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Single-particle distribution in the hydrodynamic and statistical thermodynamic models of multiparticle production

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We find that the single-particle distribution EdN/d^3p for an expanding relativistic gas described by a distribution function obeying the Boltzmann transport equation is not of the form of an integral over collective motions of a velocity weight function times a "Lorentz-transformed" rest-frame distribution function. This casts doubt on the algorithms of Milekhin and Hagedorn for determining the single-particle distribution function in their models of particle production. For the hydrodynamic model, the correct algorithm is presented.

With the advent of new high-energy accelerators, there has been a revival of interest in many-body approaches to particle production. In particular, the statistical thermodynamic model of Hagedorn[1] and Landau's hydrodynamic model[2] have had considerable success in fitting single-particle inclusive data. Recent review papers have summarized the history and successes of these models.[3-5] In both models, one assumes that the collision process yields a distribution of collective motions. In Hagedorn's approach these collective motions are called fireballs; in Landau's approach the collective motions are that of the hadronic fluid and one has an entropy and energy distribution in terms of the fluid velocity. In both models one assumes that in the local rest frame the distribution of momenta is isotropic and is described by either a Bose or a Fermi distribution of the observed particle.

The question to which we address ourselves is whether the momentum distribution in the center-of-mass frame is given by the probability of finding a particle with collective velocity $\vec{v}$ times the Lorentz-boosted thermal distribution normalized to the total number of particles. The invariant single-particle distribution that follows from this assumption is[1b,6]

$$E\frac{dN}{d^3p} \overset{?}{=} \int \frac{dN}{d^3v}\,\frac{g(\bar{E},\bar{T}(\vec{v}))}{\bar{n}(\bar{T}(\vec{v}))}\,\bar{E}d^3v\,, \tag{1}$$

where $\bar{E}$ and $\bar{T}$ are, respectively, the energy and temperature in the comoving or local rest frame of the collective motion, and

$$g(\bar{E}, \bar{T}) = g(2\pi)^{-3}[\exp(\bar{E}/k\bar{T}) \pm 1]^{-1},$$
$$\bar{n} = \int g(\bar{E}, \bar{T}) d^3\bar{p}. \tag{2}$$

We systematically use a bar to designate the value of a quantity in the comoving frame; thus,

$$\bar{E} = \gamma(\vec{v})E - \gamma(\vec{v})\vec{v} \cdot \vec{p},$$

where $\gamma(\vec{v}) = (1 - \vec{v}^2)^{-1/2}$ and E and $\vec{p}$ are the energy and momentum in the center-of-mass frame.

In the generalized statistical thermodynamic models,[3]

$$\frac{dN}{d^3\vec{v}} = M(s, \vec{v})F(s, \vec{v}, M(s, \vec{v})), \tag{3}$$

where $F(s, \vec{v}, M)$ is the probability of producing a fireball of mass M. It is difficult to criticize Eqs. (1) and (3) directly, so we will concentrate on Milekhin's[6] version of Landau's model, in which dN/d^3v is proportional to the distribution of entropy in the fluid. In a notation explained below [see Eq. (18)], Milekhin's expression is

$$\frac{dN}{d^3v} = \bar{n}(\vec{v})u^\mu \frac{\partial \sigma_\mu}{\partial^3 v}. \tag{4}$$

Equations (1) and (4) can be combined to give

$$E\frac{dN}{d^3p} \stackrel{?}{=} \int_\sigma g(\bar{E}, \bar{T}(\vec{v}))\bar{E}u^\mu d\sigma_\mu. \tag{5}$$

Equation (5) yields the correct number of particles, but it is inconsistent with energy conservation [see Eq. (20)], so we are led to consider how one determines EdN/d^3p for the simplest system, an expanding ideal gas.

The transport theory of a relativistic gas has been well studied.[7,8] We consider one type of particle of mass m and picture a many-body system as a collection of world lines that have local binary collisions and branching to describe particle creation and annihilation. In the neighborhood of a space-time point x^μ, the net number of lines making positive transit across an element $d\sigma_\mu$ of a 3-surface whose tangents lie within

$$Dp \equiv 2\delta^+(p^2 - m^2)d^4p \tag{6}$$

about p^μ serves to define a Lorentz-invariant distribution function $f(x, p)$,

$$dN(\sigma) = f(x, p)p^\mu d\sigma_\mu Dp. \tag{7}$$

The integral

$$N(\sigma) = \int Dp \int_\sigma f(x, p)p^\mu d\sigma_\mu \tag{8}$$

counts the net number of lines intersecting a given σ, and is equal to the number of particles at time t if we choose $d\sigma_\mu = (d^3x, \vec{0})$. The invariant single-particle distribution in momentum space, of those particles on σ, is

$$E\frac{dN}{d^3p} = \int_\sigma f(x, p)p^\mu d\sigma_\mu. \tag{9}$$

Equation (9) is to be compared with Eq. (5) under the assumption that the fluid is locally in thermodynamic equilibrium,

$$f(x, p) = g(\bar{E}(v(x)), T(x)). \tag{10}$$

The contrast between Eqs. (5) and (9) is that p^μ has been replaced by $\bar{E}u^\mu$ in Eq. (5). To choose between them, we make a further excursion into transport theory.

The Boltzmann equation is

$$p^\mu \partial_\mu f(x, p) = \Delta\Gamma(x, p), \tag{11}$$

where $\Delta\Gamma$ is the rate of change in f due to collisions. The stress-energy tensor defined by

$$T^{\mu\nu}(x) = \int p^\mu p^\nu f(x, p)Dp \tag{12}$$

is conserved by virtue of energy-momentum conservation in individual collisions,

$$\partial_\nu T^{\mu\nu}(x) = \int p^\mu \Delta\Gamma Dp = 0. \tag{13}$$

The collective velocity 4-vector $u^\mu(x) = (\gamma(x), \gamma(x)\vec{v}(x))$ is defined by

$$n(x)u^\mu(x) = \int p^\mu f(x, p)Dp. \tag{14}$$

The quantity $n(x)u^\mu(x)$ is a number current density and can be used to count the net number of particles on σ,

$$\int_\sigma n(x)u^\mu(x)d\sigma_\mu = \int Dp \int_\sigma f(x, p)p^\mu d\sigma_\mu = N(\sigma), \tag{15}$$

but there is no reason in general for it to be conserved,

$$\partial_\mu(n(x)u^\mu(x)) = \int \Delta\Gamma Dp \not\equiv 0.$$

Now $n(x)$ is a Lorentz scalar. Its meaning is established by using the Lorentz transformation to the comoving frame as a change of variables in evaluating the integral in Eq. (14). The transformation is

$$p^\mu = L^\mu_\nu(\vec{v})\bar{p}^\nu = L^\mu_0(\vec{v})\bar{E} + L^\mu_i(\vec{v})\bar{p}^i, \tag{16}$$

where

$$L_0^\mu(\vec{v}) = u^\mu,$$
$$L_i^\mu(\vec{v}) = -\delta^{\mu 0}u_i + (1-\delta^{\mu 0})[\delta_i^\mu - (\gamma-1)u^\mu u_i/v^2\gamma^2],$$

and (17)

$$\sum_{i=1}^{3} L_i^\mu L_i^\nu = u^\mu u^\nu - g^{\mu\nu}.$$

The evaluation of Eq. (14) is

$$\int p^\mu f(x,p)Dp = u^\mu(x)\int f(\bar{x},\bar{p})d^3\bar{p} + L_i^\mu \int \bar{p}^i f(\bar{x},\bar{p})D\bar{p} = \bar{n}(\bar{x})u^\mu(x),$$

where the second term vanishes because we are assuming that $f(\bar{x},\bar{p})$ is locally isotropic in the comoving frame. Hence, $n(x)=\bar{n}(\bar{x})$, the particle density in the local rest frame.

The stress tensor can also be evaluated under the assumption of local isotropy:

$$T^{\mu\nu}(x) = \int (\bar{E}u^\mu + L_i^\mu\bar{p}^i)(\bar{E}u^\nu + L_j^\nu\bar{p}^j)f(\bar{x},\bar{p})\frac{d^3\bar{p}}{\bar{E}}$$
$$= \int \left[\bar{E}u^\mu u^\nu + \left(\frac{\bar{p}^2}{3\bar{E}}\right)L_i^\mu L_i^\nu\right]f(\bar{x},\bar{p})d^3\bar{p}$$
$$= (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{18}$$

where ϵ and p are, respectively, the energy density and pressure in the comoving frame. In the second step in Eq. (18) we have used $\bar{p}^i\bar{p}^j = \frac{1}{3}\delta^{ij}\bar{p}^2$, and in the third step we have used Eq. (17).

For a sufficiently simple fluid motion, the function $u^\mu(x)$ can be inverted and the given surface parameterized by $\vec{v}$. Equation (15) gives the number of particles on σ having fluid velocity $\vec{v}$ in the interval d^3v as

$$\frac{dN}{d^3v} = n(\vec{v})u^\mu\frac{\partial\sigma_\mu}{\partial^3 v}, \tag{19}$$

where $n(\vec{v}) = n(x(\vec{v}))$ and $\partial\sigma_\mu/\partial^3 v$ is the Jacobian of the reparameterization of σ. Equation (19) is precisely Milekhin's expression (4) and the meaning of Eq. (5) is now established.

To see where Eq. (5) goes wrong, we use it to calculate the total energy of the gas,

$$E_{\text{tot}} = \int EdN$$
$$\overset{?}{=} \int EDp\int_\sigma g(\bar{E},\bar{T})\bar{E}u^\mu d\sigma_\mu$$
$$= \int d^3\bar{p}\int_\sigma g(\bar{E},\bar{T})(\bar{E}u^0 + L_i^0\bar{p}^i)u^\mu d\sigma_\mu$$
$$= \int_\sigma \epsilon(x)u^0u^\mu d\sigma_\mu$$
$$\neq \int_\sigma T^{0\mu}(x)d\sigma_\mu. \tag{20}$$

The corresponding calculation for Eq. (9) is

$$E_{\text{tot}} = \int EdN$$
$$= \int EDp\int_\sigma g(\bar{E},\bar{T})p^\mu d\sigma_\mu$$
$$= \int D\bar{p}\int_\sigma g(\bar{E},\bar{T})(\bar{E}u^0 + L_i^0\bar{p}^i) \times(\bar{E}u^\mu + L_j^\mu\bar{p}^j)d\sigma_\mu$$
$$= \int_\sigma T^{0\mu}(x)d\sigma_\mu. \tag{21}$$

In both Eqs. (20) and (21) we have used the assumption of local isotropy. Since Eq. (4) is well established by the considerations leading to Eq. (19), we conclude that the inconsistency lies in Eq. (1).

In the application of hydrodynamics to particle production, it is assumed that the hadronic fluid undergoes a change of composition leading to a gas consisting purely of pions when the temperature falls to a critical temperature $T_c \approx m_\pi c^2/k$. It is further assumed that the momentum distribution is essentially unchanged by the subsequent expansion of the pion gas. Therefore, EdN/d^3p is evaluated on the surface of condensation σ defined by $\bar{T}(x) = T_c$. The isotherms are calculated from the fluid equations of motion (13) using a phenomenological equation of state. Such surfaces have both spacelike and timelike portions, but that does not affect the validity of the foregoing considerations.

If the surface σ is taken to be an isotherm, Eqs. (20) and (21) can be cast in a sharper form:

$$E_{\text{tot}} \overset{?}{=} \epsilon_c V_c\langle\gamma\rangle, \tag{20'}$$

$$E_{\text{tot}} = (\epsilon_c + p_c)V_c\langle\gamma\rangle - p_cV_0, \tag{21'}$$

where

$$V_c = \int_\sigma u^\mu d\sigma_\mu, \tag{22}$$

$$\langle\gamma\rangle = \int_\sigma u^0u^\mu\frac{d\sigma_\mu}{V_c}, \tag{23}$$

$$V_0 = \int_\sigma d\sigma_0. \tag{24}$$

For very-high-energy collisions, $V_c \gg V_0$ and $\langle\gamma\rangle \gg 1$, so a considerable amount of energy is omitted in Eq. (20′) unless $p_c = 0$. Thus, the Milekhin algorithm Eq. (5) yields too few fast particles. The effect is less pronounced in Hagedorn's model because the speed of sound squared ($= dp/d\epsilon$) is much lower in his case, because of the importance of the exponentially rising mass spectrum: i.e., the

$\delta^+(p^2 - m^2)$ in Eq. (6) is replaced by an integral over the hadron mass spectrum.

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APPENDIX: LORENTZ-TRANSFORMING A REST-FRAME BOSE DISTRIBUTION

We would like to show explicitly how to Lorentz-transform a distribution which is a Bose distribution in the local rest frame. For simplicity we will restrict ourselves to one space dimension and one time dimension. The contribution to $EdN/dp = \bar{E}dN/d\bar{p}$ of a region $d\bar{x}$ in the local rest frame for which $d\bar{t}=0$ (constant time in local rest frame) is

$$\bar{E}\frac{dN}{d\bar{p}} = \frac{g}{(2\pi)^3}\frac{\bar{E}}{e^{\bar{E}/k\bar{T}(\bar{x})}-1}d\bar{x}. \qquad \text{(A1)}$$

Using

$$\bar{E} = (E - pv)\gamma ,$$

$$d\bar{x} = (dx - vdt)\gamma ,$$

and

$$d\bar{t} = (dt - vdx)\gamma \equiv 0 ,$$

one obtains

$$\begin{aligned} E\frac{dN}{dp} &= \frac{g}{(2\pi)^3}\frac{(E-pv)}{e^{\bar{E}/k\bar{T}}-1}dx \\ &= \frac{g}{(2\pi)^3}\frac{\bar{E}}{e^{\bar{E}/k\bar{T}}-1}(1-v^2)^{1/2}dx \\ &= \frac{g}{(2\pi)^3}\frac{Edx-pdt}{e^{\bar{E}/k\bar{T}}-1} \\ &= \frac{g}{(2\pi)^3}\frac{p^\mu d\sigma_\mu}{e^{\bar{E}/k\bar{T}}-1}. \end{aligned} \qquad \text{(A2)}$$

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