\tilde{\Gamma}_5\tilde{S}, \quad (13)$$

with $\tilde{\Gamma}_5=Z_5\Gamma_5$, $\tilde{S}=S/Z_2$, $\tilde{\Gamma}_\mu=Z_1\Gamma_\mu$ ($Z_1=Z_2$ because of the Ward identity), $g^2=g_0{}^2(Z_2/Z_5)^2$, and $\tilde{\Gamma}_s=Z_s\Gamma_s$ (recall also $Z_\pi=1$, here). In Eq. (13) the inverse of the photon and the pion propagators is given by $D_\gamma{}^{-1}=q^2+\lambda^2$ and $D_\pi{}^{-1}=q^2+\mu^2$, where λ is a fictitious nonzero mass for the photon. The renormalizability of (13) is easily given by an induction proof.[8] One may write

$$\tilde{K}=e^2\tilde{K}_0+g^2\tilde{K}_1 \quad (14)$$

to emphasize how the lowest-order term starts in each of the parts in (14). Let us scale the couplings $e^2, g^2\to\kappa e^2, \kappa g^2$ and make an expansion in powers of κ and finally set the latter parameter equal to unity. Let $Z_5^{[0,1]}$ denote the contribution of Z_5 to lowest order in e^2 and zeroth order in g^2. An elementary computation (in the Feynman gauge) shows that $\tilde{\Gamma}_5^{[0,1]}$ is ultraviolet-cutoff-independent with $Z_5^{[0,1]}\sim 1-(e^2/4\pi^2)\ln(\Lambda^2/m^2)$ (where Λ^2 is an ultraviolet cutoff introduced in some covariant manner). Similarly, $\tilde{\Gamma}_5^{[1,0]}$ is Λ^2-independent with $Z_5^{[1,0]}\sim 1+(g^2/16\pi^2)\ln(\Lambda^2/m^2)$. From this we conclude that $\tilde{\Gamma}_5^{[1]}$ is Λ^2-independent to first order in κ with

$$Z_5^{[1]}\sim 1+[(g^2/16\pi^2)-(e^2/4\pi^2)]\ln(\Lambda^2/m^2).$$

A simultaneous study of $\tilde{\Gamma}_\mu^{[0,1]}$ and $\tilde{\Gamma}_\mu^{[1,0]}$ shows that $\tilde{\Gamma}_\mu^{[1]}$ is Λ^2-independent with

$$Z_2^{[1]}\sim 1-[(g^2/32\pi^2)+(e^2/16\pi^2)]\ln(\Lambda^2/m^2)$$

(in the Feynman gauge). Now we suppose that for some n, $\tilde{\Gamma}_5^{[n]}$, $\tilde{\Gamma}_\mu^{[n]}$ are Λ^2-independent. We easily see then from Eqs. (13) and (14) that the cutoff independence of $\tilde{\Gamma}_5^{[n+1]}$, $\tilde{\Gamma}_\mu^{[n+1]}$ is also true from the very definition of $Z_5^{[n+1]}$ and $Z_2^{[n+1]}$ as over-all subtraction constants. The renormalizability of (13) and (14) then follows for any n. A similar analysis may be also carried out for the more elementary object $\tilde{\Gamma}_s$. The above renormalizability discussion was given only for completeness.

In Appendix A the gauge transformation law for the renormalization constants Z_5, Z_2, Z_s is given. It is explicitly shown there (the expected results) that the objects (Z_5/Z_2), (Z_2/Z_s), and (Z_5/Z_s) are all *gauge-invariant*.

III. CALCULATION OF THE GAUGE FUNCTIONS

We note that the photon propagator appears always in the combination $e^2D_{\mu\nu}$ in the various integral equations for the vertices and the propagators. We may, quite generally, write

$$e^2D_{\mu\nu}(q)=e^2\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{(q^2+\lambda^2)}+\bar{G}(q^2)\frac{q_\mu q_\nu}{q^2}\frac{1}{(q^2+\lambda^2)}, \quad (15)$$

where the parameter λ is the fictitious nonzero mass for the photon introduced to avoid difficulties associated with the infrared-divergence problem. It is quite clear from the above-mentioned integral equations that one may develop a perturbation expansion of Γ_5, for example, in g^2 and e^2 and simultaneously make an expansion of the gauge function in the form

$$\bar{G}(q^2)=\sum_{m+n\geq 1}(g^2)^m(e^2)^nG_{m,n}(q^2). \quad (16)$$

A more convenient perturbative scheme is to scale the couplings $g^2, e^2\to\kappa g^2, \kappa e^2$ and make an expansion in κ:

$$\bar{G}(q^2)=\sum_{l\geq 1}(\kappa)^lG_l(q^2), \quad (17)$$

and finally set $\kappa=1$ as done in Sec. II. In this section we wish to calculate such a gauge function $\bar{G}(q^2)$ which makes Z_5, for example, ultraviolet-cutoff-independent order by order in perturbation theory. This we shall carry out, here only, in the so-called intermediate renormalization[8] by normalizing $\Gamma_5(p,p)$ at $p=0$ (rather than on the mass shell) so as to be able to work in the Euclidean region. We shall see that the gauge functions $\bar{G}(q^2)$ should then have logarithmic growth as $q^2\to\infty$. This will be also explicitly seen in the next section, in an elementary fashion, when discussing the Callan-Symanzik scaling equations. In particular we shall see that the sum in (16) reduces to a sum of terms each having a *well-defined* limit as $q^2 \to \infty$.

From Eqs. (13) and (14) we may rewrite the integral equation for Γ_5 in the form

$$\Gamma_5 = \gamma_5 + e^2 \int \tilde{K}_0 \tilde{S} \Gamma_5 \tilde{S} + g^2 \int \tilde{K}_1 \tilde{S} \Gamma_5 \tilde{S}, \tag{18}$$

where the "tilde" as usual is to denote renormalized objects. The $\tilde{K}$'s, for example, are expanded in g^2 rather than in g_0^2. We now consider the zero momentum transfer of (18) [see also Eq. (11)]. In any gauge Weinberg's theorem[9] says that (in spacelike direction) in perturbation theory

$$\tilde{S}(p) \underset{p^2 \to \infty}{\sim} \frac{1}{\gamma p} \times (\text{powers of } \ln p^2) + \frac{m}{p^2} \times (\text{powers of } \ln p^2)$$

and (19)

$$\tilde{K}_{0,1}(p, p; p+q, p+q) \underset{\substack{q^2 \to \infty \\ p = \text{fixed}}}{\sim} \frac{1}{q^2} \times (\text{powers of } \ln q^2),$$

where we strongly emphasize that *none* of the masses have been set equal to zero above. Now let us consider the lowest-order term in g^2, $\Gamma_5^{[1,0]}(p, p)$, and set $p = 0$ (intermediate renormalization[8]). A straightforward analysis shows that $(1/Z_5)^{[1,0]}$ is Λ^2-cutoff-independent in the gauge $G_{1,0}(q^2) = 1$. [It should already be clear at this stage that an orthodox perturbation expansion of Γ_5 $(1/Z_5)$ and $\overline{G}(q^2)$ may be carried out in a straightforward manner.] Similarly, $(1/Z_5)^{[0,1]}$ is finite in the gauge $G_{0,1}(q^2) = -3$. In the notation of (17) we then conclude that $(1/Z_5)^{[1]}$ is finite in the gauge $G_1(q^2)$,

$$G_1(q^2) = g^2 - 3e^2. \tag{20}$$

Now we proceed by induction[8] and assume that there exists a gauge function $G_n = \{G_{n,0}, \ldots, G_{0,n}\}$ which makes $(1/Z_5)^{[n]}$ ultraviolet-cutoff-independent to show that we can always find a gauge function G_{n+1} to make $(1/Z_5)^{n+1}$ also finite. In this case the first piece in Eq. (18) has from the kernel $\tilde{K}_0$ a contribution of the form

$$-i \int \frac{(dq)}{(2\pi)^4} (\kappa)^{n+1} G_{n+1}(q^2) \frac{\gamma^\mu q_\mu q_\nu \gamma^\nu}{q^2(q^2+\lambda^2)} \frac{1}{(\gamma q + m)} \gamma_5 \frac{1}{(\gamma q + m)} \tag{21}$$

to $(1/Z_5)^{[n+1]}$ coming from the diagram in Fig. 3 in $\tilde{K}_0$. By inspection we see that the remaining terms coming from $\tilde{K}_0$ and $\tilde{K}_1$ can depend only on the gauge functions $\{G_{n,0}, G_{n-1,1}, \ldots, G_{0,n}\}$ which have supposedly been determined before. Let $e^2(\tilde{K}_0')^{[n]}$ denote $e^2(\tilde{K}_0)^{[n]}$ subtracted from it the term

$$-i(\kappa)^{n+1} G_{n+1}(q^2)[\gamma^\mu q_\mu q_\nu \gamma^\nu / q^2(q^2+\lambda^2)]$$

in Eq. (21). Now the $\tilde{S}^{[n]}$'s have been all computed in the gauge G_n and have the behavior in Eq. (19) in perturbation theory. Clearly then the gauge function $G_{n+1}(q^2)$ is given by an arbitrary function which asymptotically behaves like (with the right-hand side formally averaged over angles first)

$$G_{n+1}(q^2) \underset{q^2 \to \infty}{\sim} [\, i(e^2 \tilde{K}_0' \tilde{S}\tilde{S})_{n+1} + i(g^2 \tilde{K}_1 \tilde{S}\tilde{S})_{n+1}]\, q^4, \tag{22}$$

where again we emphasize that *no* masses have been set equal to zero above. From Weinberg's theorem[9] we see then that

$$G_{n+1}(q^2) \underset{q^2 \to \infty}{\sim} C_{n+1}' + \text{powers of } \ln q^2, \tag{23}$$

where the right-hand side is determined in terms of the gauge functions $\{G_{n,0}, \ldots, G_{0,n}\}$. The constant C_{n+1}' may depend on g^2, e^2, and possibly on mass ratios in the theory. Since n is arbitrary, our result is *true* for all n. From the work of the next section we shall see that Z_s is finite in the same gauge as Z_5 and since (Z_5/Z_s) is gauge-invariant (see Appendix A), it follows that Z_s and Z_5 are equal (up to a finite multiplicative constant) in any gauge.

A similar analysis as the one above may be also carried out for a finite Z_2 and we may find a corresponding gauge function $\overline{G}(q^2)$ order by order in κ. This gauge function does not necessarily coincide with the corresponding one for a finite Z_5. Needless to say that with a gauge function having a property as in (23) (in perturbation theory), the theory (in the present context) is still renormalizable in the usual sense. For example, in pure electrodynamics the renormalized photon self-energy part has a behavior as in (23) in perturbation theory and the theory is renormalizable. The asymptotic behavior of $\tilde{S}(p)$ quoted in (19) is also true with a behavior of $\overline{G}$ of the type in (23) by definition of renormalizability. The cutoff independence of Z_5 (for example), of course, is not guaranteed unless the gauge functions G_n are suitably chosen as discussed above. The structure of the gauge functions discussed above will be clarified even further from the work of the next section, where the internal consistency of our treatment

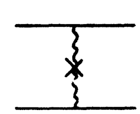

FIG. 3. The typical diagram contributing to the kernel $e^2\tilde{K}_0$ in the $(\kappa)^{n+1}$ order coming from the gauge term of the (free) photon propagator (denoted with the crossed wavy line) given by $-i(\kappa)^{n+1}G_{n+1}(q^2)[q_\mu q_\nu/q^2(q^2+\lambda^2)]$.

will be also shown.

It should be quite clear that the role the gauge term has taken above is a well-defined one even in the absence of electrodynamics. For example, in pure ps-ps dynamics, the Lagrangian density remains invariant if we redefine the Fermi field $\Psi(x)\to\Psi_\chi(x)=\Psi(x)\exp[-i\chi(x)]$ (thus parametrizing it by χ) and simultaneously introduce an additional (trivial) coupling of the form $\overline{\Psi}_\chi\gamma_\mu(\partial^\mu\chi/i)\Psi_\chi$, where χ is arbitrary and may be chosen at will. We may then quite arbitrarily introduce a function $F(x-x')=i\delta\langle(\chi(x)\chi(x'))_+\rangle$. By following such a procedure (to our knowledge, unconventional) one does not change the value of quantities, such as cross sections and self-masses (and the pion wave-function renormalization constant) since χ does not "participate" in the dynamics. Z_2 and Z_5, however, will depend on F with the Fermi field parametrized by F. The great value of such a trick is that the function F may be arbitrarily chosen at will (such as in making Z_5, for example, finite as discussed above for a specific choice of F) to facilitate the computation of physical quantities (with an underlying positive-definite metric) which are independent of F. The "classical" Green's function $G(x,x';\Phi^{\rm ext})$ in the presence of an external potential $\Phi^{\rm ext}$, appearing in $\langle 0\,\text{out}|0\,\text{in}\rangle$ (which in turn becomes parametrized by F), will be simply multiplied by $\exp\{i[F(x-x')-F(0)]\}$ as usual (see Appendix A).

IV. CALLAN-SYMANZIK SCALING EQUATIONS

To derive the Callan-Symanzik scaling equations for the various propagators of the theory we introduce an ultraviolet cutoff in the *free* pion and photon propagators as follows:

$$\begin{aligned} D_\pi(q^2)&=\frac{1}{(q^2+\mu^2)}\left(\frac{\Lambda^2}{q^2+\Lambda^2}\right),\\ e^2D_{\mu\nu}(q)&=e^2\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{(q^2+\lambda^2)}\,\frac{\Lambda^2}{(q^2+\Lambda^2)}\\ &\quad+\overline{G}(q^2)\,\frac{q_\mu q_\nu}{q^2}\,\frac{1}{(q^2+\lambda^2)}\frac{\Lambda^2}{(q^2+\Lambda^2)}, \end{aligned}\tag{24}$$

where λ is the fictitious mass for the photon. We make independent variations in the theory with respect to the proton (renormalized) mass m, the pion mass (squared) μ^2, and the photon mass (squared) λ^2, while keeping the unrenormalized (ps) coupling (squared) $g_0{}^2$ (and e^2), and the ultraviolet cutoff Λ fixed.[1,3] The gauge function $\overline{G}(q^2)$ generally depends on g^2, e^2 as well as on the masses m, μ, λ. (See also remark made at the end of Sec. III.)

Defining

$$\begin{aligned} &(1/Z_2)[L]\,Z_2=-\gamma_2,\\ &(1/m_0)[L]m_0=1+\hat\delta,\\ &(Z_2/Z_5)^2[L](Z_5/Z_2)^2=-\beta,\\ &\tilde\Gamma(p,p)=Z_2\left[\frac{\partial}{\partial\mu^2}S^{-1}(p)+\frac{\lambda^2}{\mu^2}\,\frac{\partial}{\partial\lambda^2}S^{-1}(p)\right],\\ &(1/\overline{G}(q^2))[L]\,\overline{G}(q^2)=\tilde F(q^2), \end{aligned}\tag{25}$$

and

$$[L]\equiv\left[m\frac{d}{dm}+\mu^2\frac{d}{d\mu^2}+\lambda^2\frac{d}{d\lambda^2}\right],\tag{26}$$

we easily obtain the scaling equation for the renormalized proton propagator to be in the usual manner, by applying the operator $[L]$ to $\tilde S^{-1}(p)$ and rearranging terms,

$$\begin{aligned}\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]\tilde S^{-1}(p)&=-\left[\gamma_2+\int(dq)\tilde F(q^2)\overline{G}(q^2)\left(\frac{\delta}{\delta\overline{G}(q^2)}\ln Z_2\right)\right]\tilde S^{-1}(p)\\ &\quad+[1+\hat\delta](m_0Z_2/Z_s)\tilde\Gamma_s(p,p)+\mu^2\tilde\Gamma(p,p),\end{aligned}\tag{27}$$

where the vertex function $\tilde\Gamma_s(p,p)$ has been introduced in Sec. II. Clearly, $\tilde\Gamma(p,p)$ is a renormalized cutoff-independent quantity. It vanishes like $(1/p)\times$[powers of $\ln(p^2)$] as $p^2\to\infty$ in perturbation theory[9] and it may be neglected in this limit. Now we rather strongly emphasize that because of the explicit functional differentiation appearing above with respect to $\overline{G}(q^2)$, the vertices $\tilde\Gamma_s$ and $\tilde\Gamma$ [and also throughout, for example, Eqs. (31) and (32)] are defined in terms of undifferentiated gauge functions. From the definition of $[L]$ in (26) and of γ_2 in (25) we easily see that the first square bracket on the right-hand side of Eq. (27) is given by

$$-\frac{1}{Z_2}\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]Z_2\equiv\hat\gamma_2,\tag{28}$$

and the functional derivative with respect to $\overline{G}(q^2)$ [in (27)] cancels out as expected since the latter is "already" renormalized [recall also $Z_3=1$ with no closed fermion loops; in this respect compare also Eqs. (11) and (18) which remind us that the vertex functions are multiplicatively renormalized]. The gauge function may be then kept fixed. Now we assume an expansion of the parameters β and $a\equiv[1+\hat\delta](m_0Z_2/Z_s)$ in the form

$$\beta = -\beta_0 + \beta_1 g^2 + \cdots , \tag{29}$$

$$a = a_0 + a_1 g^2 + \cdots \tag{30}$$

to show that all $\beta_0, \beta_1, \ldots ; a_0, a_1, \ldots$ (which are exactly treated in the electromagnetic coupling) are ultraviolet-cutoff-independent. To do this we derive also the scaling equation for the object $\tilde{S}(p)\tilde{\Gamma}_5(p,p)$, which is given by

$$\left[m\frac{\partial}{\partial m} + \mu^2\frac{\partial}{\partial \mu^2} + \lambda^2\frac{\partial}{\partial\lambda^2} + g^2\beta\frac{\partial}{\partial g^2}\right]\tilde{S}(p)\tilde{\Gamma}_5(p,p) = -(\beta/2)\tilde{S}(p)\tilde{\Gamma}_5(p,p) - [1+\hat{\delta}](m_0 Z_2/Z_s)\tilde{S}(p)\tilde{\Gamma}_s(p,p)\tilde{S}(p)\tilde{\Gamma}_5(p,p) + [1+\hat{\delta}](m_0 Z_2/Z_s)\tilde{S}(p)\tilde{\Gamma}_{5,s}(p,p) + \mu^2\tilde{\Gamma}'(p,p), \tag{31}$$

where

$$\tilde{\Gamma}_{5,s} \equiv \left(\frac{Z_s Z_5}{Z_2}\right)\frac{\partial}{\partial m_0}\Gamma_5(p,p)$$

and (32)

$$\tilde{\Gamma}'(p,p) \equiv \left(\frac{Z_5}{Z_2}\right)\left[\frac{\partial}{\partial\mu^2} + \left(\frac{\lambda^2}{\mu^2}\right)\frac{\partial}{\partial\lambda^2}\right] S(p)\Gamma_5(p,p).$$

The pulling out of the coefficients on the right-hand side of Eq. (31) to write the various expressions in terms of renormalized quantities is straightforward and easily obtained by drawing a few diagrams contributing to them, for example, to $\tilde{\Gamma}_{5,s}(p,p)$, and by making the necessary multiplicative renormalizations. We shall see that the self-consistency condition for the finiteness of the self-mass in pure electrodynamics (which we assume) requires that the parameter $\beta_0 > 0$ in Eq. (29). In pure electrodynamics $[m_0(Z_2/Z_5)]$ is finite (Ref. 10), and from the gauge invariance of (Z_2/Z_5) we easily see, for example, from Ref. 1 that

$$\beta_0(\alpha)/2 = \delta(\alpha) \tag{33}$$

and is cutoff-independent, where[7] ($\alpha \equiv e^2/4\pi$), $\delta(\alpha) = 3(\alpha/2\pi) + (\frac{3}{4})(\alpha/2\pi)^2 + \cdots$ [with the self-consistent range[7] for $\delta(\alpha)$: $0 < \delta(\alpha) < 2$—it is understood here that contributions from closed fermion loops are to be omitted]. We now proceed to discuss the problem of the cutoff independence of all the coefficients in (29) and (30). We first keep the ultraviolet cutoff fixed. The objects (Z_2/Z_s), (Z_2/Z_5) (see Appendix A) and hence β [see Eq. (A8)], m_0 (Ref. 3), and $\hat{\delta}$ [see Eq. (49)] are all gauge-invariant. To prove the cutoff independence of the parameters in (29) and (30) we may then work in any gauge we wish. Accordingly, we may work in the gauge in which Z_2 is Λ-independent (the analysis carried out in Sec. III shows that this can be achieved. This will in turn be shown to be self-consistent when we solve the scaling equations in this section). Also to discuss the cutoff independence of the parameters in (29) and (30) we do *not* have to consider the large-p behavior of Eqs. (27) and (31) and we may then work in a nonasymptotic region for such a proof. We rely on the fact that all the quantities with a tilde (renormalized) in Eqs. (27) and (31) are cutoff-independent and we assume that this is also the case when we sum up to all orders in the electromagnetic region (still working in a nonasymptotic region). If the coefficients in (29) and (30), which are treated exactly in the electromagnetic coupling from the beginning, are cutoff-independent, we may take the limit $\Lambda^2 \to \infty$ in (27) and (31) and drop all possible terms which vanish in this limit (and otherwise keep terms which do not vanish in this limit). Using the expansions in (29) and (30) and working in a nonasymptotic region in p, as mentioned above, we obtain for (27) expansions of the form

$$\begin{aligned}
&[L'\tilde{S}^{-1}]_0 - [L'\ln Z_2]_0\tilde{S}_0^{-1} = a_0(\tilde{\Gamma}_s)_0 + (\mu^2\tilde{\Gamma})_0 ,\\
&[(L'-\beta_0)(\tilde{S}^{-1})_1] - [(L'-\beta_0)(\ln Z_2)_1](\tilde{S}^{-1})_0 - [L'(\ln Z_2)_0](\tilde{S}^{-1})_1 = a_1(\tilde{\Gamma}_s)_0 + a_0(\tilde{\Gamma}_s)_1 + (\mu^2\tilde{\Gamma})_1 ,\\
&\cdots = \cdots ,
\end{aligned} \tag{34}$$

where

$$L' \equiv m\frac{\partial}{\partial m} + \mu^2\frac{\partial}{\partial\mu^2} + \lambda^2\frac{\partial}{\partial\lambda^2} . \tag{35}$$

Similarly, we obtain for (31) an expansion in the form

$$\begin{aligned}
&[L' - (\beta_0/2)](\tilde{S}\tilde{\Gamma}_5)_0 = -a_0(\tilde{S}\tilde{\Gamma}_s\tilde{S}\tilde{\Gamma}_5 - \tilde{S}\tilde{\Gamma}_{5,s})_0 + (\mu^2\tilde{\Gamma}')_0 ,\\
&[L' - (3\beta_0/2)](\tilde{S}\tilde{\Gamma}_5)_1 = -(\beta_1/2)(\tilde{S}\tilde{\Gamma}_5)_0 - a_1(\tilde{S}\tilde{\Gamma}_s\tilde{S}\tilde{\Gamma}_5 - \tilde{S}\tilde{\Gamma}_{5,s})_0 - a_0(\tilde{S}\tilde{\Gamma}_s\tilde{S}\tilde{\Gamma}_5 - \tilde{S}\tilde{\Gamma}_{5,s})_1 + (\mu^2\tilde{\Gamma}')_1 .\\
&\cdots = \cdots ,
\end{aligned} \tag{36}$$

From the first equation in (34) we see that a_0 is (Λ) cutoff-independent and the limit $\Lambda^2 \to \infty$ exists as is well known. From the second equation in (34) we also obtain that a_1 is cutoff-independent since β_0 is so. In turn we see from the second equation in (36) that β_1 is also cutoff-independent since β_0, a_0, and a_1 are so. The proof of the cutoff independence of the coefficients in (29) and (30) now easily follows by an elementary induction proof. This essentially follows since to any order n in g^2, Eq. (34) depends only on $(\beta_{n-1}, \ldots, \beta_0; a_n, \ldots, a_0)$ (i.e., not β_n) and Eq. (36) depends on $(\beta_n, \ldots, \beta_0; a_n, \ldots, a_0)$. Now that we have established the cutoff independence of the coefficients (which are treated exactly in the electromagnetic coupling) of the gauge-invariant objects β and a in (29) and (30), we may go back to Eq. (27), use Eq. (28), and infer that $\hat{\gamma}_2$ is cutoff-independent in *any* gauge. We may take the orthodox attitude and choose a gauge such that $\hat{\gamma}_2 = 0$ and hence have

$$\left[m \frac{\partial}{\partial m} + \mu^2 \frac{\partial}{\partial \mu^2} + \lambda^2 \frac{\partial}{\partial \lambda^2} + g^2 \beta \frac{\partial}{\partial g^2} \right] Z_2 = 0 , \quad (37)$$

or we may make an expansion of the form

$$\hat{\gamma}_2 = \gamma + \gamma_1 g^2 + \cdots \quad (38)$$

and put a restriction on γ, for example, by fixing the nonvanishing part of the gauge function when $g^2 \to 0$ [see Sec. III and Eq. (16)]. In the Feynman gauge, for example,[3] $\gamma = -(\alpha/2\pi) + \cdots$, and in the generalized Landau gauge[7] $\gamma = 0$.

Upon writing

$$\tilde{S}^{-1}(p) = \gamma p F(p^2) + m G(p^2) \quad (39)$$

and making an elementary use of Weinberg's theorem[9] to drop terms in the renormalized objects (with a tilde) which vanish in perturbation theory in the limit of large p and keep terms which do not vanish in the usual manner, we obtain the scaling equations[1,3] [see Eq. (27)]

$$\left[m \frac{\partial}{\partial m} + \mu^2 \frac{\partial}{\partial \mu^2} + \lambda^2 \frac{\partial}{\partial \lambda^2} + g^2 \beta \frac{\partial}{\partial g^2} + \hat{\gamma}_2 \right] F(p^2) \underset{p^2 \to \infty}{\sim} 0 \quad (40)$$

and

$$\left[m \frac{\partial}{\partial m} + \mu^2 \frac{\partial}{\partial \mu^2} + \lambda^2 \frac{\partial}{\partial \lambda^2} + g^2 \beta \frac{\partial}{\partial g^2} + \hat{\gamma}_2 \right] m G(p^2) \underset{p^2 \to \infty}{\sim} a J(p^2) , \quad (41)$$

where $J(p^2)$ satisfies the equation[3]

$$\left[m \frac{\partial}{\partial m} + \mu^2 \frac{\partial}{\partial \mu^2} + \lambda^2 \frac{\partial}{\partial \lambda^2} + g^2 \beta \frac{\partial}{\partial g^2} + \hat{\gamma}_2 - \hat{\delta} \right] J(p^2) \underset{p^2 \to \infty}{\sim} 0 \quad (42)$$

(it follows also that $\hat{\delta}$ is cutoff-independent). Equations (40) and (42) have the same structure. We first solve Eq. (40). We use the expansions in (29) and (38) to obtain

$$\left[m \frac{\partial}{\partial m} + \mu^2 \frac{\partial}{\partial \mu^2} + \lambda^2 \frac{\partial}{\partial \lambda^2} + \gamma - (\beta_0/2) \right] J_0 \sim 0 \quad (43a)$$

and for $n \geq 1$,

$$\left[\frac{m}{n} \frac{\partial}{\partial m} + \frac{\mu^2}{n} \frac{\partial}{\partial \mu^2} + \frac{\lambda^2}{n} \frac{\partial}{\partial \lambda^2} - \beta_0 + \frac{\gamma - (\beta_0/2)}{n} \right] J_n + \sum_{\substack{r+s=n \\ 1 \leq r < n}} \left[\beta_s \left(\frac{r}{n} \right) + \frac{(\gamma_s - \delta_s)}{n} \right] J_r \sim 0 . \quad (43b)$$

The solution to (43) is elementary and is given by ($e^2 \neq 0$)

$$J(p^2) \underset{p^2 \to \infty}{\sim} \left(\frac{p^2}{m^2} \right)^{[\gamma - (\beta_0/2)]/2} \left[C_0 + C_1 \left(\frac{p^2}{m^2} \right)^{-\beta_0/2} + C_2 \left(\frac{p^2}{m^2} \right)^{-2(\beta_0/2)} + \cdots \right] , \quad (44)$$

where $C_n \equiv C_n(\mu^2/m^2, \lambda^2/m^2; g^2, e^2)$ for $n = 1, 2, \ldots$. Equation (40) has a similar solution with β_0 set equal to zero outside the square bracket in (44). Upon substituting the solution (44) in (41) and using the definition (39) we finally obtain ($e^2 \neq 0$)

$$\tilde{S}^{-1}(p) \underset{p^2 \to \infty}{\sim} \gamma p \left(\frac{p^2}{m^2} \right)^{\gamma/2} \left[f_0 + O\left(\left(\frac{p^2}{m^2} \right)^{-\beta_0/2} \right) \right] + m \left(\frac{p^2}{m^2} \right)^{[\gamma - (\beta_0/2)]/2} \left[C_0 + O\left(\left(\frac{p^2}{m^2} \right)^{-\beta_0/2} \right) \right] , \quad (45)$$

where the coefficients f_0, C_0 (and the coefficients of the corrections as well) have similar dependence on the mass ratios and g^2, e^2 as the C_n's in (44). The parameters γ and β_0 in Eq. (45) are infrared-independent.[3] From Eq. (28) we also obtain ($e^2 \neq 0$)

$$Z_2 \underset{\Lambda^2 \to \infty}{\sim} \left(\frac{\Lambda^2}{m^2} \right)^{\gamma/2} \left[z_0 + O\left(\left(\frac{\Lambda^2}{m^2} \right)^{-\beta_0/2} \right) \right] . \quad (46)$$

The result (46) is quite interesting and states that when the theory is *summed up to all orders* (in the manner discussed above), Z_2 is finite in the generalized Landau gauge,[7] i.e., for $\gamma = 0$. This in turn means that when the gauge function $\overline{G}(q^2)$ which makes Z_2 finite (see Sec. III) is summed to all orders and when the limit $q^2 \to \infty$ is taken, the just-mentioned function reduces to a sum of functions of the form inside the square bracket in Eq. (44) (with a nontrivial dependence on g^2), with each being well defined in this limit. More specifically, $\overline{G}(q^2)$ which makes Z_2 finite is

$$\overline{G}(q^2) = -e^2 G_L + O((m^2/q^2)^{\beta_0/2}),$$

where G_L is the generalized Landau gauge[7] $G_L = -(3e^2/32\pi^2) + \cdots$. We make no attempt to discuss the convergence of a sum of the form in (44) [or (45)], even for $p^2 \gg m^2(g^2)^{2/\beta_0}$ [note that $C_n \sim O((g^2)^n)$ for $n \geq 1$, etc.]. One may of course keep vanishing terms in $\overline{G}(q^2)$ for $q^2 \to \infty$ if one wishes. As mentioned above, for example, Z_2 is finite in the gauge $-e^2 G_L$ plus corrections of the form $O((m^2/q^2)^{\beta_0/2})$, whose coefficients may be so adjusted to cancel *even* the vanishing terms inside the square bracket in (46) (in this connection see also Appendix A for the gauge transformation property of Z_2), etc. Equation (37) has also a solution of the form in (46) (with $\gamma \to 0$) for $\hat{\gamma}_2 = 0$. This is clearly expected since the effective strong coupling $g^2(m^2/p^2)^{\beta_0/2}$ vanishes at high energies $[(|p^2|)^{1/2} \gg m(g^2)^{1/\beta_0}]$ and does not contribute to the gauge function, asymptotically, when the theory is summed up to all orders as discussed above. In this latter case the coefficient z_0 and the coefficients of the terms neglected inside the square bracket in (46) will be simply changed to some other finite constant (s) as can be easily checked. To obtain the expression for Z_2 in any gauge, one may make use of its well-known transformation law given in Appendix A. The asymptotic behavior of $\tilde{\Gamma}_5(p,p)$ is similarly derived as for $\tilde{S}^{-1}(p)$, for example from Eq. (31), to be $(e^2 \neq 0)$

$$\tilde{\Gamma}_5(p,p) \underset{p^2\to\infty}{\sim} \gamma_5 \left(\frac{p^2}{m^2}\right)^{[\gamma-(\beta_0/2)]/2} \left[l_0 + O\left(\left(\frac{p^2}{m^2}\right)^{-\beta_0/2}\right)\right] \tag{47}$$

[up to a possible, but finite, multiplicative factor of the form $(1 + (l_0'/l_0)\gamma p/p)$ for $p^2 \to \infty$] and for Z_5 we have $(e^2 \neq 0)$

$$Z_5 \underset{\Lambda^2\to\infty}{\sim} \left(\frac{\Lambda^2}{m^2}\right)^{[\gamma-(\beta_0/2)]/2} \left[z_0' + O\left(\left(\frac{\Lambda^2}{m^2}\right)^{-\beta_0/2}\right)\right]. \tag{48}$$

Equation (48) states that when the theory is summed up to all orders as discussed above and with the generalized Landau gauge $\gamma = 0$, Z_5 is identically equal to zero with *no* ultraviolet cutoff. Z_5, however, is finite for $\gamma = \beta_0/2$ (again in the summed up theory in the manner discussed above and *then* the limit $\Lambda^2 \to \infty$ taken). An elementary analysis shows that this corresponds to the gauge function

$$\overline{G}(q^2) = -e^2[G_L - (4\pi^2\beta_0/e^2)] + O((m^2/q^2)^{\beta_0/2})$$

(see also Sec. III),

$$[G_L - (4\pi^2\beta_0/e^2)] = 3 + O(e^4).$$

The gauge transformation law for Z_5 is given in Appendix A. We should warn the reader that the statements just made for Z_2 and Z_5 are based on summed up expressions and then the limit $\Lambda^2 \to \infty$ is taken and note that the coefficients z_0, z_0' [and the coefficients of the small corrections inside the square brackets in Eq. (46) and (48)] have nontrivial dependence on g^2 as well. For example, *if* we expand the corrections inside the square bracket in Eq. (48) in powers of β_0 we then have to generate a dependence on g^2 in the corresponding gauge function for ensuring the finiteness of Z_5 as also shown in Eq. (20). So one should be very careful when making contact with perturbation theory results.

Finally we also have from (25) for m_0

$$\left[L' + g^2\beta \frac{\partial}{\partial g^2} - \hat{\delta} \right]\left(\frac{m_0}{m}\right) = 0, \tag{49}$$

with the solution $(e^2 \neq 0)$

$$m_0 \underset{\Lambda^2\to\infty}{\sim} m\left(\frac{\Lambda^2}{m^2}\right)^{-\beta_0/4} \left[u_0 + O\left(\left(\frac{\Lambda^2}{m^2}\right)^{-\beta_0/2}\right)\right], \tag{50}$$

with u_0 having a similar dependence on the other parameters as $C_0, C_1, \ldots$ in (44), and the self-mass $\delta m \equiv m - m_0$ is asymptotically finite. Needless to say all the various corrections occurring inside the square brackets, for example, in Eq. (45) cannot be necessarily kept which may be *beyond* the range of validity of the accuracy of our treatment (the main thing, of course, here is their very interesting damping property). For example, in the just-mentioned corrections, terms with a sufficiently large n subscript (e.g., coefficients of $f_0, \ldots, f_n$), say, should be omitted which may be of the same order as various terms omitted in perturbation theory (with vanishing properties) for the renormalized objects (with a tilde) occurring asymptotically, for example, in pure electrodynamics. From Eqs. (45), (46), and (50) we may also write the expression for the *unrenormalized* $S^{-1}(p)$ for both p^2, $\Lambda^2 \to \infty$. We may temporarily work in a class of constant gauges independent of the various masses. Upon taking the partial derivative with respect to μ^2 keeping m_0, g_0^2, λ^2 (and Λ^2) fixed and agreeing to drop terms which vanish rapidly enough in the above-mentioned limit in *unrenormalized* perturbation theory, together with the above-mentioned equations, we easily derive that

$$(\partial/\partial\mu^2)(f_0/z_0)=0=(\partial/\partial\mu^2)(C_0/z_0u_0).$$

Repeating the same procedure by keeping m_0, g_0^2, μ^2 (and Λ^2) fixed and varying λ^2, we also obtain

$$(\partial/\partial\lambda^2)(f_0/z_0)=0=(\partial/\partial\lambda^2)(C_0/z_0u_0).$$

These two results then relate the *explicit* μ and λ dependence of the just-mentioned integration constants as in Ref. 3. Similarly, we obtain $(\partial/\partial\mu^2)(l_0/z_0')=0$, $(\partial/\partial\lambda^2)(l_0/z_0')=0$ for $\Gamma_5(p,p)$ and Z_5. Gauge transformation properties (to arbitrary gauges) of these constants may be also derived from Appendix A as in Ref. 3.

We should remind the reader of the very clear but important fact that the asymptotic behavior of the various objects given above is *true only in the presence of the electromagnetic interaction* (i.e., $e^2\neq 0$). Otherwise, we obtain the well-known perturbation expansion in pure ps-ps coupling (in the present context with no closed fermion loops) upon integrating the Callan-Symanzik scaling equations. Finally, we note that due to the absence of an analog to the Ward identity leading to $Z_1=Z_2$ for Z_5 the derivative with respect to g^2 appears in the scaling equations even in the absence of closed fermion loops. Needless to say the above scaling equations for the various amplitudes above may be also solved by Symanzik's original method by defining an effective coupling g_{eff}^2 first and then integrating the various equations in the usual manner.[4] One then obtains *identical* results as above upon expanding the exponents—multiplying the various amplitudes written as functions of g_{eff}^2 rather than g^2—in powers of g_{eff}^2. We have preferred the above method by expanding in powers of g^2 first, integrating the various equations, and then identifying the effective strong coupling in an elementary fashion. The final results are, of course, the same.

V. THE STABILITY OF THE EIGENVALUE CONDITION FOR THE FINE-STRUCTURE α

Now we study the photon self-energy part with no closed fermion loops with the exception, of course, of the over-all closed loop defining the self-energy part. Let $\alpha\pi^{[1]}(q^2)$ denote the unrenormalized photon self-energy part after having extracted the two powers of the momentum q to define the vacuum-polarization tensor. $\pi^{[1]}$ is given by

$$\pi^{[1]}(q^2)=Z_2 i\int\frac{(dp)}{(2\pi)^4}\mathrm{Tr}[\gamma^\mu\tilde S(p+q)\tilde\Gamma_\mu(p+q,p)\tilde S(p)],\tag{51}$$

where *it is understood that two powers of q are to be removed in (51)*. Now we rely on the gauge invariance of the photon self-energy part and work in the gauge with a finite Z_2, for which $\hat\gamma_2=0$ [in (28)], which is effectively nothing but the generalized Landau gauge with the coefficients $(z_0, z_1, \ldots)$ in (46) simply changed to some other coefficients as clearly discussed in the previous section. We consider the expression [to be more precise we multiply the integrand in Eq. (51) by a form factor $\Lambda^2/(p^2+\Lambda^2)$ for the moment (the limit $\Lambda^2\to\infty$ will be taken later) to render the over-all integration in (51) meaningful]

$$\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]\pi^{[1]}(q^2)\equiv\chi^{[1]}(q^2),\tag{52}$$

and make use of (37). A typical contributing factor in the integrand of $\chi^{[1]}(q^2)$ is for $p^2\to\infty$

$$\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]\tilde S(p)$$
$$=-\tilde S(p)[a\tilde\Gamma_s(p,p)+\mu^2\tilde\Gamma(p,p)]\tilde S(p),\tag{53}$$

and an elementary consideration as in the previous section shows that the right-hand side of Eq. (53) (for $p^2\to\infty$) vanishes like

$$\left(\frac{1}{p^2}\right)\left[\left(\frac{m^2}{p^2}\right)^{\beta_0/2}+O\left(\left(\frac{m^2}{p^2}\right)^{2(\beta_0/2)}\right)\right].\tag{54}$$

A similar analysis also holds when considering the factor $\tilde\Gamma_\mu(p,p)$ for $p^2\to\infty$. [The so-called overlap divergence problem is of no significance here in $\chi^{[1]}$ and in the integral (51) in a summed up theory with a finite Z_2.] By simple power counting we then see that (remembering the two powers of q that are to be subtracted) at worst

$$\chi^{[1]}(q^2)\underset{q^2\to\infty}{\sim}\frac{1}{q}\left[\left(\frac{m^2}{q^2}\right)^{\beta_0/2}+\cdots\right],\tag{55}$$

and hence the left-hand side of (52) vanishes asymptotically and is Λ-cutoff-independent as we let $\Lambda^2\to\infty$.

Defining the single-loop contribution

$$\rho^{[1]}=\left[(1/Z_3)\left(m\frac{d}{dm}+\mu^2\frac{d}{d\mu^2}+\lambda^2\frac{d}{d\lambda^2}\right)Z_3\right]^{[1]}\tag{56}$$

and using the definition (52) we immediately obtain the scaling equation for the renormalized single closed fermion-loop contribution $\alpha\pi_c^{[1]}$ to the renormalized photon self-energy part to be

$$\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]\alpha\pi_c^{[1]}(q^2)=\rho^{[1]}+\alpha\chi^{[1]}(q^2) \quad (57)$$

coming from

$$L(\alpha\pi_c^{[1]})=L(\alpha\pi^{[1]})-L(1)+[(1/Z_3)LZ_3]^{[1]}Z_3^{[0]},$$

where L is the operator defined on the left-hand side of the above equation—cf. Eqs. (7a) and (10) in Ref. 1. When $q^2\to\infty$ we can omit the second term on the right-hand side of Eq. (57) and immediately infer that $\rho^{[1]}$ is Λ-cutoff-independent. Asymptotically we then have

$$\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}\right]\alpha\pi_c^{[1]}(q^2)=\rho^{[1]}, \quad q^2\to\infty. \quad (58)$$

We expand

$$\rho^{[1]}=-2\alpha F^{[1]}(\alpha)+\rho_1 g^2+\cdots \quad (59)$$

to obtain the solution of (58) to be simply ($e^2\neq 0$)

$$\alpha\pi_c^{[1]}(q^2)\underset{q^2\to\infty}{\sim}C^{[1]}+\alpha F^{[1]}(\alpha)\ln(q^2/m^2)+O((m^2/q^2)^{\beta_0/2}), \quad (60)$$

and hence the eigenvalue condition $F^{[1]}(\alpha)=0$ is not altered, i.e., a possible zero of $F^{[1]}(x)$ does *not* "move" in the presence of the strong coupling. The limit $\lambda\to 0$ can be taken in the gauge-invariant object Z_3.[1,3,11]

Before closing this section, we wish to make a final and important remark. If we expand the corrections in Eq. (60) for $\pi_c^{[1]}(q^2)$ ($q^2\to\infty$) in powers of β_0, we see that we generate not only *a single power of* $\ln q^2$, *which "modifies" the eigenvalue condition, but also arbitrary powers of the latter.* The coefficient of the single power of $\ln q^2$ is easily seen in this case to be a function of e^2 *and* g^2 as well—a readily checked perturbation-theory result. Accordingly, one should be very careful when making contact with perturbation-theory results. Questions of this sort originated the investigation carried out in this work.

VI. CONCLUSION

We have made a study of quantum electrodynamics in the presence of neutral-meson theory with ps-ps coupling at short distances without closed fermion loops. We have expanded the Callan-Symanzik scaling function in powers of g^2 (and exactly treated in the electromagnetic coupling), proved their cutoff independence, and then solved the various scaling equations in Sec. IV in an elementary fashion by resumming back in g^2. It was also seen that the effective strong coupling at very high energies $(|p^2|)^{1/2}\gg m(g^2)^{1/\beta_0(\alpha)}$ is $g^2(m^2/p^2)^{\beta_0(\alpha)/2}$ [rather than $g^2\ln(p^2/m^2)$] and vanishes asymptotically. Clearly, a typical π^0-π^0 scattering box graph with a strong vertex in (47) (in the generalized Landau gauge) in a skeleton expansion (in the present context) is automatically *finite*. We have then shown the stability of the eigenvalue condition $F^{[1]}(\alpha)=0$ in the presence of the strong coupling. Another type of "solution" to the problem, in the present context, is discussed in Appendix B which, however is neither *physically* nor *technically* very attractive for various mentioned reasons. The rule for checking the stability of the eigenvalue condition for α in the sense discussed in Sec. V in other field theories as well is simple. One has to check the vanishing (sufficiently fast) of a corresponding expression to Z_5/Z_2 in pure electrodynamics [see Eqs. (46), (48), (59), and (60)] in the limit of infinite cutoff (and not otherwise).

The single-closed-fermion-loop contribution to the pion self-energy part and the implication of our present results on the finiteness problem of the full theory will be discussed in the following paper.

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APPENDIX A: GAUGE TRANSFORMATIONS

We consider general gauge transformations of the (free) photon propagator of the form[12]

$$D_{\mu\nu}(x)\to D_{\mu\nu}(x)+\partial_\mu\partial_\nu M(x), \quad (A1)$$

where $M(x)$ is an arbitrary function with a cutoff Λ^2 as defined quite generally in Eq. (24) and if, for example, $D_{\mu\nu}(x)$ on the right-hand side of Eq. (A1) denotes the propagator in the Landau gauge (with a cutoff) then

$$e^2M(x)=-\frac{i}{32\pi^4}\int_0^\infty\frac{dk^2d\Omega_k}{(k^2+\lambda^2)}\frac{\Lambda^2}{(k^2+\Lambda^2)}\overline{G}(k^2)e^{ikx}, \quad (A2)$$

and converges for $\overline{G}(k^2)$ increasing with arbitrary powers of $\ln k^2$ as $k^2\to\infty$ with a fixed Λ^2. We may arbitrarily choose $\overline{G}(k^2)$ to be finite for $k^2\to 0$ such that $M(x)\to 0$ as $x^2\to\infty$. We note that for $\xi\equiv(5,s)$, we have the corresponding transformation

$$\langle(\Psi(x)\phi_\xi(z)\overline{\Psi}(y))_+\rangle=\left[-i\frac{\delta}{\delta J^\xi(z)}\langle(\Psi(x)\overline{\Psi}(y))_+\rangle\right]_{\text{sources},\ \lambda=0}$$
$$\to\exp\{ie^2[M(x-y)-M(0)]\}\langle(\Psi(x)\phi_\xi(z)\overline{\Psi}(y))_+\rangle, \tag{A3}$$

since[12]

$$\langle(\Psi(x)\overline{\Psi}(y))_+\rangle \to \exp\{ie^2[M(x-y)-M(0)]\}\langle(\Psi(x)\overline{\Psi}(y))_+\rangle. \tag{A4}$$

From the very definition of $\Gamma_5(p,p')$ and $\Gamma_s(p,p')$ we easily see that when we set p and p' on the mass shell with $\xi\equiv(5,s)$, $\gamma_\xi=(\gamma_5,\underline{1})$

$$\overline{u}(p)\Gamma_\xi(p,p')u(p')=[\overline{u}(p)\gamma_\xi u(p')]F_\xi(q^2),\quad q\equiv p-p' \tag{A5}$$

where $F_\xi(q^2)$ is some scalar function which has the transformation property[12]

$$F_\xi(q^2)\to\exp[ie^2M(0)]F_\xi(q^2). \tag{A6}$$

Therefore irrespective at which normalization point of $F_\xi(q^2)$, $(1/Z_5)$, and $(1/Z_s)$ are defined, the transformation law of the latter quantities is always the same, i.e.,

$$\frac{1}{Z_\eta}\to\frac{1}{Z_\eta}\exp[ie^2M(0)], \tag{A7}$$

where now we may take $\eta\equiv(2,5,s)$. Clearly the objects (Z_5/Z_2), (Z_5/Z_s), (Z_s/Z_2) are all gauge-invariant. One may define $(1/Z_5)$, $(1/Z_s)$ at zero momentum transfer, for example (or at some other convenient point). The definition of the renormalized (ps) coupling depends on the normalization point of $F_5(q^2)$ for defining $(1/Z_5)$. The quantity $(1/Z_1)$ $(Z_1=Z_2)$ is, of course, defined at zero momentum transfer. For discussing renormalization problems it is more convenient to make such normalizations at zero momentum transfer as was done in Sec. II. In the gauge transformation laws given above it is assumed that the fictitious nonzero mass λ for the photon is kept fixed. We should emphasize that the class of gauge functions $\overline{G}(k^2)$ corresponding to the various renormalization constants (in which they are separately finite) are regular in the sense that they sum up, as discussed in Sec. IV, to a sum of (regular) functions which all have a well-defined limit as $k^2\to\infty$. (We do not attempt to discuss convergence problems of such sums even in the asymptotic region.)

For reference we, here, write the scaling equation for the object $(Z_5/Z_2)^2$:

$$\left[m\frac{\partial}{\partial m}+\mu^2\frac{\partial}{\partial\mu^2}+\lambda^2\frac{\partial}{\partial\lambda^2}+g^2\beta\frac{\partial}{\partial g^2}+\beta\right]\left(\frac{Z_5}{Z_2}\right)^2=0. \tag{A8}$$

APPENDIX B: ALTERNATIVE TYPE OF SOLUTION

In this appendix we wish to point out (with no rigor) a different type of solution to the problem (in the present context) which is, however, far less interesting physically and technically more obscure than the one given in the bulk of the present paper. This solution corresponds to the special case when the function β defined in Eq. (25) [see also Eq. (A8)], which depends on g^2 and e^2, vanishes [note that this is a gauge-independent statement—see Eq. (A8) and Appendix A] for some special dependence of its arguments, for example. From Eq. (A7) we immediately see in this case that both Z_5 and Z_2 become finite in the same gauge which is clearly not in the spirit of (at least) the low-order perturbation theory results (see Secs. II and III). Accordingly both $\Gamma_\mu(p,p)$ and $\Gamma_5(p,p)$ are finite in the same gauge and the effective strong coupling (here) formally defined by

$$\lim_{p^2\to\infty}\{\mathrm{Tr}[g\gamma p\gamma_5\tilde{S}(p)\tilde{\Gamma}_5(p,p)]\}^2$$

does not vanish at all short distances and the strong interaction never gets damped out, however high the energies may be. The latter is not physically very interesting. By simple power counting, one sees that a π^0-π^0 scattering box graph (in the present context) is also divergent. Finally by working in the gauge which makes Z_2 finite, we also see that $\alpha\pi_c^{[1]}(q^2)$ grows with a single power of $\ln(q^2)$ (for $q^2\to\infty$) [see (57)] whose coefficient may now depend on both g^2 and e^2 and is not the Adler-Baker-Johnson function any more. We now summarize the consequences of this alternative type of solution in the present context:

(i) Z_5 and Z_2 are finite in the *same* gauge and it is *not* in the spirit of the low-order perturbation-theory results as seen in Secs. II and III.

(ii) The effective strong coupling *does not damp out* at any energies, however high they may be, and is not physically very attractive.

(iii) A single-closed fermion loop such as in a π^0-π^0 scattering box graph is divergent [compare this with the situation occurring in the solution presented in the bulk of the present paper discussed in Sec. VI; this is also true for $Z_4(\equiv Z_\pi)$, the wave-amplitude renormalization constant for the pion, without additional restrictions].

(iv) Extension of the analysis to the full theory due to the above reasons may become quite technically obscure without more, possibly severe, restrictions and additional counterterms.

(v) The Adler-Baker-Johnson function changes to some other function which now depends on the strong coupling as well. This new function is again the coefficient of a term with a single power of $\ln(q^2)$ (for $q^2 \to \infty$) in $\alpha\pi_c^{[1]}(q^2)$. (This together with the assumption of the vanishing of β does not necessarily overdetermine the parameters of the theory.) The whole attractive idea of determining α *within* pure electrodynamics,[1] according to the present dynamics with such an alternative solution, may be destroyed.

Based on the above (i)–(v) points, the author feels that the solution given in the bulk of the present paper is far more *physically* and *technically* attractive than the alternative type of solution presented in this appendix.

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Stability of the eigenvalue condition for the fine-structure constant α and short-distance behavior in strong interaction. II

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In the preceding paper, we have made a preliminary study of the short-distance behavior of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar coupling. We have learned in particular, by an elementary summation procedure, that in the single-fermion-loop contribution to the renormalized photon self-energy part, the Adler-Baker-Johnson eigenvalue condition for the fine-structure constant α, $F^{[1]}(x)|_{x=\alpha} = 0$, remains unaltered. We extend our results, in the single-fermion-loop context, to the pion self-energy part. As a consequence of the vanishing of the effective strong coupling at high energies, π^0-π^0 scattering graphs are finite and no ϕ^4 counterterm is required. We finally infer from the above work that the photon self-energy part in the *multi*loop contribution is asymptotically finite at the eigenvalue $[F^{[1]}(x)|_{x=\alpha} = 0]$ independently of the value of the strong coupling. The point $x = \alpha$ is the assumed (infinite order) zero of the single-loop electromagnetic-current-correlation functions in mass-zero pure electrodynamics.

In the preceding paper[1] we have studied the short-distance behavior of quantum electrodynamics in the presence of neutral-meson theory with pseudoscalar-pseudoscalar (ps-ps) coupling without closed fermion loops. The scaling equations for the various components (propagators, vertices, etc.) of the theory have been solved. It was then shown that in the single-closed-fermion-loop contribution $\pi_c^{[1]}$ to the (renormalized) photon self-energy part, the Adler-Baker-Johnson eigenvalue condition[2,3] for the (renormalized) fine-structure constant[2] α, $F^{[1]}(\alpha) = 0$, remains *unaltered*. This means that a possible zero of $F^{[1]}(x)$ does *not* "move" in the presence of the strong coupling. This leads to the beautiful idea that the value of α may be possibly determined within *pure* electrodynamics[2] (i.e., electrodynamics in isolation from the rest of the world). The mechanism which is responsible for the stability of the eigenvalue condition is that the *effective* strong coupling vanishes at high energies.[1] The approach we have used for the investigation of the above problem was through