PHYSICAL REVIEW D VOLUME 10, NUMBER 6 15 SEPTEMBER 1974

Formulation of Abelian gauge theories without regulators

J. Lowenstein*
Department of Physics, New York University, New York, New York 10003

M. Weinstein†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

W. Zimmermann
Department of Physics, New York University, New York, New York 10003
(Received 22 May 1974)

In this, the first of three papers, we present the essential features of a treatment of the U(1) Higgs model based upon a regulator-free, momentum-space subtraction scheme. The principal new results which follow from this approach are that (1) the fields of the theory satisfy their classical equations of motion, (2) the source of the vector-meson field, $j^\mu(x)$, is a finite, conserved current, (3) the Higgs theory passes "smoothly" over to a Goldstone-boson theory when the vector-meson coupling constant (e) is set equal to zero, (4) the conserved current is gauge-invariant and can be used as an interpolating field for the stable one-particle states of the theory, and (5) one can define a generalized Higgs model wherein only part of the vector-meson mass comes from spontaneous breakdown; this theory has the features of the usual Higgs model, is ghost-free *but* it keeps its Goldstone boson. This first paper is devoted to stating precisely what can be proved and to establishing the relationship of our treatment to classical-field-theory ideas on the one hand and other quantum-field-theoretic treatments on the other.

INTRODUCTION

Some time has passed since the flurry of activity concerning Weinberg-Higgs-Kibble-type models for weak and electromagnetic interactions first began. While many of the original hopes that these theories could provide easy answers to heretofore unsolved problems show no immediate signs of being realized, it is by no means true that they have lost their appeal. On the contrary, there seems to be every reason to suppose that one has just begun to understand the relevance of these ideas to the phenomenology of hadron physics and to the physics of weak and electromagnetic interactions.

With the interesting work of 't Hooft and Veltman, Lee and Zinn-Justin,[1] and others, it is clear that a first pass at clarifying the essentials of how these theories work has been made. As in any first attempt, however, loose ends remain. In addition, much of what is simple about the structure of these theories has been obscured by the tremendous amount of purely technical detail which had to be handled in the discussion of their renormalization. For this reason, this series of papers is meant to serve a dual purpose. First, we wish to tie off loose ends for the Abelian Higgs model. Second, we wish to present a formal procedure, different from those used in previous discussions, to simplify the task of extracting a physical understanding of the structure of the renormalized field theory and of seeing how things differ, if at all, from one's naive notions.

The principal new results to be proved in the three papers of this series are as follows:

(1) The fields of the theory satisfy the naive equations of motion which follow from the usual classical arguments applied to the Higgs Lagrangian.

(2) The current, $j^\mu(x)$, which appears as the source of the vector-meson field in the equations of motion is both *finite* and *conserved*.

(3) If one takes the limit in which the vector-meson coupling constant, e, is set equal to zero, the Green's functions of the theory pass over to those of a Goldstone-boson theory (defined by taking the $e=0$ limit of the Higgs Lagrangian).

(4) In this same limit the current, $(1/e)j^\mu(x)$, passes over to the corresponding conserved current of the Goldstone theory.

(5) The fact that the fields satisfy simple equations of motion and the source of the vector-meson field is a finite conserved current is equivalent to the Ward identities first discussed for the Abelian Higgs model by Lee.[2]

(6) In the Abelian Higgs theory the conserved current, $j^\mu(x)$, is gauge-invariant and can be used as an interpolating field for the stable physical one-particle states of the theory. [This fact makes it possible for one to give a reasonably detailed discussion of this theory in terms of a positive-metric Hilbert space and generalized unitarity relations among the currents. This useful prop-

erty of $(1/e)j^{\mu}(x)$ survives the limit $e \to 0$ and yields a field-theoretic discussion of the underlying Goldstone-boson theory which is a precise parallel of phenomenological discussions.][3]

(7) One can generalize the usual Abelian Higgs model to a theory wherein only part of the vector-meson mass comes from the spontaneous breakdown of U(1) symmetry. In this model, as in the Higgs model, the naive equations of motion and current conservation guarantee the absence of ghosts in physical on-shell amplitudes; *what is new is that the Goldstone boson of the theory persists.* A careful study of how this pre-Higgs model passes over to a true Higgs model (i.e., one in which the Goldstone boson decouples from the theory) is extremely instructive.

The simplest way to obtain these results is to a adopt a scheme for defining renormalized Green's functions which avoids the need for cumbersome ultraviolet regularization techniques. At present we are only aware of two approaches which are available for this purpose. The first is the extremely promising dimensional-regularization scheme of 't Hooft and Veltman.[4] While this scheme is very appealing, as presently formulated it would make giving a satisfactory discussion of points (1)-(7) quite difficult. The second technique, the one we adopt, is a generalized version of the momentum-space subtraction procedure for Feynman integrals introduced by one of the authors.[5] One important virtue of this scheme is that it separates the question of whether one has a finite, Lorentz-invariant set of Feynman amplitudes from the problem of proving field equations and symmetry properties.[6] Another appealing feature of this method, especially from the point of view of understanding the physical basis of the procedure involved, is that the proofs of equations of motion, etc. simply parallel the discussion of the classical Lagrangian field theory.

Our goal is to make these papers as accessible as possible to those readers unfamiliar with the momentum-space subtraction procedure which we use and the way one uses it to derive equations of motion, etc. For this reason the first paper of this series states exactly what can be proved, explains—by example—most of the subtle points which arise, and indicates by a study of specific Feynman graphs how these problems are overcome. The second paper of this series is devoted to a complete specification of that subtraction procedure which is most convenient for giving mathematical proofs, and to a discussion of the derivation of equations of motion, etc. for what we shall refer to as the explicitly broken pre-Higgs model. Finally, paper III is devoted to a discussion of various limiting cases of the general model and to a discussion of the unitarity structure of the theory.

To set the stage for the more formal discussions of papers II and III, we proceed in several steps. In Sec. I of this paper we begin by discussing a generalized version of the classical Higgs model and its various limits. Next, for the sake of completeness, we devote Sec. II to a review of the Feynman rules for such a theory. Section III is devoted to giving a precise statement of the important theorems which we shall prove, and a discussion of how these theorems are related to previous results; in particular we show how to derive the familiar forms of the Ward identities. Finally, we conclude this section by indicating how one can discuss the ghosts of the theory and whether or not they contribute to on-shell physical amplitudes.

Finally, with the general discussions of Secs. I–III behind us, we try—in Secs. IV and V—a loose but essentially correct discussion of the way one defines the momentum-space subtraction procedure and uses it to prove equations of motion, etc.

I. CLASSICAL FOUNDATIONS

There are two main reasons for beginning with a discussion of the classical version of a generalized Abelian Higgs model. The first is to insure that those readers who are unfamiliar with spontaneous symmetry breaking and its role in gauge theories will find our treatment essentially self-contained. A more fundamental reason is that there is a close parallel between the classical equations of motion and the field equations in renormalized perturbation theory, expressed in terms of normal products; thus, many of the classical results obtained in this section will be applicable, with only minor modifications to the quantized version of the model.

A. General pre-Higgs Lagrangian

Our starting point is the theory of a complex scalar field coupled to a massive vector field. The Lagrangian density is of the form

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m_0^{\,2}A_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + (D^\mu\varphi)^* D_\mu\varphi - a^2\varphi^*\varphi - he^2(\varphi^*\varphi)^2\,,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,,\quad D_\mu = \partial_\mu - ieA_\mu\,, \tag{1}$$

where m, α, e, a, and h are real parameters. Introducing two real fields φ_1 and φ_2 such that

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)\,, \tag{2}$$

one obtains from Eq. (1) the following Euler-Lagrange equations:

$$\partial_\nu F^{\mu\nu} - \frac{1}{\alpha}\partial^\mu \partial_\nu A^\nu - m_0{}^2 A^\mu = j^\mu ,$$
$$(\Box + a^2)\varphi_1 = j^{(1)} ,$$
$$(\Box + a^2)\varphi_2 = j^{(2)} ,$$

where

$$\begin{aligned} j^\mu &= ie(\varphi^* D^\mu \varphi - \varphi(D^\mu \varphi)^*) \\ &= e\varphi_2 \overleftrightarrow{\partial}^\mu \varphi_1 + e^2 A^\mu (\varphi_1{}^2 + \varphi_2{}^2) \text{ with } \overleftrightarrow{\partial}^\mu = \overrightarrow{\partial}^\mu - \overleftarrow{\partial}^\mu , \\ j^{(1)} &= -he^2 \varphi_1(\varphi_1{}^2 + \varphi_2{}^2) - eA^\mu \partial_\mu \varphi_2 \\ &\quad - e\partial_\mu (A^\mu \varphi_2) + e^2 A_\mu A^\mu \varphi_1 , \\ j^{(2)} &= -he^2 \varphi_1(\varphi_1{}^2 + \varphi_2{}^2) - eA^\mu \partial_\mu \varphi_1 \\ &\quad - e\partial_\mu (A^\mu \varphi_1) + e^2 A_\mu A^\mu \varphi_2 . \end{aligned} \tag{3}$$

Imposing boundary conditions such that a localized source produces outgoing (incoming) waves for large positive (negative) times, we may convert Eq. (3) into the following set of coupled integral equations:

$$\begin{aligned} \varphi_1(x) &= \varphi_{01}(x) + i\int d^4y\, \Delta_F(x-y; a^2) j^{(1)}(y) , \\ \varphi_2(x) &= \varphi_{02}(x) + i\int d^4y\, \Delta_F(x-y; a^2) j^{(2)}(y) , \\ A_\mu(x) &= A_{0\mu}(x) - i\int d^4y\, \Delta_F(x-y; m^2) j_\mu(y) \\ &\quad - \frac{i}{m^2}\int d^4y [\partial_\mu \Delta_F(x-y; m_0{}^2) \\ &\quad - \partial_\mu \Delta_F(x-y; \alpha m_0{}^2)]\partial^\nu j_\nu(y), \end{aligned} \tag{4}$$

where

$$\Delta_F(\xi; \kappa^2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\xi} \frac{i}{p^2 - \kappa^2 + i0} , \tag{5}$$

and the inhomogeneous terms are linear combinations of free fields:

$$\begin{aligned} &(\Box + a^2)\varphi_{01} = 0 , \\ &(\Box + a^2)\varphi_{02} = 0 , \\ &(\Box + \alpha m_0{}^2)\partial^\mu A_{0\mu} = 0 , \\ &(\Box + m_0{}^2) V_{0\mu} = 0 , \\ &V_{0\mu} = A_{0\mu} + \frac{1}{\alpha m_0{}^2}\partial_\mu \partial_\nu A_0^\nu . \end{aligned} \tag{6}$$

The expression for $A_\mu(x)$ in Eq. (4) may be simplified somewhat by noting that the current j_μ is conserved, i.e.,

$$\partial^\mu j_\mu = 0 . \tag{7}$$

(This is a straightforward consequence of the field equations.) Hence the third equation in (4) becomes

$$A_\mu(x) = A_{0\mu}(x) - i\int d^4y\, \Delta_F(x-y; m_0{}^2) j_\mu(y) , \tag{8}$$

so that the scattered waves contain no scalar mode of mass $m_0\sqrt{\alpha}$. Equation (8) tells us that the field $\partial_\mu A^\mu$ propagates freely and can be suppressed entirely by appropriate choice of initial conditions. The decoupling of the field $\partial_\mu A^\mu$ is an essential feature of the model which persists in the quantum version, where it is crucial to the elimination of negative-metric "ghosts" from the theory.

It is interesting to observe that the fact that the scalar mode of mass $m_0\sqrt{\alpha_0}$ is dynamically trivial is reflected in a symmetry property of the field equations (3). The latter are invariant under the restricted gauge transformations

$$\begin{aligned} &A_\mu \to A_\mu + \partial_\mu \Lambda , \\ &\varphi \to \varphi e^{ie\Lambda} , \end{aligned} \tag{9}$$

where

$$(\Box + \alpha m_0{}^2)\Lambda = 0 .$$

In the quantum version of the model, it will be natural to consider as observable only those fields which are invariant under these gauge transformations (or under a more general class).

B. Perturbative solutions to the field equations

In order to try and obtain some feeling for the nature of the solutions to the field equations [Eq. (3)] we observe that for positive values of a^2 and $m_0{}^2$, Eqs. (4) and (8) may be solved iteratively to arbitrary finite order in e. One hopes the resulting expansions provide asymptotic series for the fields for sufficiently small e (no claim is made about the convergence of the iteration scheme). Since the expansion parameter only multiplies terms of second or higher degree in the fields, and since the Green's function $\Delta_F(\xi)$ is well behaved asymptotically, the perturbative ("scattering") corrections to small-amplitude, freely propagating waves will be appropriately small, and will remain so for all times.

For negative a^2 (the pre-Higgs model), a satisfactory iteration of (4) is not possible, thanks to the exponential growth of solutions of the homogeneous field equations and the exponential growth of the Green's function for large times, since in this case small departures from the propagation will not remain small asymptotically. To understand why a perturbative solution about $\phi_1(x) = \phi_2(x) = A_\mu(x) = 0$ cannot work in this case, it is convenient to consider the limiting case (Goldstone model) $e \to 0$, $he^2 \to g \neq 0$. The essential point is that one expects a "small vibration" to be stable if one is expanding about a solution, $\phi_1(x)=\bar{\psi}(x)$ and $\phi_2(x)=\bar{\chi}(x)$, which is a minimum of the potential energy,

$$V(\psi,\chi)\equiv \tfrac{1}{4}g(\phi_1^2+\phi_2^2)^2+\tfrac{1}{2}a^2(\phi_1^2+\phi_2^2).$$

Clearly, for the case $a^2>0$ the solution $(\bar{\psi},\bar{\chi})=0$ is such a minimum; however, for $a^2<0$ it is easy to check that $(\bar{\psi},\bar{\chi})=0$ is a relative *maximum* of the function $V(\phi_1,\phi_2)$. Since no small-vibration expansion about a relative maximum of potential energy can be expected to be stable, it is no surprise that in our attempt to make such an expansion we encounter exponential asymptotic growth of the resulting solutions.

It is a simple matter to show that the true minima of the potential function form a one-parameter family $(\bar{\psi}_\theta,\bar{\chi}_\theta)$ such that $\bar{\psi}_\theta^2+\bar{\chi}_\theta^2=|a^2/g|$, i.e., $(\bar{\psi}_\theta,\bar{\chi}_\theta)=f(\cos\theta,\sin\theta)$, where $f^2=|a^2/g|$, and so it should be possible to formulate a stable small-vibration theory of the model by setting $\varphi_1=\psi+f$ and $\varphi_2=\chi$ and expanding the solution about $\psi=\chi=A_\mu=0$. It is not obvious that this should work for general values of the coupling constant, but let us try it anyhow. With these substitutions Eqs. (3) become

$$(\Box+2hw^2)\psi(x)=J_\psi(x),\quad w=ef$$

$$\Box\chi(x)-w\partial_\mu A^\mu(x)=J_\chi(x),\quad m^2=m_0^2+w^2$$

$$-(\Box+m^2)A_\mu+\left(1-\frac{1}{\alpha}\right)\partial_\mu\partial^\nu A_\nu+w\partial_\mu\chi=J_\mu(x),$$

where

$$J_\psi(x)=-he^2\psi(\psi^2+\chi^2)-e[A^\mu\partial_\mu\chi+\partial^\mu(A_\mu\chi)]+e^2A_\mu A^\mu\psi-he(3\psi^2+\chi^2)+ewA_\mu A^\mu,$$
$$J_\chi(x)=-he^2\chi(\psi^2+\chi^2)+e[A^\mu\partial_\mu\psi+\partial^\mu(A_\mu\psi)]+e^2A_\mu A^\mu\chi-2he\psi\chi, \tag{10}$$
$$J_\mu(x)=e(\chi\partial_\mu\psi-\psi\partial_\mu\chi)+e^2A_\mu\left(\psi^2+\chi^2+2\frac{w}{e}\psi\right).$$

With the same choice of boundary conditions as before, Eqs. (3) may be converted into the following integral equations:

$$\psi(x)=\psi_0(x)+i\int dy\,\Delta_F(x-y;2hw^2)J_\psi(y),$$
$$\chi(x)=\chi_0(x)+\frac{i}{m_0^2}\int dy\,\Delta_F(x-y;0)[m^2J_\chi(y)-w\partial^\mu J_\mu(y)]-\frac{iw}{m_0^2}\int dy\,\Delta_F(x-y;\alpha m_0^2)[wJ_\chi(y)-\partial^\mu J_\mu(y)],$$
$$A_\mu(x)=A_{0\mu}(x)-i\int dy\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y;m^2)J^\nu(y) \tag{11}$$
$$+\frac{iw}{m_0^2m^2}\int dy\,\partial_\mu\Delta_F(x-y;0)[m^2J_\chi(y)-w\partial^\nu J_\nu(y)]-\frac{i}{m_0^2}\int dy\,\partial_\mu\Delta_F(x-y;\alpha m_0^2)[wJ_\chi(y)-\partial^\nu J_\nu(y)],$$

where ψ_0, χ_0, and $A_{0\mu}$ are solutions of the respective homogeneous differential equations. Defining

$$\rho_0=\chi_0+\frac{w}{\alpha m_0^2}\partial_\mu A_0^\mu,$$
$$\tau_0=\partial_\mu A_0^\mu, \tag{12}$$
$$V_{0\mu}=A_{0\mu}-\frac{w}{m^2}\partial_\mu\chi_0+\frac{1}{\alpha m^2}\partial_\mu\partial^\nu A_{0\nu},$$

it is easy to see from the differential equations that ψ_0, as well as these linear combinations, are free fields; i.e.,

$$(\Box+2hw^2)\psi_0=0,$$
$$\Box\rho_0=0, \tag{13}$$
$$(\Box+\alpha m_0^2)\tau_0=0,$$
$$(\Box+m^2)V_{0\mu}=0.$$

Owing to the relation

$$\partial^\mu J_\mu(x)=wJ_\chi(x), \tag{14}$$

a straightforward consequence of the field equations, the last two of Eqs. (11) assume the simpler form

$$\chi(x)=\chi_0(x)+i\int dy\,\Delta_F(x-y;0)J_\chi(y),$$
$$A_\mu(x)=A_{0\mu}(x)-i\int dy\left(g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y;m^2)J^\nu(y)$$
$$+\frac{iw}{m^2}\int dy\,\partial_\mu\Delta_F(x-y;0)J_\chi(y). \tag{15}$$

Note that once again the scattered waves contain no ghosts of mass $m_0\sqrt{\alpha}$, and the scalar field $\partial_\mu A^\mu$ satisfies a free-field equation. As in the case $a^2\geqslant 0$, the decoupling of the ghost oscillations is linked to the invariance of the equations of motion under the gauge transformations of Eq. (9) where

$$\varphi=\frac{1}{\sqrt{2}}\left(\psi+i\chi+\frac{w}{e}\right). \tag{16}$$

C. Currents, gauge invariance, and special limits of the model

In the quantized version of the pre-Higgs model, it will be essential to consider as observable only those quantities which in some sense are free of ghosts. One criterion for composite fields, formed from products of the basic fields ψ, χ, and A_μ and their derivatives, to be ghost-free will turn out to be invariance under the gauge transformations defined by Eqs. (9) and (6), without the free-field restriction on Λ. Perhaps the most important of these composite fields is the original current j_μ, as given by Eq. (3), which is obviously gauge invariant and related to the current J_μ defined in Eq. (10) by

$$j_\mu = J_\mu + w^2 A_\mu - w\partial_\mu \chi_0 . \tag{17}$$

To check explicitly that the classical j_μ has no asymptotic ghost oscillations, we note that in Eq. (10) the mass $m_0\sqrt{\alpha}$ scalar mode can enter only in the zeroth-order contribution to $w^2A_\mu - w\partial_\mu\chi$. But

$$w^2 A_{0\mu} - w\partial_\mu \chi_0 = w^2 V_{0\mu} - \frac{m_0^2 w}{m^2}\partial_\mu \rho_0 , \tag{18}$$

so that we have only a mass-m vector mode and a mass-0 scalar "mode," but no scalar oscillations of mass $m_0\sqrt{\alpha}$.

Note that in the weak-coupling limit, $e \to 0$, $w/e \to f$, j_μ goes over continuously into the conserved current of the Goldstone model,

$$\frac{1}{e} j_\mu \to \chi\partial_\mu\psi - \psi\partial_\mu\chi - f\partial_\mu\chi,$$

and from (15) and (18) it is clear that only the massless Goldstone mode survives in the asymptotic behavior of the limiting current for large times. In the quantized version of the theory this corresponds to the possibility of using the current as an interpolating field for the Goldstone particles, a matter of considerable importance in the structural analysis of this and related models.

In the limit of vanishing "photon" mass, $m_0 \to 0$, the pre-Higgs current j_μ goes over continuously into the conserved current of the Higgs model. As is evident from (15) and (18), it is the zero-mass mode, the remnant of the Goldstone mode, which disappears in this limit, with a mass-w vector mode surviving in the asymptotic waves. Correspondingly, if one takes the $m_0 \to 0$ limit of the quantized pre-Higgs model, one finds that the zero-mass particles of the latter decouple, with the conserved current becoming a suitable interpolating field for mass-w particles of spin one in the limiting theory. This aspect of the so-called Higgs phenomenon will be discussed at some length in paper III.

II. SOME PRELIMINARY REMARKS

In the preceding section we studied the classical theory of a massive vector field interacting with a self-coupled scalar field. Our purpose in this section is to lay the foundations for extending the discussion to the corresponding quantum field theory. We shall assume throughout that the reader is familiar with ordinary perturbation theory, the mechanics of Feynman graphs, and the general spirit of the Dyson-Salam program for renormalizing Feynman amplitudes in momentum space. We shall not assume a familiarity with the extension of these ideas to a full, regulator-free, momentum-space renormalization procedure. The latter will be sketched, with instructive examples, in the course of this article, and the reader interested in further details is referred to Ref. 15.

The general point of view which motivates the discussion to follow is that the perturbative approach to quantum field theory amounts to a set of rules for constructing finite Feynman amplitudes (Green's functions). These rules are usually presented in two stages: One first specifies a procedure for assigning a Feynman integrand to each Feynman diagram, and then one gives a prescription for modifying this integrand so as to guarantee the integrability of the resulting expression. One is then left with the task of showing that the theory so defined has the equations of motion, covariance properties, and symmetries which one anticipated at the outset on the basis of the given Lagrangian (whose role here is merely as a formal device for specifying Feynman rules).

The classical Lagrangian of the pre-Higgs model is given by Eq. (1) with the substitution $\phi_1 = \psi + f$, $\phi_2 = \chi$. Working the Lagrangian out in terms of A_μ, ψ, χ and setting

$$a^2 = -hw^2 ,$$

we obtain (up to constant terms)

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I ,\\
\mathcal{L}_0 &= -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \tfrac{1}{2}(m_0^2 + w^2)A_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 - wA_\mu\partial^\mu\chi \\
&\quad + \tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi + \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi - hw^2\psi^2 ,\\
\mathcal{L}_I &= -\tfrac{1}{4}he^2[(\psi^2+\chi^2)^2 + 4ew\psi(\psi^2+\chi^2)] + eA^\mu\chi\overleftrightarrow{\partial}_\mu\psi + \tfrac{1}{2}e^2A_\mu A^\mu\left(\psi^2+\chi^2+2\frac{w}{e}\psi\right) .
\end{aligned} \tag{19}$$

The Feynman rules for the Green's functions are conveniently summarized by the Gell-Mann–Low expansion, and there is a close connection between the formal specifications of Feynman rules and the classical perturbation theory described in Sec. I. The free fields to be inserted into the Gell-Mann–Low formula are those associated with the unperturbed Lagrangian $\mathcal{L}_0$, Eq. (19), and will obey the homogeneous versions of the field equations in Eq. (10). Their propagators will be precisely the classical Green's functions which appear in the integral equations of Eq. (11). Specifically, their momentum-space form is

$$
\begin{aligned}
\langle T\tilde{\psi}_0(p)\psi_0(0)\rangle_0 &= \frac{i}{p^2-2hw^2+i0}\,,\\
\langle T\tilde{\chi}_0(p)\chi_0(0)\rangle_0 &= \frac{i}{m_0^{\,2}}\left(\frac{m^2}{p^2+i0}-\frac{w^2}{p^2-\alpha m_0^{\,2}+i0}\right),\\
\langle T\tilde{A}_{0\mu}(p)\chi_0(0)\rangle_0 &= \frac{wp_\mu}{m_0}\left(\frac{1}{p^2+i0}-\frac{1}{p^2-\alpha m_0^{\,2}+i0}\right),\\
\langle T\tilde{A}_{0\nu}(p)A_\mu(0)\rangle_0 &= \frac{(-g_{\mu\nu}+p_\mu p_\nu/m^2)}{p^2-m^2+i0}+\left(\frac{w}{m_0 m}\right)^2\frac{p_\mu p_\nu}{p^2+i0}\\
&\quad -\frac{1}{m_0^{\,2}}\,\frac{p_\mu p_\nu}{p^2-\alpha m_0^{\,2}+i0}\,,
\end{aligned}
\tag{20}
$$

where we have adopted the shorthand notation

$$
\langle T\tilde{\tau}_1(p_1)\cdots\tilde{\tau}_u(p_u)\tau(y)\rangle = \int \exp\left(i\sum p_i x_i\right)\langle T\tau_1(x_1)\cdots\tau_u(x_u)\tau(y)\rangle\,.
\tag{21}
$$

The poles of these propagators correspond to the particles of the theory; so we have a vector meson of mass $m=(m_0^{\,2}+w^2)^{1/2}$, a scalar particle σ of mass $M=(2hw^2)^{1/2}$, a scalar particle π of vanishing mass, and a scalar ghost particle of mass $m_0\sqrt{\alpha}$, which must be quantized with indefinite metric due to the sign of the residue in Eq. (20).

A. Some necessary modifications of our Lagrangian

With the Lagrangian given in Eq. (19) the formal Gell-Mann–Low expansion leads to ultraviolet as well as infrared divergences. Ultraviolet divergences are eliminated by making subtractions for the Feynman integrals in momentum space (see Sec. IV; the complete rules will be given in paper II). For resolving the infrared problem we first modify the Lagrangian such that all particles acquire a nonvanishing mass. To this end we add to the Lagrangian an explicit symmetry-breaking term

$$
\mu^2\frac{w}{e}\psi
\tag{22}
$$

as well as a gauge-invariant contribution of the form

$$
-\tfrac{1}{2}(\mu^2-c)\varphi^*\varphi = -\tfrac{1}{2}(\mu^2-c)\left(\psi^2+\chi^2+2\frac{w}{e}\psi\right)+\text{constant term}\,,
\tag{23}
$$

where c is chosen to enforce the vanishing of the vacuum expectation value of ψ. The model so obtained will be called the explicitly broken pre-Higgs model. Its Green's functions are free of ultraviolet and infrared divergences. Eventually the limits $\mu\to 0$ (pre-Higgs model), $\alpha\to 0$ (Landau gauge), and $m\to 0$ (Higgs model) will be taken. That the Green's function remain finite in these limits will be shown in paper III.

Since the subtractions will be made at zero external momenta there is no guarantee that the full propagators continue to have poles at the same values of p^2 for which the unperturbed propagators are singular. For unstable particles, such as the σ in the pre-Higgs model, the propagator poles must of course move into the complex plane. For the stable particles of the pre-Higgs or Higgs model, however, we want the mass parameters of the free Lagrangian to be the exact values of the physical masses. Moreover, the Green's functions should be normalized in a convenient manner. For these reasons, it is necessary to introduce renormalized fields and parameters by means of the substitutions

$$
\begin{aligned}
&\psi\to z_2^{\,1/2}\psi,\quad \alpha\to z_3\alpha\,,\quad h\to z_1 z_2^{\,-2}z_3 h\,,\\
&\chi\to z_2^{\,1/2}\chi,\quad e\to z_3^{\,-1/2}e\,,\quad m\to z_3^{\,-1/2}m\,,\\
&A_\mu\to z_3^{\,1/2}A_\mu\,,\quad w\to z_2^{\,1/2}z_3^{\,-1/2}w\,.
\end{aligned}
\tag{24}
$$

For $\mu\neq 0$ the renormalization factors z are power series in e with finite coefficients; moreover, we find it useful to make the substitutions

$$
w\to sw\,,\quad c\to s^2c\,.
$$

The parameter s will only be used for formulating the subtraction scheme and eventually s will be set equal to one. The renormalization constants z_i and c are assumed to be independent of s. Since $s=0$ corresponds to exact symmetry the parameter s measures the strength of the symmetry breaking. With the substitutions of Eq. (24) and the insertion of the additional terms of Eqs. (22) and (23), the Lagrangian of the explicitly broken pre-Higgs model becomes

$$\mathcal{L} = -\tfrac{1}{4}z_3 F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m_0{}^2 A_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + z_2(D^\mu\varphi)^* D_\mu\varphi + z_1 hs^2w^2\varphi^*\varphi - he^2(\varphi^*\varphi)^2 - (\mu^2 - cs^2)\varphi^*\varphi + \mu^2 s\frac{w}{e}\psi\,, \tag{25}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \quad D_\mu = \partial_\mu - ieA_\mu\,,$$

$$\varphi = \frac{1}{\sqrt{2}}\left(\psi + i\chi + s\frac{w}{e}\right).$$

Explicitly in terms of A_μ, ψ, and χ we have

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I\,,\\
\mathcal{L}_0 &= -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \tfrac{1}{2}(m_0{}^2 + s^2w^2)A_\mu A^\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2\\
&\quad - swA_\mu\partial^\mu\chi + \tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi + \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi - \tfrac{1}{2}(\mu^2 + 2hs^2w^2)\psi^2 - \tfrac{1}{2}\mu^2\chi^2\,,\\
\mathcal{L}_I &= -\tfrac{1}{4}z_1 he^2(\psi^2+\chi^2)^2 - z_1 ehsw\psi(\psi^2+\chi^2) - (z_1 - 1)hs^2w^2\psi^2\\
&\quad + z_2 eA_\mu\chi\overleftrightarrow{\partial}^\mu\psi - (z_2 - 1)swA_\mu\partial^\mu\chi + \tfrac{1}{2}z_2 e^2 A_\mu A^\mu\left(\psi^2 + \chi^2 + 2s\frac{w}{e}\psi\right)\\
&\quad + \tfrac{1}{2}(z_2 - 1)s^2w^2A_\mu A^\mu + \tfrac{1}{2}(z_2 - 1)(\partial_\mu\psi\partial^\mu\psi + \partial_\mu\chi\partial^\mu\chi)\\
&\quad + \tfrac{1}{2}cs^2\left(\psi^2 + \chi^2 + 2s\frac{w}{e}\psi\right) - \tfrac{1}{4}(z_3 - 1)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\,.
\end{aligned} \tag{26}$$

Adopting this notation the field equation for the vector field A_μ, following from the classical Lagrangian of Eq. (26), now becomes

$$\partial_\nu F^{\mu\nu} - \frac{1}{\alpha}\partial^\mu\partial_\nu A^\nu - m_0{}^2 A^\mu = j^\mu_{\rm cl}\,, \tag{27}$$

with the classical current given by

$$\begin{aligned}
j^\mu_{\rm cl} &= iez_2(\varphi^* D^\mu\varphi - \varphi(D^\mu\varphi)^*) - (z_3 - 1)\partial_\nu F^{\mu\nu}\\
&= z_2 e[\chi\overleftrightarrow{\partial}^\mu\psi - e^{-1}sw\partial^\mu\chi + eA^\mu(\psi^2+\chi^2) + 2eswA^\mu\psi + s^2w^2A^\mu] + (z_3 - 1)(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu\,.
\end{aligned} \tag{28}$$

Note that now when $\mu \neq 0$ the classical current is only partially conserved:

$$\partial_\mu j^\mu_{\rm cl} = \mu^2 sw\chi\,. \tag{29}$$

Solving the "free field" equations appropriate to $\mathcal{L}_0$ as given in Eq. (26) we find the propagators are

$$\begin{aligned}
\langle T\tilde{\psi}_0(p)\psi_0(0)\rangle &= \frac{i}{p^2 - \mu^2 - 2hs^2w^2 + i0}\,,\\
\langle T\tilde{\chi}_0(p)\chi_0(0)\rangle &= \frac{i(p^2 - \alpha m_0{}^2) - i\alpha s^2w^2}{D_0(p^2 + i0)}\,,\\
\langle T\tilde{A}_{0\mu}(p)\chi_0(0)\rangle &= \frac{-\alpha swp_\mu}{D_0(p^2 + i0)}\,,\\
\langle T\tilde{A}_{0\mu}(p)A_\nu(0)\rangle &= -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{i}{p^2 - m_0{}^2 - s^2w^2 + i0}\\
&\quad - \frac{p_\mu p_\nu}{p^2}\,\frac{i\alpha(p^2 - \mu^2)}{D_0(p^2 + i0)}\,,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
D_0(p^2) &= (p^2 - \kappa^2)(p^2 - \lambda^2)\\
&= (p^2 - \mu^2)(p^2 - \alpha m_0{}^2) + \alpha\mu^2 s^2w^2\,.
\end{aligned} \tag{31}$$

The masses κ and λ are given by

$$\begin{aligned}
\kappa^2 &= \tfrac{1}{2}(\alpha m_0{}^2 + \mu^2)\\
&\quad - \tfrac{1}{2}[(\alpha m_0{}^2 + \mu^2)^2 - 4\alpha\mu^2(m_0{}^2 + s^2w^2)]^{1/2}\,,\\
\lambda^2 &= \tfrac{1}{2}(\alpha m_0{}^2 + \mu^2)\\
&\quad + \tfrac{1}{2}[(\alpha m_0{}^2 + \mu^2)^2 - 4\alpha\mu^2(m_0{}^2 + s^2w^2)]^{1/2}\,.
\end{aligned} \tag{32}$$

Consideration of the results reveals that the free Lagrangian $\mathcal{L}_0$ describes the following kinds of particles:

(i) a vector meson of mass $m = (m_0{}^2 + s^2w^2)^{1/2}$ (three degrees of freedom),

(ii) a scalar ψ particle of mass $M = (\mu^2 + 2hs^2w^2)^{1/2}$,

(iii) a scalar χ particle of mass κ, and

(iv) a scalar ghost particle of mass λ.

(The decomposition of $A_{0\mu}$ and χ_0 with respect to conventional free fields will be given in paper II, Sec. II.) In the limit $\mu\to 0$ the mass κ approaches zero and the χ particle becomes the Goldstone particle of the pre-Higgs model.

B. Specification of Feynman rules

The renormalized time-ordered functions are constructed from the Gell-Mann–Low expansion

$$\langle TX\rangle = \mathrm{Fin}\left\langle TX^{(0)}\exp\left[i\int dz\,\mathcal{L}_I^{(0)}(z)\right]\right\rangle^{\text{norm}}, \qquad (33)$$

with the interaction Lagrangian of Eq. (26), where X denotes an arbitrary product of field operators,

$$X=\prod_{j=1}^{n_A} A_{\mu_j}(x_j)\prod_{k=1}^{n_\psi}\psi(y_k)\prod_{l=1}^{n_\chi}\chi(z_l). \qquad (34)$$

We have adopted the notation $X^{(0)}$ and $\mathcal{L}_I^{(0)}$ to signify that the corresponding expressions with the free fields are to be inserted. In the usual way the time-ordered functions on the right-hand side of Eq. (33) are to be expanded as a sum of products of free propagators, each corresponding to a Feynman diagram. The superscript "norm" indicates that contributions of diagrams having disconnected parts with no external lines should not be included. "Fin" denotes the finite part which one is to take for each Feynman integral, as discussed in Sec. IV. Finally, the parameters z_1, z_2, and z_3 are chosen to guarantee the desired mass and normalization conditions.

According to the Feynman rules following from Eqs. (33) and (34), all Green's functions,

$$\left\langle T\prod_{j=1}^{n_A} A_{\mu_j}(x_j)\prod_{k=1}^{n_\psi}\psi(y_k)\prod_{l=1}^{n_\chi}\chi(z_l)\right\rangle=0 \qquad (n_A+n_\chi=\text{odd}), \qquad (35)$$

for which the total number of A_μ and χ fields is odd, vanish. This property reflects an exact symmetry of the theory under the substitution

$$A_\mu\to -A_\mu,\quad \chi\to-\chi. \qquad (36)$$

The current of the quantized theory is defined through its Green's functions which are constructed by the Gell-Mann–Low expansion

$$\langle Tj^\mu(x)X\rangle = \mathrm{Fin}\left\langle Tj_{\text{cl}}^{(0)\mu}(x)X^{(0)}\exp\left[i\int dz\,\mathcal{L}_I^{(0)}(z)\right]\right\rangle^{\text{norm}}, \qquad (37)$$

where $j_{\text{cl}}^{(0)\mu}$ denotes the classical current [Eq. (28)] with the free-field operators inserted. For the precise definition of the finite part indicated in Eq. (37) we refer again to the more detailed discussion in Sec. IV. The current given by Eq. (37) will be denoted as

$$j^\mu = N_3[j_{\text{cl}}^\mu],$$

where the symbol N_3 stands for normal product of degree 3. The matrix elements of the current between incoming and outgoing states are obtained from Green's functions of the form indicated in Eq. (37) by applying the reduction formulas to the field operators. It should be noted that only those fields associated with stable particles contribute to the asymptotic limits.

C. Remarks about the Goldstone limit of this model

The Lagrangian of the explicitly broken Goldstone model is obtained from Eq. (26) by taking the limit $e\to 0$, $w/e\to f$, and $h^{1/2}e\to g$. In this limit it is convenient to use the χ mass μ, the ψ mass M, related to f and g by

$$fg=\left(\frac{M^2-\mu^2}{2}\right)^{1/2}, \qquad (38)$$

as independent variables. With these parameters the Lagrangian of the explicitly broken Goldstone model becomes

$$\mathcal{L}=z_2\partial^\mu\varphi^*\partial_\mu\varphi+z_1\frac{M^2-\mu^2}{2}s^2\varphi^*\varphi - z_1g^2(\varphi^*\varphi)^2-(\mu^2-s^2)\varphi^*\varphi,$$

$$\varphi=\frac{1}{\sqrt{2}}(\psi+i\chi+sf) \qquad (39)$$

or, explicitly,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I,$$

$$0=\tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi+\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi-\tfrac{1}{2}[\mu^2+s^2(M^2-\mu^2)]\psi^2-\tfrac{1}{2}\mu^2\chi^2,$$

$$\mathcal{L}_I=-\tfrac{1}{4}z_1g^2(\psi^2+\chi^2)-z_1g\left(\frac{M^2-\mu^2}{2}\right)^{1/2}s\psi(\psi^2+\chi^2) \qquad (40)$$

$$-(z_1-1)\frac{M^2-\mu^2}{2}s^2\psi^2+\tfrac{1}{2}(z_2-1)(\partial_\mu\psi\partial^\mu\psi+\partial_\mu\chi\partial^\mu\chi)+\tfrac{1}{2}cs^2\left[\psi^2+\chi^2+2\left(\frac{M^2-\mu^2}{2}\right)^{1/2}sg^{-1}\psi\right],$$

and the classical current is

$$j^{\mu}_{\rm cl}=iz_2\varphi^*\overleftrightarrow{\partial}^{\mu}\varphi$$
$$=z_2\chi\overleftrightarrow{\partial}^{\mu}\psi-\left(\frac{M^2-\mu^2}{2}\right)^{1/2}sg^{-1}\partial^{\mu}\chi\,. \qquad (41)$$

The free propagators for this case are

$$\langle T\tilde{\psi}_0(p)\psi_0(0)\rangle=\frac{i}{p^2-\mu^2-s^2(M^2-\mu^2)+i0}\,,$$
$$\langle T\tilde{\chi}_0(p)\chi_0(0)\rangle=\frac{i}{p^2-\mu^2+i0}\,.$$

The Green's functions of the fields and the current are constructed from the Gell-Mann–Low expansion with the interaction Lagrangian, $\mathcal{L}_I$, given in Eq. (41); however, in this limit the symbol X of Eqs. (33) and (34) stands for general expressions of the form

$$X=\prod_{k=1}^{n_\psi}\psi(y_k)\prod_{l=1}^{n_\chi}\chi(z_l)\,. \qquad (42)$$

This completes our preliminary remarks concerning the specification of the pre-Higgs Lagrangian. In the next section we turn to the statement of important general results.

III. STATEMENT OF GENERAL RESULTS

Most of the salient features of the explicitly broken Higgs model and its various limiting cases are consequences of two fundamental equations relating the Green's functions of the basic fields to those of the current j_μ. The derivation of these relations will be discussed briefly in Sec. V, with a systematic treatment postponed until paper II.

A. Important equations

The first of these key equations is the equation of motion for the vector potential:

$$\left\langle T\left[\partial_\nu F^{\mu\nu}(x)-\frac{1}{\alpha}\partial^\mu\partial_\nu A^\nu(x)-m_0{}^2A^\mu(x)\right]X\right\rangle=\langle Tj^\mu(x)X\rangle-i\sum_n\delta(x-x_n)\delta^\mu_{\mu_n}\left\langle T\prod_{j\neq n}A_{\mu_j}(x_j)\prod_k\psi(y_k)\prod_l\chi(z_l)\right\rangle. \qquad (43)$$

Here X denotes an arbitrary expression of the form

$$X=\prod_{j=1}^{n_A}A_{\mu_j}(x_j)\prod_{k=1}^{n_\psi}\psi(y_k)\prod_{l=1}^{n_\chi}\chi(z_l).$$

Applying a reduction formula to obtain matrix elements between incoming and outgoing states, we may transform Eq. (43) into the following operator field equation:

$$\partial_\nu F^{\mu\nu}-\frac{1}{\alpha}\partial^\mu\partial_\nu A^\nu-m_0{}^2A^\mu=j^\mu\,. \qquad (44)$$

This is identical to the classical equation of motion derived via the variational principle from the Lagrangian of Eq. (40).

The second important equation is the Ward-Takahashi identity for the current's Green's functions, namely,

$$\partial^\mu_x\langle Tj_\mu(x)X\rangle=\mu^2w\langle T\chi(x)X\rangle-ie\sum_n\delta(x-y_n)\left\langle T\chi(y_n)\prod_j A_{\mu_j}(x_j)\prod_{k\neq n}\psi(y_k)\prod_l\chi(z_l)\right\rangle$$
$$+ie\sum_n\delta(x-z_n)\left\langle T\left[\psi(z_n)+\frac{w}{e}\right]\prod_j A_{\mu_j}(x_j)\prod_k\psi(y_k)\prod_{l\neq n}\chi(z_l)\right\rangle. \qquad (45)$$

Once again applying the reduction formula, we obtain the law of partial current conservation in operator form,

$$\partial^\mu j_\mu=\mu^2w\chi\,. \qquad (46)$$

It is significant that Eqs. (45) and (46) would follow in precisely the same form from naive application of the formal equations of motion and equal-time commutation relation. Equation (46) is just the partial conservation law for the classical current, and the terms involving δ functions in Eq. (45) correspond to the equal-time commutators which one usually finds as a result of commuting the time derivatives with the time ordering. [Our derivation in paper II will not make use of these naive manipulations, which cannot be justified in perturbation theory. Actually the T product used here represents a renormalized version of the T^* product for which differentiations always commute with time ordering. In our way of deriving Eq. (45), covariant contact terms arise directly from the Feynman rules and are not due to the differentiation of step functions.]

Taking the divergence of Eq. (43) and inserting Eq. (45), we obtain

$$\frac{1}{\alpha}\langle T(\Box+\alpha m_0{}^2)\partial^\mu A_\mu(x)X\rangle+\mu^2 w\langle T\chi(x)X\rangle=i\sum_n \partial_{\mu_n}\delta(x-x_n)\Big\langle T\prod_{j\neq n}A_{\mu_j}(x_j)\prod_k\psi(y_k)\prod_l\chi(z_l)\Big\rangle$$
$$+ie\sum_n\delta(x-y_n)\Big\langle T\chi(y_n)\prod_j A_{\mu_j}(x_j)\prod_{k\neq n}\psi(y_k)\prod_l\chi(z_l)\Big\rangle$$
$$-ie\sum_n\delta(x-z_n)\Big\langle T\Big[\psi(z_n)+\frac{w}{e}\Big]\prod_j A_{\mu_j}(x_j)\prod_k\psi(y_k)\prod_{l\neq n}\chi(z_l)\Big\rangle, \tag{47}$$

which may be interpreted either as the equation of motion of the ghost-particle field, $\partial_\mu A^\mu$, or as the Ward-Takahashi identity corresponding to localized gauge transformations of the second kind.

In its role as field equation, Eq. (47) leads, via the reduction formula, to the operator relation

$$\frac{1}{\alpha}(\Box+\alpha m_0{}^2)\partial^\mu A_\mu+\mu^2 w\chi=0; \tag{48}$$

thus $\partial_\mu A^\mu$ becomes a free field in the limit $\mu\to 0$. Note that in this limit the ghost particles do not interact with the physical particles of the pre-Higgs model, and this should make possible the construction of a unitary S matrix.

B. Connections to other formalisms

The consequences of Eq. (47) as a Ward-Takahashi identity are most conveniently expressed in terms of the vertex functions rather than the full Green's functions. To do this, we first rewrite Eq. (47) in terms of the generating functional for connected Green's functions:

$$-\frac{i}{\alpha}(\Box+\alpha m_0{}^2)\partial^\mu\frac{\delta G}{\delta J^\mu(x)}=-\partial^\mu J_\mu(x)+wJ^\chi(x)+i\mu^2 w\frac{\delta G}{\delta J^\chi(x)}+ie\Big[J^\psi(x)\frac{\delta G}{\delta J^\chi(x)}-J^\chi(x)\frac{\delta G}{\delta J^\psi(x)}\Big], \tag{49}$$

where

$$G[J^0,\ldots,J^5]=\sum_{n=1}^\infty\sum_{a_1\cdots a_n}\frac{i^n}{n!}\int J^{a_1}(x_1)\cdots J^{a_n}(x_n)\Big\langle T\prod_{j=1}^n A_{a_j}(x_j)\Big\rangle^{\text{conn}}dx_1\cdots dx_n. \tag{50}$$

The generating functional for vertex functions

$$\Gamma[K^0,\ldots,K^5]=\sum_{k=1}^\infty\sum_{a_1\cdots a_n}\frac{1}{n!}\int K_{a_1}(x_1)\cdots K_{a_n}(x_n)\Big\langle T\prod_{j=1}^n A^{a_j}(x_j)\Big\rangle^{\text{prop}}dx_1\cdots dx_n \tag{51}$$

is then introduced by means of the Legendre transformation,

$$\Gamma[K^0,\ldots,K^5]=G[J^0,\ldots,J^5]-i\int dx\sum_{b=0}^5 J^b(x)K_b(x), \tag{52}$$

where

$$K_a(x)=\frac{1}{i}\frac{\delta G}{\delta J^a(x)}$$

and hence

$$J^a(x)=i\frac{\delta\Gamma}{\delta K_a(x)}.$$

Equation (49) thus becomes

$$\frac{1}{\alpha}(\Box+\alpha m_0{}^2)\partial^\mu K_\mu(x)+\alpha\mu^2 wK_\chi(x)=-i\Big[\partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)}-w\frac{\delta\Gamma}{\delta K_\chi(x)}\Big]-ie\Big[K_\chi(x)\frac{\delta\Gamma}{\delta K_\psi(x)}-K_\psi(x)\frac{\delta\Gamma}{\delta K_\chi(x)}\Big], \tag{53}$$

which yields via functional differentiation an infinite set of identities relating the vertex functions. For $\mu=0$ and $m_0=0$, this coincides with the Ward-Takahashi identity employed by Lee in the Higgs model.[2] Note that as usual the vertex functions with more than two arguments are given by sums over proper (one-particle irreducible) diagrams.

C. Some simple consequences of these identities

For the sake of completeness we now include a discussion of how one can use Eqs. (49) and (53) to obtain information about the full propagators of the theory which are related to the two-point vertex functions by

$$\sum_{c=0}^5\Gamma^{ac}(p)G_{ab}(p)=-\delta^a_b, \tag{54}$$

where

$$G_{ab}(p)=\int dx\, e^{ipx}\langle TA_a(\tfrac{1}{2}x)A_b(-\tfrac{1}{2}x)\rangle,$$
$$\Gamma^{ab}(p)=\int dx\, e^{ipx}\langle TA^a(\tfrac{1}{2}x)A^b(-\tfrac{1}{2}x)\rangle^{\text{prop}}. \tag{55}$$

Equation (54) may be simplified considerably with the aid of Lorentz invariance, the discrete symmetry [Eq. (36)], and the relations

$$G_{ba}(p)=G_{ab}(-p),\quad \Gamma_{ba}(p)=\Gamma_{ab}(-p). \tag{56}$$

From these considerations it follows that the mixed two-point functions involving only one ψ field vanish, and that we may write the remaining ones in the form

$$G_{\mu\nu}(p)=\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)G^T_{AA}(p^2)+\frac{p_\mu p_\nu}{p^2}G^L_{AA}(p^2),$$
$$\Gamma_{\mu\nu}(p)=\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\Gamma^T_{AA}(p^2)+\frac{p_\mu p_\nu}{p^2}\Gamma^L_{AA}(p^2),$$
$$G_{\mu\chi}(p)=G_{\mu5}(p)=-G_{5\mu}(p)=p_\mu G_{A\chi}(p^2), \tag{57}$$
$$\Gamma_{\mu\chi}(p)=\Gamma_{\mu5}(p)=-\Gamma_{5\mu}(p)=p_\mu\Gamma_{A\chi}(p^2),$$
$$G_{44}(p)=G_{\psi\psi}(p^2),\quad G_{55}(p)=G_{\chi\chi}(p^2),$$
$$\Gamma_{44}(p)=\Gamma_{\psi\psi}(p^2),\quad \Gamma_{55}(p)=\Gamma_{\chi\chi}(p^2),$$

and in addition, it is useful to separate out the "trivial" part of the two-point vertex functions by introducing

$$\Gamma^T_{AA}(p^2)=-i[p^2-m^2-\Pi_A(p^2)],$$
$$\Gamma_{A\chi}(p^2)=-w\Gamma(p^2), \tag{58}$$
$$\Gamma_{\psi\psi}(p^2)=i[p^2-M^2-\Pi_\psi(p^2)].$$

In terms of the functions of p^2, Eq. (54) becomes

$$\begin{pmatrix}\Gamma^L_{AA} & \sqrt{p^2}\,\Gamma_{A\chi}\\ -\sqrt{p^2}\,\Gamma_{A\chi} & \Gamma_{\chi\chi}\end{pmatrix}\begin{pmatrix}G^L_{AA} & \sqrt{p^2}\,G_{A\chi}\\ -\sqrt{p^2}\,G_{A\chi} & G_{\chi\chi}\end{pmatrix}=-1,$$
$$\Gamma^T_{AA}G^T_{AA}=-1,\quad \Gamma_{\psi\psi}G_{\psi\psi}=-1, \tag{59}$$

which may be inverted to yield

$$G^T_{AA}=-(\Gamma^T_{AA})^{-1},\quad G_{\psi\psi}=-(\Gamma_{\psi\psi})^{-1}$$
$$G^L_{AA}=-\frac{\alpha\Gamma_{\chi\chi}}{D},\quad G_{\chi\chi}=\frac{\alpha\Gamma_{A\chi}}{D},\quad G_{\chi\chi}=-\frac{\alpha\Gamma^L_{AA}}{D},$$
$$D=\alpha(\Gamma^L_{AA}\Gamma_{\chi\chi}+p^2\Gamma_{A\chi}{}^2). \tag{60}$$

The Ward-Takahashi identity, Eq. (50), gives two identities among the two-point vertex functions. With the aid of Eqs. (57) and (58) these may be written

$$\Gamma^L_{AA}(p^2)-iw\Gamma_{A\chi}(p^2)=-\frac{i}{\alpha}(p^2-\alpha m_0{}^2),$$
$$p^2\Gamma_{A\chi}(p^2)-iw\Gamma_{\chi\chi}(p^2)=0, \tag{61}$$

or alternatively

$$\Gamma^L_{AA}=-\frac{i}{\alpha}(p^2-\alpha m_0{}^2)+iw^2\Gamma,$$
$$\Gamma_{\chi\chi}=i(p^2\Gamma-\mu^2). \tag{62}$$

The implications of the Ward-Takahashi identity for the two-point Green's functions may now be obtained by inserting Eq. (62) in Eq. (60). One then obtains the following general form for the propagators of the explicitly broken Higgs model:

$$G_{\mu\nu}(p)=\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{i}{p^2-m^2-\Pi_A(p^2)}+\frac{p_\mu p_\nu}{p^2}\,\frac{(-i\alpha)(p^2\Gamma-\mu^2)}{D},$$
$$G_{\mu\chi}(p)=-p_\mu\frac{\alpha w\Gamma}{D}, \tag{63}$$
$$G_{\psi\psi}(p^2)=\frac{i}{p^2-M^2-\Pi_\psi(p^2)},$$
$$G_{\chi\chi}(p^2)=\frac{i(p^2-\alpha m_0{}^2)-i\alpha w^2\Gamma}{D},$$

where

$$D=[p^2(p^2-\alpha m_0{}^2)+\alpha\mu^2w^2]\Gamma-\mu^2(p^2-\alpha m_0{}^2).$$

In the limit $\mu\to 0$ the propagators assume the form

$$G_{\mu\nu}(p)=\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{-i}{p^2-m^2-\Pi_A(p^2)}+\left(\frac{p_\mu p_\nu}{p^2}\right)\frac{-i\alpha}{p^2-\alpha m_0{}^2},$$
$$G_{\mu\chi}(p)=-\frac{\alpha w p_\mu}{p^2(p^2-\alpha m_0{}^2)}, \tag{64}$$
$$G_{\psi\psi}(p^2)=\frac{i}{p^2-M^2-\Pi_\psi(p^2)},$$
$$G_{\chi\chi}(p^2)=\frac{i}{p^2\Gamma(p^2)}-\frac{i\alpha w^2}{p^2(p^2-\alpha m_0{}^2)}.$$

Note that $G_{\mu\nu}$, $G_{\mu\chi}$, and $G_{\chi\chi}$ all have poles at $p^2=\alpha m_0{}^2$ with residues coinciding with their unperturbed values. This is not surprising, since we have already seen that such poles correspond to noninteracting ghosts.

The expressions given in Eq. (64) are not complete without a specification of normalization conditions for $\Pi_A(p^2)$, $\Pi_\psi(p^2)$, and $\Gamma(p^2)$. These emerge most naturally from a discussion of the pre-Higgs mass spectrum based on Eq. (64). Most importantly $G_{\mu\nu}$, $G_{\mu\chi}$, and $G_{\chi\chi}$ should all have poles at $p^2=0$, corresponding to the scalar χ particle (Goldstone boson). These will clearly be present, provided that $\Pi_A(0)$ and $\Gamma(0)$ are finite and independent of e. In order to fulfill the latter requirements, we choose as one of the defining normalization conditions of the model (already for $\mu \neq 0$)

$$\Pi_A(0)=0\,. \tag{65}$$

Since in perturbation theory the vector propagator does not develop a singularity at $p^2=0$ (if $\mu\neq 0$), the coefficient of $p_\mu p_\nu/p^2$ in the expression for $G_{\mu\nu}$ in Eq. (63) should vanish at $p^2=0$; hence we have

$$\Gamma(0)=1+w^{-2}\Pi_A(0)\,. \tag{66}$$

Thus, imposing Eq. (65) automatically entails

$$\Gamma(0)=1\,. \tag{67}$$

As Eqs. (66) and (67) remain valid in the $\mu\to 0$ limit, they fix the (finite) residues of the Goldstone poles of $G_{\mu\nu}$ and $G_{\chi\chi}$ in the pre-Higgs model.[7]

The χ is the only stable (physical) particle in the pre-Higgs model. The massive spin-one vector boson and spin-zero ψ particle of the unperturbed theory become unstable in the interacting theory, with the vector boson decaying into an odd number and the ψ decaying into an even number of χ's. The instability of these particles is accompanied, via the usual unitarity arguments, by nonzero imaginary parts of $\Pi_A(p^2)$ and $\Pi_\psi(p^2)$ for positive p^2. Thus $G_{\mu\nu}$ and $G_{\psi\psi}$ will not have simple poles at $p^2=m^2$ and $p^2=M^2$, respectively (except in zeroth order). In a nonperturbative formulation of the model, one might expect to find a pole off the real axis on an "unphysical" sheet of each of the Π functions, corresponding physically to a resonance in the scattering amplitudes for particles. To pin down the positions of these resonances, one may specify the points at which the real parts of Π_A and Π_ψ vanish to all orders:

$$\begin{aligned}&\mathrm{Re}\Pi_A(w^2)=0\,,\\&\mathrm{Re}\Pi_\psi(M^2)=0\,.\end{aligned} \tag{68}$$

In imposing Eqs. (68) we have in mind, of course, the eventual Higgs limit, $m_0\to 0$, in which the vector boson, and perhaps the ψ, become stable, with respective masses w and M.

To see how the normalization conditions (65), and Eqs. (68), as well as the vanishing of $\langle\psi\rangle$ to all orders, determine the counterterms (z_j-1) $(j=1,2,3)$ and c in the effective Lagrangian of the explicitly broken Higgs model, let us write each of the quantities $\langle\psi\rangle$, $\Pi_A(p^2)$, and $\Pi_\psi(p^2)$ as a sum of a trivial part and a part from diagrams with at least one loop:

$$\begin{aligned}&\langle\psi\rangle=\frac{c}{hew}+\hat{T}\,,\\&\Pi_A(p^2)=(z_2-1)w^2-(z_3-1)p^2+\hat{\Pi}_A(p^2)\,,\\&\Pi_\psi(p^2)=(z_1-1)M^2-(z_2-1)p^2-c+\hat{\Pi}_\psi(p^2)\,.\end{aligned} \tag{69}$$

From Eq. (69) and the normalization conditions, it follows that

$$\begin{aligned}&c=-hew\hat{T}\,,\\&z_1-1=-M^2[\,2h\,\mathrm{Re}\hat{\Pi}_A(0)+hew\hat{T}+\mathrm{Re}\hat{\Pi}_\psi(M^2)]\,,\\&z_2-1=-w^{-2}\hat{\Pi}_A(0)\,,\\&z_3-1=w^{-2}\,\mathrm{Re}[\,\hat{\Pi}_A(w^2)-\hat{\Pi}_A(0)]\,,\end{aligned} \tag{70}$$

which may be solved recursively to arbitrary order in perturbation theory.

It is interesting to observe that although the Green's functions normalized as in Eqs. (65) and (68) approach finite limits when $\mu\to 0$,[8] the renormalization counterterms diverge logarithmically in that limit. This is of course the motivation for introducing the explicitly broken Higgs model, which in itself is of little interest, but which allows us to take as our starting point a well-defined effective Lagrangian.

D. Some observations concerning the Higgs limit of this model

Turning now to the Higgs limit, we note that the pre-Higgs Green's functions will not in general approach finite limits when $m_0\to 0$ tends to zero. In particular, the double-pole term of $G_{\chi\chi}$, which is already present in zeroth order, develops a logarithmic divergence concentrated at $p=0$.[18] One way of resolving this problem is to restrict oneself to the Landau gauge, $\alpha=0$, before passing to the limit $m_0\to 0$. Then the double pole will be absent and the Higgs limit can be shown to exist. The two-point functions will then assume the form given by Lee,[9] namely,

$$\begin{aligned}&G_{\mu\nu}(p)=\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{-i}{p^2-w^2-\Pi_A(p^2)}\,,\\&G_{\mu\chi}(p)=0\,,\\&G_{\psi\psi}(p^2)=\frac{i}{p^2-M^2-\Pi_\psi(p^2)}\,;\\&G_{\chi\chi}=\frac{i}{p^2\Gamma(p^2)}\,,\end{aligned} \tag{71}$$

with the normalization conditions

$$\begin{aligned}&\Pi_A(0)=\Pi_A(w^2)=\mathrm{Re}\Pi_\psi(M^2)=0\,,\\&\Gamma(0)=1\,.\end{aligned} \tag{72}$$

Note that zero-mass poles are still present in the Landau-gauge propagators of the Higgs model.

These correspond to positive- and negative-metric ghosts which do not participate in physical processes. With the decoupling of the zero-mass χ particle, the spin-one vector boson becomes stable, and our normalization condition on $\Pi_A(p^2)$ fixes a pole in $G_{\mu\nu}$ at $p^2=w^2$. If in addition $m<2w$, the spin-zero ψ particle also becomes stable, in which case $G_{\psi\psi}$ has a pole at $p^2=M^2$.

Actually, it is not necessary to resort to Landau gauge in order to extract the physical content of the Higgs model. As will be shown in paper III, all physical S-matrix elements of both the pre-Higgs and Higgs models may be expressed, via reduction formulas, in terms of the gauge-invariant Green's functions of the conserved current, j_μ. The gauge invariance and unitarity of the S matrix then follow easily from the corresponding properties of these Green's functions. In addition, the latter provide one possible specification of the observable off-mass-shell content of the theory. These ideas will be explored more fully in paper III.

IV. SUBTRACTION RULES FOR THE EXPLICITLY BROKEN GOLDSTONE AND PRE-HIGGS MODELS

The purpose of the remainder of this paper is to provide the background for the discussions in paper II. Of course, it is not possible to include all the details of the renormalization procedure and derivation of field equations in this brief account, but fortunately a knowledge of the fine points is not a prerequisite for understanding the key features.

The first point we wish to make is that conventional methods of renormalization fail for the Lagrangian of Eq. (40) as it stands, because—as will be shown in paper II—the application of Bogoliubov's prescription[10] for defining regularized Green's functions in its standard form leads to anomalies in the equation of partial current conservation. As a consequence, the current of the Higgs and pre-Higgs models will not be conserved in the limit $\mu^2\to 0$, and so the S matrix becomes nonunitary since the ghost particles do not decouple from the system.

Two alternative methods are presently available for resolving this difficulty. One possibility is to include in the Lagrangian additional counterterms which are nonlinear and not invariant under the gauge transformations of Eqs. (9) and (16). The coefficients of the Lagrangian must then be correlated in such a way that the field equations and the Ward identities continue to hold. If one does this, it is then possible to apply Bogoliubov's method without modification. This discussion was first carried through by Symanzik[11] for the Goldstone model and applied to the Higgs model by Lee.[12]

In the second method the subtraction scheme is modified while the Lagrangian is not changed, and so it is possible to maintain the gauge invariance of the nonlinear part of the Lagrangian. This gauge-invariant renormalization method was used by Lee in his treatment of the Goldstone model[13] and later extended by Lee and Zinn-Justin[14] to non-Abelian models with spontaneous symmetry breaking.

In the present work the second method of gauge-invariant quantization is developed without introducing the regularization which was essential for Lee's[13] treatment. Instead we employ Dyson's technique of making subtractions in momentum space in order to extract the finite part of a Feynman integral. We will, however, change the subtraction scheme significantly by including subtractions with respect to the symmetry-breaking parameters. Already the examples given in Ref. 15 suggest how to modify the subtraction terms by inserting the propagators of the fully symmetric theory. In the general case the situation is more involved. In order to remove linear or quadratic divergences in a consistent manner one must also allow for first- and second-order symmetry breaking in some of the subtraction terms.

A. A heuristic approach

In order to motivate the modified subtraction procedure, it is worthwhile to describe first a somewhat artificial formulation of the explicitly broken pre-Higgs model, in which both e and w are treated as perturbation parameters. (For the corresponding formulation of the explicitly broken Goldstone model[17] one uses g and $M^2-\mu^2$ as expansion parameters.) The Green's functions in this approach may be calculated to arbitrary order in each of the two parameters by means of the Gell-Mann–Low expansion, with the unperturbed theory defined by

$$\mathcal{L}_0=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}m_0{}^2A_\mu A^\mu-\frac{1}{2\alpha}(\partial_\mu A^\mu)$$
$$+\tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi+\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi-\tfrac{1}{2}\mu^2(\psi^2+\chi^2) \qquad (73)$$

for the explicitly broken pre-Higgs model and

$$\mathcal{L}_0=\tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi+\tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi-\tfrac{1}{2}\mu^2(\psi^2+\chi^2) \qquad (74)$$

for the explicitly broken Goldstone model. The interaction Lagrangian is given by $\mathcal{L}_I=\mathcal{L}-\mathcal{L}_0$, with the same as in Eq. (26) or Eq. (39), respectively.

Suppose that

$$\int dk_1\cdots dk_m I_\Gamma(k_1\cdots k_m;p_1\cdots p_n;s) \qquad (75)$$

is the unrenormalized integral associated with some one-particle irreducible Feynman diagram Γ. The *degree* (or superficial divergence) of Γ is defined to be

$$d(\Gamma)=4-n-\sum_{V\in\Gamma}(4-d_V), \tag{76}$$

where n is the number of external lines of Γ, the summation is over all vertices of Γ, and d_V is the canonical dimension of the product of fields associated with the vertex V. The expression for $d(\Gamma)$ can be simplified somewhat by noting that each term in $\mathcal{L}_I$ of d_V contains a factor s^{4-d_V}, so that

$$d(\Gamma)=4-n-\nu, \tag{77}$$

where ν is the overall power of s in I_Γ.

For a primitive diagram (i.e., a one-particle irreducible diagram all of whose proper subdiagrams have negative degree) Γ with negative degree, the Feynman integral is convergent without subtractions. For $d(\Gamma)\geq 0$, the finite part is defined as the integral of

$$R_\Gamma=(1-t^{d(\Gamma)}_{p_1\cdots p_n})I(k_1\cdots k_m;p_1\cdots p_n;s)|_{s=1}, \tag{78}$$

where $t^{d(\Gamma)}_{p_1\cdots p_n}$ denotes the Taylor expansion to order $d(\Gamma)$ in $p_1\cdots p_n$ about $p_1=p_2=\cdots=p_n=0$. For nonprimitive diagrams, a subtraction must be made corresponding to each set of nonoverlapping subdiagrams of nonnegative degree. For the precise formulation of the general finite-part prescription, based on Bogoliubov's combinatorial technique, the reader is referred to Ref. 16. The resulting renormalized Feynman integrals may be shown to be convergent by an application of Weinberg's power-counting theorem.

Thus far, the parameter s has been used only as an aid in calculating $d(\Gamma)$. It assumes a more important role if we exploit the commutation property of the Taylor operator,

$$t^d_x x^n f(x)=x^n t^{d-n}_x f(x), \tag{79}$$

to rewrite R_Γ for primitive Γ as

$$R_\Gamma=(1-t^{4-n}_{p_1\cdots p_n s})I_\Gamma(k_1\cdots k_m;p_1\cdots p_n;s)|_{s=1}, \tag{80}$$

where $t^{4-n}_{p_1\cdots p_n s}$ is now a joint Taylor series in $p_1\cdots p_n$ *and* s about $p_j=s=0$. *Observe that we now have a uniform subtraction degree which depends only on the number of external lines of a diagram, as in simpler renormalizable models.*

In the alternative version of these models in which one uses $\mathcal{L}_0$ as defined in Eqs. (73) and (74) the subtraction prescriptions of Eqs. (78) and (80) are entirely equivalent and it is purely a matter of taste which one adopts. If, on the other hand, one applies these definitions of the renormalized Feynman integral to our version of the theory, with $\mathcal{L}_0$ as given in Eq. (26), one finds that the two expressions are not equivalent: It is obvious that the s-dependent free propagators will be treated differently in the two subtraction formulas. At this point we will merely note that if one wishes to use the manifestly symmetrical form of the Lagrangian for the theory then it is the prescription in Eq. (80) which leads to a conserved current, and thence to the Ward-Takahashi identities of Sec. III, whereas the prescription of Eq. (78) inevitably produces a nonconserved current. We therefore adopt Eq. (80) as the definition of our renormalized integral. (The proof of convergence of the resulting integrals requires a generalization of that given in Ref. 5, but this can be done in a straightforward manner.[16])

Formally, our version of the explicitly broken pre-Higgs model, including the correct subtraction scheme, may be obtained by defining R_Γ as in Eqs. (76) and (78) and then for each diagram Γ summing over all diagrams which differ from it only by the insertion—in all possible ways—of the vertices

$$\mathcal{L}_{0s}=\tfrac{1}{2}s^2w^2A_\mu A^\mu-swA_\mu\partial^\mu\chi-hs^2w^2\psi^2.$$

This correspondence is not a rigorous one, however, since the sums, which involve only geometric series, are not convergent for all ranges of the integration variables $k_1,k_2,\ldots,k_m$.

We conclude this section with some simple examples of the subtraction procedure sketched above.

B. Explicitly broken Goldstone model

Example 1 (see Fig. 1).

$$I_{\Gamma_1}=\frac{g^2}{[(p-k)^2-\mu^2+i0](k^2-\mu^2+i0)},$$

$$R_{\Gamma_1}=(1-t^0_{p,s})I_{\Gamma_1}(k,p)|_{s=1}$$

$$=g^2\left[\frac{1}{[(p-k)^2-\mu^2+i0](k^2-\mu^2+i0)}-\frac{1}{(k^2-\mu^2+i0)^2}\right].$$

Example 2 (see Fig. 2).

$$I_{\Gamma_2}=\frac{g^2}{[(p-k)^2-\mu^2-s^2(M^2-\mu^2)+i0][k^2-\mu^2-s^2(M^2-\mu^2)+i0]},$$

$$R_{\Gamma_2}=g^2\left[\frac{1}{[(p-k)^2-M^2+i0](k^2-M^2+i0)}-\frac{1}{(k^2-\mu^2+i0)^2}\right].$$

C. Explicitly broken pre-Higgs model

Example 1 (see Fig. 3).

$$I_{\Gamma_1}=\frac{he^2}{k^2-\mu^2-2hw^2s^2+i0},$$

$$R_{\Gamma_1}=(1-t^2_{p,s})I_\Gamma(k,p,s)|_{s=1}$$

$$=he^2\left(\frac{1}{k^2-\mu^2-2hw^2+i0}-\frac{1}{k^2-\mu^2+i0}-\frac{2hw^2}{(k^2-\mu^2+i0)^2}\right).$$

Example 2 (see Fig. 4).

$$I_{\Gamma_2}=(2ehw)^2s^2\frac{k^2-\alpha m_0^2-\alpha w^2s^2}{[(p-k)^2-\mu^2-2hw^2s^2+i0][(k^2-\mu^2+i0)(k^2-\alpha m_0^2+i0)+\alpha\mu^2w^2s^2]},$$

$$R_{\Gamma_2}=(1-t^2_{p,s})I_{\Gamma_2}(k,p,s)|_{s=1}$$

$$=(2ehw^2)\left[\frac{k^2-\alpha(m_0^2+w^2)}{[(p-k)^2-\mu^2-2hw^2+i0](k^2-x^2+i0)(k^2-\lambda^2+i0)}-\frac{1}{(k^2-\mu^2)^2}\right].$$

Example 3 (see Fig. 5).

$$I_{\Gamma_3}=\frac{3hews}{k^2-\mu^2-2hw^2s^2+i0},$$

$$R_{\Gamma_3}=(1-t^3_{p,s})I_{\Gamma_3}|_{s=1}$$

$$=3\frac{w}{e}R_{\Gamma_1}.$$

D. Remarks about the limits $\mu^2\to 0$ and $m_0\to 0$

We conclude this section with some comments on the examples given for the explicitly broken pre-Higgs model. The tadpole diagram Γ_3 contributes a term

$$c_3=\frac{3i}{e}\int\frac{dk}{(2\pi)^4}R_{\Gamma_1}(k)$$

to the counterterm c of the Lagrangian defined in Eq. (26). Note that the contributions of Γ_1 and Γ_2 to Π_χ are both logarithmically divergent when $\mu\to 0$, thanks to the subtraction terms with the denominators $(k^2-\mu^2)^2$. The same is true of the contribution of the trivial diagram with coefficient ic_3e. Nevertheless, the sum over these three terms,

$$\sum=ic_3e+\int\frac{dk}{(2\pi)^4}R_{\Gamma_1}(k)+\int\frac{dk}{(2\pi)^4}R_{\Gamma_2}(k,p)$$

$$=(2ehw)^2\int\frac{dk}{(2\pi)^4}\left[\frac{1}{[(p-k)^2-\mu^2-2hw^2+i0](k^2-\kappa^2+i0)}-\frac{1}{(k^2-\mu^2-2hw^2+i0)(k^2-\mu^2+i0)}\right.$$

$$\left.-\frac{\alpha w^2+(\lambda^2-\alpha m_0^2)}{[(p-k)^2-\mu^2-2hw^2+i0](k^2-\kappa^2+i0)(k^2-\lambda^2+i0)}\right],$$

is finite in the limit, illustrating the sort of cancellations which enable the full self-energy part to be finite. In the Higgs limit, $m_0\to 0$, the combination of the first two terms in $\sum$ approaches a finite limit,

$$\sum\nolimits_{\text{Higgs}}=(2hew)^2\int\frac{dk}{(2\pi)^4}\frac{1}{k^2+i0}\left(\frac{1}{(p-k)^2-2hw^2+i0}-\frac{1}{k^2-2hw^2+i0}\right).$$

The third term, on the other hand, acquires a logarithmic divergence (as $\kappa^2\to 0$, $\lambda^2\to 0$) except in Landau gauge ($\alpha=0$), where it vanishes. As discussed in Sec. III, this behavior is typical of the $m_0\to 0$ limit of the pre-Higgs Green's function.

V. NORMAL PRODUCTS AND FIELD EQUATIONS; A FEW BRIEF REMARKS

The subtraction procedure outlined in the preceding section may be readily adapted to the defi-

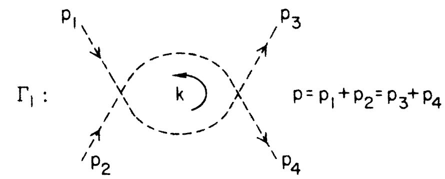

FIG. 1. This diagram is a typical contribution to the χ four-point function. Throughout, dashed lines represent χ propagators and solid lines represent ψ propagators.

nition of finite Green's functions of a normal product, $N_\delta[Q]$, where Q is a formal product of basic fields and their derivatives, where the degree δ is an integer at least as large as the canonical dimension d of Q. Such a Green's function is given formally by a Gell-Mann–Low expansion in which each Feynman diagram contains a distinguished vertex V_Q corresponding to the normal product. The finite-part prescription for a primitive diagram Γ then assumes the form

$$R_\Gamma = (1 - t^{\delta(\Gamma)}_{p_1 \cdots p_n s}) I_\Gamma |_{s=1}, \tag{81}$$

where

$$\delta(\Gamma) = \begin{cases} 4-n \text{ if } V_Q \notin \Gamma, \\ \delta - n \text{ if } V_Q \in \Gamma, \end{cases}$$

with a corresponding modification of the rules for nonprimitive diagrams.

Some of the most frequently encountered normal products in this (or any) model are those associated with the amputation of an external line (free propagator),

$$\left\langle TA_a(x) \prod_j^n A_{a_j}(x_j) \right\rangle = \sum_{b=0}^{5} \int dy\, G^0_{ab}(x-y) \left\langle TA_b(y) \,\middle|\, \prod_j^n A_{a_j}(x_j) \right\rangle, \tag{82}$$

where G^0_{ab} is the free two-point function whose inverse is obtained by applying the Euler derivative to $\mathcal{L}_0$. Hence

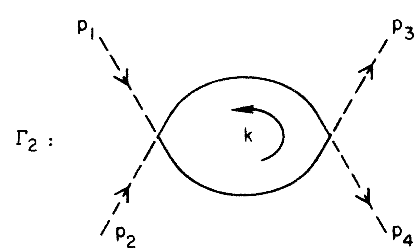

FIG. 2. Another contribution to the χ four-point function.

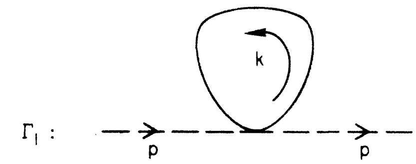

FIG. 3. A typical self-energy correction to the χ propagator which would not exist in the unmodified subtraction scheme.

$$\left\langle TA_b(y) \,\middle|\, \prod_j^n A_{a_j}(x_j) \right\rangle = -i \left\langle T \frac{\hat\partial \mathcal{L}_0}{\hat\partial A_a}(y) \prod_j^n A_{a_j}(x_j) \right\rangle, \tag{83}$$

where

$$\frac{\hat\partial}{\hat\partial A_a} = \frac{\partial}{\partial A_a} - \partial^\mu \frac{\partial}{\partial(\partial^\mu A_a)}.$$

On the other hand, the amputated function may also be expressed as a linear combination of normal products by considering the various types of interaction vertices at which the amputated propagator terminates. Suppose that a line originating at an external vertex of type a ends at a vertex corresponding to a term in $\mathcal{L}_I$ proportional to

$$P(y) = \prod_{i=1}^n \partial^{(i)} A_{a_i}(y), \tag{84}$$

where $\partial^{(i)}$ is a k_ith-order differential operator; then, in particular, one of the factors of $P(y)$, say the jth, with $a_j = b$, is contracted to form $\partial_y^{(j)} G^0_{ab}(x-y)$, leaving the remainder of the diagram with a distinguished vertex corresponding to the field product with $M-1$ factors,

$$P_j'(y) = \prod_{i \neq j}^m \partial^{(i)} A_{a_i}(y). \tag{85}$$

To arrive at the contribution to the amputated diagram as defined implicitly in (82), we integrate by parts, so that

$$\partial_y^{(j)} G^0_{ab}(x-y) \to G^0_{ab}(x-y),$$

$$P_j'(y) \to P_j(y) = (-1)^{k_j} \partial^{(j)} \prod_{i \neq j}^m \partial^{(i)} A_{a_i}(y).$$

Summing over all possible contractions with the factors of $P(y)$ yields

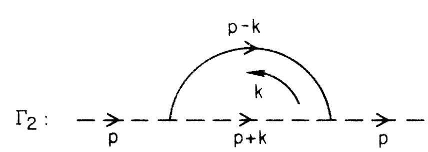

FIG. 4. Another self-energy correction.

Γ_3 :

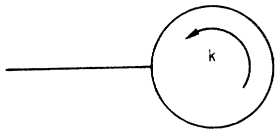

FIG. 5. A ψ-tadpole graph; this would also vanish in the unmodified subtraction scheme.

$$\sum_j \delta_{ba_j} P_j(y) = \frac{\hat{\partial} P}{\delta A_a}(y). \tag{86}$$

Computation of the number of subtractions for subgraphs containing the special vertex is easy, since for the original Green's function each interaction vertex is assigned a degree equal to its dimension (once the s factors have been commuted through). Hence the contribution to the amputated function of diagrams in which the distinguished vertex is of type $P(y)$ is proportional to

$$\left\langle TN_{d_p-1}\left[\frac{\delta P}{\delta A_b}(y)\right]\prod_i^m A_{a_i}(x_i)\right\rangle, \tag{87}$$

where

$$d_p = M + \sum_i^m k_i\,.$$

In case the external vertex is not contracted with any interaction vertex, but rather directly with the kth external vertex, the contribution to the amputated function is obviously given by

$$\delta(y-x_k)\prod_{j\neq k} A_{a_j}(x_j). \tag{88}$$

Summing over all possibilities, we obtain

$$\left\langle TA_b(y)\,\middle|\,\prod_j A_{a_j}(x_j)\right\rangle = i\left\langle TN\left[\frac{\hat{\partial}\mathcal{L}_I}{\delta A_b}(y)\right]\prod_j^n A_{a_j}(x_j)\right\rangle + \sum_{k=1}^n \delta_{ba_k}\delta(y-x_k)\left\langle T\prod_{j\neq k}^n A_{a_j}(x_j)\right\rangle, \tag{89}$$

where N without a subscript denotes the normal product with minimal degree assignment. Comparison of (89) with (83) then yields the *field equations*

$$\left\langle TN\left[\frac{\hat{\partial}\mathcal{L}}{\delta A_b}(y)\right]\prod_{j=1}^N A_{a_j}(x_j)\right\rangle = i\sum_{n=1}^N \delta_{ba_n}\delta(y-x_n)\left\langle T\prod_{j\neq n} A_{a_j}(x_j)\right\rangle. \tag{90}$$

Apart from the normal-product symbol and the δ-function terms, these are precisely the classical equations of motion discussed in Sec. II. In the case of the vector field, Eq. (90) may be written explicitly as

$$-\left\langle T\left[\,z_3\partial_\nu F^{\mu\nu}(x) - \frac{1}{\alpha}\partial^\mu\partial_\nu A^\nu(x) - (m_0^{\,2} + z_2 w^2)A^\mu(x)\,\right]X\right\rangle$$

$$= \langle Tz_2 N[\,e\chi\overleftrightarrow{\partial}^\mu\psi - w\partial^\mu\chi + e^2A^\mu(\psi^2+\chi^2) + 2wA^\mu\psi](x)X\rangle + i\sum_{n=1}\delta^\mu_{\mu_n}\delta(x-x_n)\left\langle T\prod_{j\neq n}A_{\mu_j}(x_j)\prod_k\psi(y_k)\prod_l\chi(z_l)\right\rangle. \tag{91}$$

Shifting

$$\langle T[\,z_2 w^2 A^\mu - (z_3-1)\partial_\nu F^{\mu\nu}\,]X\rangle$$

to the right-hand side then gives Eq. (43).

The same sort of graphical arguments used to establish Eq. (90) may be applied to obtain field equations of the type

$$\left\langle TN\left[A_a\frac{\hat{\partial}\mathcal{L}}{\delta A_b}\right](x)\prod_j^N A_{a_j}(x_j)\right\rangle = i\sum_{k=1}^N \delta_{ba_k}\delta(y-x_k)\left\langle TA_a(x_k)\prod_{j\neq k}A_{a_j}(x_j)\right\rangle. \tag{92}$$

Such equations are crucial to establishing the fundamental Ward-Takahashi identity of the current of Eq. (45). For further details the reader is referred to paper II.

ACKNOWLEDGMENTS

The authors would like to thank P. Dürr and L. Van Hove for their hospitality at the Max-Planck-Institut für Physik und Astrophysik, München, during the time part of this work was done.

We gratefully acknowledge helpful discussions with N. Christ, B. W. Lee, T. D. Lee, A. Mueller, N. Papanicolaou, B. Schroer, R. Stora, and G. 't Hooft.

*Work supported in part by funds from National Science Foundation.

†Work supported by the U. S. Atomic Energy Commission.

[1]G. 't Hooft, Nucl. Phys. B33, 173 (1971); B35, 167 (1971); B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 3121 (1972); 5, 3137 (1972); 5, 3155 (1972); 7, 1049 (1973); 8, 4654(E) (1973).

[2]B. W. Lee, Phys. Rev. D 5, 823 (1972).

[3]These results provide the basis for the discussions given in M. Weinstein, Phys. Rev. D 7, 1854 (1973); 8, 2511 (1973).

[4]G. 't Hooft and T. Veltman, Nucl. Phys. B44, 189 (1972).

[5]W. Zimmermann, Commun. Math. Phys. 15, 208 (1969).

[6]This method has already been applied to the construction of a finite energy-momentum tensor, the discussion of explicitly broken symmetries, and the anomalies of partially conserved currents. See J. Lowenstein, Phys. Rev. D 4, 2281 (1971); Univ. of Maryland Report No. 73-068, 1972 (unpublished); A. Rouet and R. Stora, Nuovo Cimento Lett. 4, 136 (1972); 4, 139 (1972); J. Lowenstein and B. Schroer, Phys. Rev. D 6, 1553 (1972); 7, 1929 (1973).

[7]A proof of this point will be discussed in part III.

[8]To be discussed in paper III.

[9]See Ref. 2.

[10]N. N. Bogoliubov and D. W. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959); K. Hepp, Commun. Math. Phys. 2, 301 (1965); Zimmerman (Ref. 5).

[11]K. Symanzik, Commun. Math. Phys. 16, 48 (1970).

[12]See Ref. 2.

[13]B. W. Lee, Nucl. Phys. B9, 649 (1969).

[14]B. W. Lee and J. Zinn-Justin, Phys. Rev. D 5, 823 (1972); 5, 3121 (1972); 5, 3137 (1972); 8, 4654(E) (1973).

[15]A. Rouet and R. Stora, Nuovo Cimento Lett. 4, 139 (1972).

[16]M. Gomes, J. Lowenstein, and W. Zimmermann, Commun. Math. Phys. (to be published).

[17]For a detailed treatment of this model see F. Jegerlehner and B. Schroer, Nucl. Phys. B68, 461 (1974).

[18]I. M. Gel'Fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964).