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Nonperturbative effective potential for $\lambda\phi^4$ theory in the many-field limit*

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A $\lambda\phi^4$ theory of self-interacting boson fields with O(N) symmetry is considered. A nonperturbative method is developed to compute the effective potential, which becomes exact in the large-N limit. The stability of the vacuum and possible spontaneous symmetry breakdown of the theory are studied in this limit. The existence of a local mimimum requires $0 < 1 + (N\lambda)^{-1}96\pi^2$, which may be either a true bound or an indication of the breakdown of the N^{-1} expansion. The function $\beta(\lambda)$ that appears in the Callan-Symanzik equations is calculated in the large-N limit of the massless theory without reference to a perturbation expansion in λ. It is observed that the presence of a nontrivial zero of $\beta(\lambda)$ is a necessary condition for a spontaneously broken symmetry in the massless theory. On this basis, it is shown that this $\beta(\lambda)$ is unlikely to have a nontrivial zero. The leading-logarithm approximation to the effective potential is discussed within the above framework.

I. INTRODUCTION

A number of the basic questions of quantum field theory all too often seem to have only tentative answers based on a perturbation expansion in the coupling constant. One such issue dealt with in recent applications of field theory is that of the spontaneous breakdown of symmetry.[1] Another is the short-distance behavior of renormalizable field theories, as expressed by the renormalization group or Callan-Symanzik equations.[2] A complete understanding of questions of this sort is lacking, in part due to the limitations of calculational methods.

We have tried to develop a calculational scheme which does not involve a perturbation expansion in the coupling constant. For the moment we have studied the somewhat academic case of $\lambda\phi^4$ theory with O(N) symmetry. A nonperturbative method, which becomes exact in the large-N limit, has been developed for the calculation of the effective potential of the theory. This enables us to examine questions of the stability of the vacuum, spontaneous breakdown of symmetry, and possible nontrivial zeros of $\beta(\lambda)$, the function that appears in the Callan-Symanzik equation, without recourse to an expansion in λ.

In this paper this scheme is presented, together with simple applications of the results. In order to provide the framework for this discussion, we rederive the familiar loop expansion of the generating functional[3-5] for the one-particle irreducible (1PI) Green's functions by a somewhat unfamiliar method. This allows us to identify the terms in the functional differential equation for the 1PI generating functional which are dominant in the large-N limit.[6] The result is a coupled pair of functional equations, which are closed in this limit. When the generating functional is specialized to constant external field, the functional equations reduce to ordinary equations for the effective potential of the theory. It turns out that these equations can be solved exactly in the large-N limit, with the solution expressed implicitly in terms of a transcendental equation. This transcendental equation lends itself to a qualitative analysis, which we discuss, as well as a numerical analysis, which is not presented here.

Initially, our system of equations involves unrenormalized quantities and a cutoff. The renormalization of these nonlinear, implicit equations is carried out, resulting in a coupled pair of equations for the effective potential, which is expressed in terms of renormalized quantities only, and is finite. The qualitative behavior of the effective potential is studied, leading to conclusions as to the stability of the vacuum, and possible spontaneous symmetry breaking in the theory. Since the effective potential for the massless theory satisfies a homogeneous Callan-Symanzik[2,7] (CS) equation, we are able to compute the function $\beta(\lambda)$, which appears in the CS equation, exactly in the large-N limit, independent of a perturbation expansion in λ. It is shown that a *necessary* condition[8] for spontaneous symmetry breaking in the *massless* theory is that $\beta(\lambda)$ have a nontrivial zero. As a result, we argue that it is unlikely that this $\beta(\lambda)$ has a nontrivial zero.

The paper concludes with two appendixes; one gives the details of the renormalization, and the other describes the truncation of our equations required to produce the leading-logarithm approximation.[6]

II. THE LOOP EXPANSION

A. Functional differential equations

Consider N self-interacting scalar bosons in the presence of an external source, with O(N)-symmetric interaction, described by the Lagrangian

$$\mathcal{L}(\tilde{\phi}_a, j) = \tfrac{1}{2}[(\partial_\mu \tilde{\phi}_a)^2 - \mu^2 \tilde{\phi}^2] - \frac{\lambda}{4!}\tilde{\phi}^4 + j_a(x)\tilde{\phi}_a(x) , \tag{2.1}$$

where $\tilde{\phi}^2 = \tilde{\phi}_a\tilde{\phi}_a$, $\tilde{\phi}^4 = (\tilde{\phi}^2)^2$, and repeated internal indices are summed. The equation of motion for the field is

$$(\Box_x + \mu^2)\tilde{\phi}_a + \frac{\lambda}{3!}\tilde{\phi}^2\tilde{\phi}_a = j_a(x) . \tag{2.2}$$

The generating functional for the vacuum amplitude in the presence of the external source[3] is

$$\langle 0^+ | 0^- \rangle_j = W(j) = (\text{const}) \int [d\tilde{\phi}_a] \exp\left[i \int d^4x\, \mathcal{L}(\tilde{\phi}_a, j)\right] . \tag{2.3}$$

The functional $W(j)$ satisfies the functional differential equation

$$(\Box_x + \mu^2)\left[\frac{\delta W(j)}{\delta j_a(x)}\right] - \frac{\lambda}{3!}\left[\frac{\delta^3 W(j)}{\delta j_a(x)\,\delta j_b(y)\,\delta j_b(z)}\right]_{x=y=z} = i j_a(x) W(j) . \tag{2.4}$$

The generating functional for the connected diagrams[3] of the theory, $Z(j)$, is defined by

$$Z(j) = -i \ln W(j) . \tag{2.5}$$

This functional satisfies the functional differential equation

$$(\Box_x + \mu^2)\left[\frac{\delta Z(j)}{\delta j_a(x)}\right] = \frac{\lambda}{3!}\left[\frac{\delta^3 Z(j)}{\delta j_a(x)\,\delta j_b(y)\,\delta j_b(z)}\right]_{x=y=z} + \frac{i\lambda}{3!}\left\{2\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(y)}\right]\left[\frac{\delta Z(j)}{\delta j_b(z)}\right] + \left[\frac{\delta^2 Z(j)}{\delta j_b(y)\,\delta j_b(z)}\right]\left[\frac{\delta Z(j)}{\delta j_a(x)}\right]\right\}_{x=y=z} - \frac{\lambda}{3!}\left[\frac{\delta Z(j)}{\delta j_a(x)}\right]\left[\frac{\delta Z(j)}{\delta j_b(x)}\right]^2 + j_a(x) . \tag{2.6}$$

The generating function $\Gamma(\phi)$ for the 1PI Green's functions[4] is obtained from the Legendre transformation

$$Z(j) = \Gamma(\phi) + \int d^4x\, j_a(x)\,\phi_a(x) , \tag{2.7a}$$

with

$$\frac{\delta Z(j)}{\delta j_a(x)} = \phi_a(x) , \qquad \frac{\delta \Gamma(\phi)}{\delta \phi_a(x)} = -j_a(x) . \tag{2.7b}$$

From this one obtains

$$\frac{\delta^2 \Gamma(\phi)}{\delta\phi_a(x)\,\delta\phi_b(y)} = -\frac{\delta j_a(x)}{\delta\phi_b(y)} , \tag{2.8}$$

$$\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(y)} = \frac{\delta\phi_b(y)}{\delta j_a(x)} = -\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_a(x)\,\delta\phi_b(y)}\right]^{-1} , \tag{2.9}$$

and

$$\frac{\delta^3 Z(j)}{\delta j_a(x)\,\delta j_b(y)\,\delta j_c(z)} = \frac{\delta^2\phi_a(x)}{\delta j_b(y)\,\delta j_c(z)} \tag{2.10a}$$

$$= -\int d^4r\, d^4w\, d^4t \left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_c(z)\,\delta\phi_f(r)}\right]^{-1}\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_b(y)\,\delta\phi_e(t)}\right]^{-1}\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_a(x)\,\delta\phi_d(w)}\right]^{-1} \times \left[\frac{\delta^3\Gamma(\phi)}{\delta\phi_d(w)\,\delta\phi_e(t)\,\delta\phi_f(r)}\right] . \tag{2.10b}$$

From Eqs. (2.6)–(2.10), $\Gamma(\phi)$ is found to satisfy the functional equation

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)} + (\Box_x + \mu^2)\phi_a(x) + \frac{\lambda}{3!}[\phi_b(x)]^2\phi_a(x) = \frac{i\lambda}{3!}\left\{2\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(y)}\right]\phi_b(x) + \left[\frac{\delta^2 Z(j)}{\delta j_b(x)\,\delta j_b(y)}\right]\phi_a(x)\right\}_{x=y} + \frac{\lambda}{3!}\left[\frac{\delta^3 Z(j)}{\delta j_a(x)\,\delta j_b(y)\,\delta j_b(z)}\right]_{x=y=z} , \tag{2.11}$$

where the functional derivatives of $Z(j)$ appearing in (2.11) are given implicitly as functionals of ϕ by means of Eqs. (2.8)–(2.10).

We give a graphical representation in Figs. 1 and 2 for the terms that appear in Eqs. (2.8)–(2.11). Since $Z(j)$ is the generating function for *connected* graphs, $[\delta^2 Z(j)/\delta j(x)\,\delta j(x)]$ must contain *at least* one closed loop. For the same reason $[\delta^3 Z(j)/\delta j(x)\,\delta j(x)\,\delta j(x)]$ contains terms with *at least* two loops. By contrast, each of the terms on the left-hand side of (2.11) involves tree graphs, as well as (of course) graphs with an arbitrary number of loops. The observations of this paragraph[9] enable us to construct a loop expansion[5] for $\Gamma(\phi)$ which proceeds by induction.

B. The loop expansion

Expand

$$\Gamma(\phi)=\sum_{n=0}^{\infty} a^n\,\Gamma_n(\phi) \tag{2.12a}$$

and

$$Z(j)=\sum_{n=0}^{\infty} a^n Z_n(j)\ , \tag{2.12b}$$

where a is a dimensionless parameter which "counts loops," and $\Gamma_n(\phi)$ and $Z_n(j)$ are functionals which generate n-loop graphs *only*. One substitutes (2.12) into (2.11), identifies equal powers of a, and then sets $a=1$. In light of the observations at the end of Sec. II A, we have

$$\frac{\delta^2 Z(j)}{\delta j(x)\,\delta j(y)}=\sum_{n=0}^{\infty} a^n\,\frac{\delta^2 Z_n(j)}{\delta j(x)\,\delta j(y)}\ , \tag{2.13}$$

so that

$$\left[\frac{\delta^2 Z(j)}{\delta j(x)\,\delta j(y)}\right]_{x=y}=a\sum_{n=0}^{\infty} a^n\left[\frac{\delta^2 Z_n(j)}{\delta j(x)\,\delta j(y)}\right]_{x=y} \tag{2.14}$$

and

$$\left[\frac{\delta^3 Z(j)}{\delta j(x)\,\delta j(y)\,\delta j(z)}\right]_{x=y=z}=a^2\sum_{n=0}^{\infty} a^n\left[\frac{\delta^3 Z_n(j)}{\delta j(x)\,\delta j(y)\,\delta j(z)}\right]_{x=y=z}\ , \tag{2.15}$$

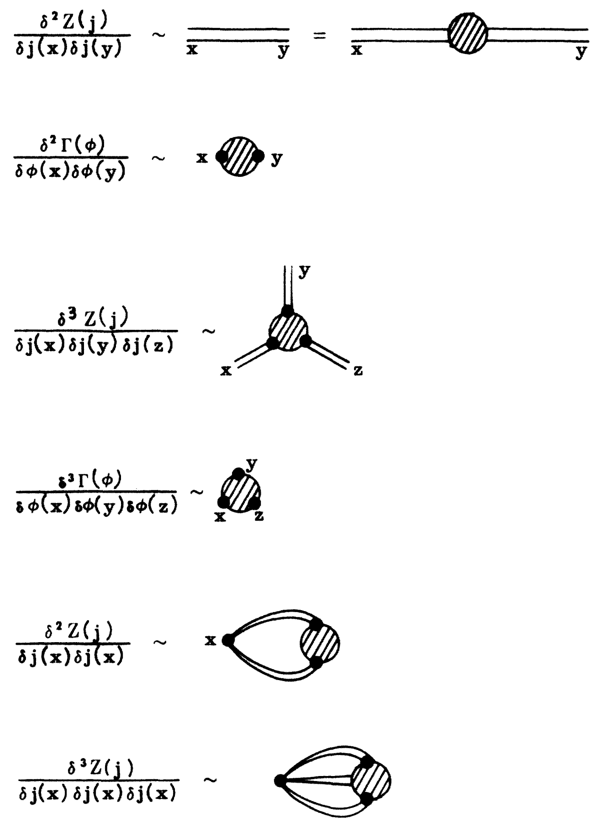

FIG. 1. Graphical representation of the individual terms that enter the functional differential equation for $\Gamma(\phi)$. The double lines represent the complete two-point Green's function in the presence of an arbitrary external field. The shaded blob represents one-particle irreducible parts in the presence of an arbitrary external field.

etc.

The equations generating diagrams with a fixed number of loops are

$$\frac{\delta\Gamma_0(\phi)}{\delta\phi_a(x)}=-(\Box_x+\mu^2)\,\phi_a(x)-\frac{\lambda}{3!}\,[\phi(x)]^2\,\phi_a(x)\quad\text{for } n=0 \tag{2.16}$$

and

$$\frac{\delta\Gamma_n(\phi)}{\delta\phi_a(x)}=\frac{i\lambda}{3!}\left\{2\left[\frac{\delta^2 Z_{n-1}(j)}{\delta j_a(x)\,\delta j_b(x)}\right]\phi_b(x)+\left[\frac{\delta^2 Z_{n-1}(j)}{\delta j_b(x)\,\delta j_b(x)}\right]\phi_a(x)\right\}+\frac{\lambda}{3!}\left[\frac{\delta^3 Z_{n-2}(j)}{\delta j_a(x)\,\delta j_b(x)\,\delta j_b(x)}\right],\quad\text{for } n\geq 1\ , \tag{2.17}$$

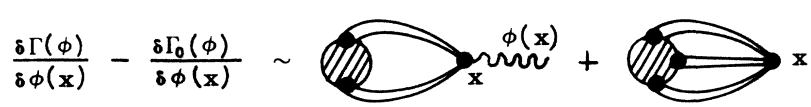

FIG. 2. Graphical representation for the $\delta\Gamma(\phi)/\delta\phi(x)$, with the same conventions as Fig. 1. The wavy line represents explicit dependence on the external field, interacting locally at the point vertex.

where the last term in (2.17) appears only for $n \geq 2$. Equation (2.17) determines the n-loop approximation, $\delta\Gamma_n(\phi)/\delta\phi(x)$, entirely in terms of functionals with $(n-1)$ or fewer loops. It is this feature which makes the inductive determination of $\Gamma(\phi)$ possible. To make this practical, so as to integrate (formally) the first-order functional differential equation for $\Gamma_n(\phi)$, one must evaluate the terms on the right-hand side of (2.17) explicitly in terms of ϕ by means of Eqs. (2.8)–(2.10). We illustrate the induction scheme by an explicit evaluation of $\Gamma_0(\phi)$, $\Gamma_1(\phi)$, and $\Gamma_2(\phi)$. In addition to demonstrating the nature of the loop expansion for the 1PI functional, these calculations will provide crucial observations for the construction of $\Gamma(\phi)$ exact in the large-N limit.

C. The zero-loop and one-loop approximations

The functional equation satisfied by the tree approximation, (2.16), is easily integrated to give

$$\Gamma_0(\phi) = \int d^4x \left\{ \tfrac{1}{2}[\partial_\mu \phi_a(x)]^2 - \tfrac{1}{2}\mu^2[\phi_a(x)]^2 - \frac{\lambda}{4!}[\phi_a{}^2(x)]^2 \right\} + \text{constant} . \qquad (2.18)$$

Of course $\Gamma_0(\phi)$ is identical to the classical action, so that (2.18) agrees with the usual result[5] given for the generating functional of the 1PI tree graphs.

Proceeding with the induction scheme, the one-loop functional is given by

$$\frac{\delta\Gamma_1(\phi)}{\delta\phi_a(x)} = -\frac{i\lambda}{3!}\left\{ 2\left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi_a(x)\,\delta\phi_b(x)}\right]^{-1}\phi_b(x) + \left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi_b(x)\,\delta\phi_b(x)}\right]^{-1}\phi_a(x)\right\} . \qquad (2.19)$$

However, from (2.18)

$$\frac{\delta^2\Gamma_0(\phi)}{\delta\phi_a(x)\,\delta\phi_b(y)} = -\left[(\Box_x + \mu^2)\,\delta_{ab} + \frac{\lambda}{3!}\,\delta_{ab}\,\phi^2(x) + \frac{\lambda}{3}\,\phi_a(x)\,\phi_b(x)\right]\delta^4(x-y) . \qquad (2.20)$$

The inverse of this functional defines the *free-particle* Green's function in the presence of an external field $\phi_a(x)$,

$$G_{ab}(x,y) = -\left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi_a(x)\,\delta\phi_b(y)}\right]^{-1} . \qquad (2.21)$$

It is sometimes convenient to write

$$G_{ab}(x,y) = \langle y|\,G_{ab}\,|x\rangle = \langle y, b|\,G\,|x, a\rangle , \qquad (2.22)$$

where G is a formal operator and $|x\rangle$ is a formal vector in the appropriate function space. Therefore,

$$\int d^4z \left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi_a(x)\,\delta\phi_b(z)}\right] G_{bc}(z,y) = -\delta_{ac}\,\delta^4(x-y) . \qquad (2.23)$$

In terms of the definition (2.21), (2.19) becomes

$$\frac{\delta\Gamma_1(\phi)}{\delta\phi_a(x)} = \frac{i\lambda}{3!}\left[2G_{ab}(x,x)\,\phi_b(x) + G_{bb}(x,x)\,\phi_a(x)\right] . \qquad (2.24)$$

This equation is easily integrated by noting that

$$\frac{\delta G^{-1}{}_{bc}(y,z)}{\delta\phi_a(x)}\,G_{cb}(z,y) = \tfrac{1}{3}\lambda\left[2G_{ab}(z,y)\,\phi_b(x) + G_{bb}(z,y)\,\phi_a(x)\right] \times \delta^4(y-z)\,\delta^4(x-z) . \qquad (2.25)$$

Then

$$\frac{\delta\Gamma_1(\phi)}{\delta\phi_a(x)} = \frac{i}{2}\int d^4y \left\langle y \left| \left(\frac{\delta}{\delta\phi_a(x)}\,G^{-1}{}_{bc}\right) G_{cb} \right| y \right\rangle = \frac{i}{2}\,\frac{\delta}{\delta\phi_a(x)}\int d^4y \langle y|\,[\ln G^{-1}]_{bb}\,|y\rangle = \frac{i}{2}\,\frac{\delta}{\delta\phi_a(x)}\,\mathrm{Tr}\ln G^{-1} , \qquad (2.26)$$

where the trace is over both internal and space-time indices. Thus,

$$\Gamma_1(\phi) = \tfrac{1}{2}\,i\,\mathrm{Tr}\ln G^{-1} + \text{constant}, \qquad (2.27)$$

where G is the formal operator defined in (2.22). Once again, our results agree[5] with other determinations of the one-loop functional.

D. The two-loop functional

From Eq. (2.17) we have

$$\frac{\delta\Gamma_2(\phi)}{\delta\phi_a(x)} = \frac{i\lambda}{3!}\left\{ 2\left[\frac{\delta^2 Z_1(j)}{\delta j_a(x)\,\delta j_b(x)}\right]\phi_b(x) + \left[\frac{\delta^2 Z_1(j)}{\delta j_b(x)\,\delta j_b(x)}\right]\phi_a(x)\right\} + \frac{\lambda}{3!}\left[\frac{\delta^3 Z_0(j)}{\delta j_a(x)\,\delta j_b(x)\,\delta j_b(x)}\right] . \qquad (2.28)$$

To proceed with (2.28) in particular, and the n-loop equation in general, one must compute the functional derivatives that appear on the right-hand side of (2.28), and of (2.17) in the general case, in terms of ϕ. We sketch this procedure for the general case, and give the details for (2.28).

From Eqs. (2.9) and (2.12), suppressing internal indices,

$$\int d^4z \sum_{n=0} a^n \frac{\delta^2 Z_n(j)}{\delta j(x)\,\delta j(z)} \sum_{m=0} a^m \frac{\delta^2 \Gamma_m(\phi)}{\delta\phi(z)\,\delta\phi(y)} = -\delta^4(x-y)\ . \tag{2.29}$$

Identifying equal powers of a, one obtains

$$\int d^4z \left[\frac{\delta^2 Z_0(j)}{\delta j(x)\,\delta j(z)}\right]\left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi(z)\,\delta\phi(y)}\right] = -\delta^4(x-y)\ , \tag{2.30a}$$

$$\int d^4z \left\{\left[\frac{\delta^2 Z_0(j)}{\delta j(x)\,\delta j(z)}\right]\left[\frac{\delta^2\Gamma_1(\phi)}{\delta\phi(z)\,\delta\phi(y)}\right] + \left[\frac{\delta^2 Z_1(j)}{\delta j(x)\,\delta j(z)}\right]\left[\frac{\delta^2\Gamma_0(\phi)}{\delta\phi(z)\,\delta\phi(y)}\right]\right\} = 0\ , \tag{2.30b}$$

etc. Therefore,

$$\frac{\delta^2 Z_1(j)}{\delta j_a(x)\,\delta j_b(y)} = \int d^4z\, d^4w\, G_{ac}(x,z)\left[\frac{\delta^2\Gamma_1(\phi)}{\delta\phi_c(z)\,\delta\phi_d(w)}\right] G_{db}(w,y)\ , \tag{2.31}$$

with $G_{ab}(x,y)$ defined by (2.21). Similarly, from Eq. (2.10b) we obtain a similar result for $\delta^3 Z_n(j)/\delta j(x)\,\delta j(y)\,\delta j(z)$. In particular,

$$\frac{\delta^3 Z_0(j)}{\delta j_a(x)\,\delta j_b(y)\,\delta j_c(z)} = \int d^4r\, d^4w\, d^4t\, G_{cd}(z,r)\, G_{be}(y,t)\, G_{af}(x,w)\left[\frac{\delta^3\Gamma_0(\phi)}{\delta\phi_f(w)\,\delta\phi_e(t)\,\delta\phi_d(r)}\right]\ . \tag{2.32}$$

Making use of Eqs. (2.31), (2.24), and (2.25), we obtain

$$\begin{aligned}\frac{\delta^2 Z_1(j)}{\delta j_a(x)\,\delta j_b(x)} &= \int d^4z\, d^4w\, G_{ac}(x,z)\, G_{db}(w,x)\, \frac{\delta}{\delta\phi_c(z)}\left[\frac{\delta\Gamma_1(\phi)}{\delta\phi_d(w)}\right]\\
&= \frac{i\lambda}{3!}\int d^4z\, G_{ac}(x,z)\, G_{db}(z,x)\,[2G_{cd}(z,z) + \delta_{cd}\, G_{ff}(z,z)]\\
&\quad -\frac{i\lambda^2}{18}\int d^4z\, d^4w\, G_{ac}(x,z)\, G_{db}(w,x)\\
&\qquad \times[2G_{cd}(z,w)\,\phi_f(w)\, G_{fg}(w,z)\,\phi_g(z) + 2\phi_c(z)\, G_{df}(w,z)\, G_{fg}(z,w)\,\phi_g(w)\\
&\qquad + 2\phi_d(w)\, G_{cf}(z,w)\, G_{fg}(w,z)\,\phi_g(z) + \phi_c(z)\,\phi_d(w)\, G_{gf}(z,w)\, G_{fg}(w,z)\\
&\qquad + 2G_{cf}(z,w)\,\phi_f(w)\, G_{dg}(w,z)\,\phi_g(z)]\ .\end{aligned} \tag{2.33}$$

Similarly, from Eqs. (2.20) and (2.32)

$$\frac{\delta^3 Z_0(j)}{\delta j_a(x)\,\delta j_b(x)\,\delta j_c(x)} = -\frac{\lambda}{3}\int d^4z\, G_{cd}(x,z)\, G_{be}(x,z)\, G_{af}(x,z)\,[\delta_{de}\,\phi_f(z) + \delta_{ef}\,\phi_d(z) + \delta_{fd}\,\phi_e(z)]\ . \tag{2.34}$$

This completes the construction of $\delta\Gamma_2(\phi)/\delta\phi(x)$ in terms of $\phi(x)$ and the free-particle propagator in the presence of an external field. The (formal) integration of the first-order functional differential equation (2.28) is not difficult. To this end consider the functional

$$I(\phi) = \int d^4z\,[G_{bb}(z,z)\, G_{cc}(z,z) + 2G_{bc}(z,z)\, G_{cb}(z,z)]\ , \tag{2.35}$$

which means

$$\begin{aligned}\frac{\delta I(\phi)}{\delta\phi_a(x)} = &-\frac{4\lambda}{3}\,\phi_b(x)\int d^4z\,[2G_{cd}(z,z)\, G_{ac}(x,z)\, G_{db}(z,x) + G_{ff}(z,z)\, G_{ac}(x,z)\, G_{cb}(z,x)]\\
&-\frac{2\lambda}{3}\,\phi_a(x)\int d^4z\,[2G_{bc}(z,z)\, G_{db}(x,z)\, G_{cd}(z,x) + G_{ff}(z,z)\, G_{bc}(x,z)\, G_{cb}(z,x)]\ .\end{aligned} \tag{2.36}$$

Comparing (2.33) with (2.36) one concludes that

$$2\left[\frac{\delta^2 Z_1(j)}{\delta j_a(x)\,\delta j_b(x)}\right]\phi_b(x) + \left[\frac{\delta^2 Z_1(j)}{\delta j_b(x)\,\delta j_b(x)}\right]\phi_a(x) = -\frac{i}{4}\,\frac{\delta I(\phi)}{\delta\phi_a(x)} + \int d^4z\, d^4w\,[\cdots]\ . \tag{2.37}$$

The remaining terms of (2.37) combine simply with (2.34) to form a single functional. Define the functional

$$J(\phi)=\int d^4z\, d^4w\, \phi_b(z)\,\phi_c(w)\,[2G_{bd}(z,w)\,G_{ed}(z,w)\,G_{ec}(z,w)+G_{de}(z,w)\,G_{de}(z,w)\,G_{bc}(z,w)]\ . \qquad (2.38)$$

It is straightforward to compute $\delta J(\phi)/\delta\phi_a(x)$ and to compare it with (2.34) and the remainder of (2.37). As a result, one obtains

$$\frac{\delta\Gamma_2(\phi)}{\delta\phi_a(x)}=\frac{\lambda}{4!}\frac{\delta I(\phi)}{\delta\phi_a(x)}-\frac{\lambda^2}{36}\frac{\delta J(\phi)}{\delta\phi_a(x)}\ , \qquad (2.39)$$

so that

$$\Gamma_2(\phi)=\frac{\lambda}{4!}I(\phi)-\frac{\lambda^2}{36}J(\phi)+\text{constant}. \qquad (2.40)$$

Our construction of $\Gamma_2(\phi)$ agrees with earlier determinations[5] of the two-loop 1PI functional by other methods.

A graphical representation of the function $I(\phi)$ and $J(\phi)$ is given in Fig. 3. We now make several observations which are essential to the development of the large-N expansion presented in Sec. III. First note that $I(\phi)$ is represented graphically by a two-loop *bubble* graph with a single four-point vertex and free propagators in the presence of an external field. By contrast, $J(\phi)$ is represented graphically by a nonbubble graph with two three-point vertices, each proportional to the external field ϕ. Stated differently, the *bubble* term $I(\phi)$ *only* arises from terms of the form

$$\phi(x)[\delta^2Z_1(j)/\delta j(x)\delta j(x)]$$

in the functional equation for $\delta\Gamma_2(\phi)/\delta\phi(x)$, while the *nonbubble* term $J(\phi)$ originates from *both*

$$\phi(x)[\delta^2Z_1(j)/\delta j(x)\delta j(x)]$$

and

$$[\delta^3Z_0(j)/\delta j(x)\delta j(x)\delta j(x)]$$

in the functional differential equation. In other words, the term

$$[\delta^3Z_0(j)/\delta j(x)\delta j(x)\delta j(x)]$$

does *not* contribute to the bubble graphs. Finally, notice from Eqs. (2.28), (2.32), and (2.34) and the above discussion that [aside from the dependence of the propagator $G(x,y)$ on ϕ] the terms in $\delta\Gamma_2(\phi)/\delta\phi(x)$ that generate the bubble graphs are *linear* in $\phi(x)$, while the terms that generate the two-loop nonbubble graphs are *cubic* in ϕ, due to the presence of two three-point vertices. These distinctions, which are easily generalized to n loops, will play an essential role in understanding Secs. II F and III.

One may continue the loop expansion by induction, as sketched above, to determine $\Gamma_3(\phi),\ldots,\Gamma_n(\phi),\ldots$. However, we proceed no further in this direction.

E. The renormalized loop expansion

So far we have only considered the loop expansion for the unrenormalized 1PI Green's functions. Since the renormalization of the loop expansion has been treated by other workers,[1,5] we only make a few remarks to enable the reader to translate earlier work into our language.

The Lagrangian density (2.1) may be modified to read

$$\mathcal{L}(\tilde\phi,j)=\tfrac12[(\partial_\mu\tilde\phi)^2-\bar\mu^2\tilde\phi^2]-\frac{\bar\lambda}{4!}\tilde\phi^4+\tfrac12A(\partial_\mu\tilde\phi)^2-\tfrac12B\tilde\phi^2-\frac{1}{4!}C\tilde\phi^4+j_a(x)\,\tilde\phi_a(x)\ , \qquad (2.41)$$

where A, B, and C are wave-function, mass, and coupling-constant counterterms, and $\bar\mu^2$ and $\bar\lambda$ are finite, renormalized quantities. Analogously, (2.11) may be replaced by

$$\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)}=-[\Box_x+\bar\mu^2]\,\phi_a(x)-\frac{(\bar\lambda+C)}{3!}\,\phi^2(x)\,\phi_a(x)+i\,\frac{(\bar\lambda+C)}{3!}\left\{2\left[\frac{\delta^2Z(j)}{\delta j_a(x)\,\delta j_b(x)}\right]\phi_b(x)+\left[\frac{\delta^2Z(j)}{\delta j_b(x)\,\delta j_b(x)}\right]\phi_a(x)\right\}$$
$$+\frac{(\bar\lambda+C)}{3!}\left[\frac{\delta^3Z(j)}{\delta j_a(x)\,\delta j_b(x)\,\delta j_b(x)}\right]-[A\Box_x+B]\,\phi_a(x)\ . \qquad (2.42)$$

For the purposes of the loop expansion, write

$$A=\sum_{n=1}^{\infty}a^nA_n\ ,$$
$$B=\sum_{n=1}^{\infty}a^nB_n\ , \qquad (2.43)$$

and

$$C=\sum_{n=1}^{\infty}a^nC_n\ .$$

One constructs the loop expansion as before, taking note of the loop dependence of the counterterms by means of Eqs. (2.42) and (2.43). One determines the renormalization constants A_n, B_n, and C_n by means of the usual renormalization conditions, described in detail by Coleman and Weinberg[5] and Jackiw.[5] We have carried out the *renormalized* loop expansion by our methods in detail up to the two-loop functional. Our results agree with earlier work, so that we omit the details here, as they are not essential to the mainstream of our development.

F. The bubble approximation

The purpose of this section is to develop the bubble approximation to the functional $\Gamma(\phi)$, which in $\lambda\phi^4$ theory is characterized by those vacuum graphs which contain *only* four-point vertices proportional to λ and free propagators in the presence of an external field. The nonbubble vacuum graphs contributing to $\Gamma(\phi)$ contain two or more three-point vertices (each proportional to $\lambda\phi$) as well as four-point vertices, and free propagators.

It is easy to understand this characterization graphically. The loop expansion for $\Gamma(\phi)$ is represented in Fig. 4, while the loop expansions for $[\Gamma(\phi)]_{\text{bubble}}$ and $[\Gamma(\phi)]_{\text{nonbubble}}$ are shown in Figs. 5 and 6, respectively. Note that the ϕ dependence of $[\Gamma(\phi)]_{\text{bubble}}$ can be attributed *entirely* to the propagators, while the ϕ dependence of $[\Gamma(\phi)]_{\text{nonbubble}}$ arises from the three-point vertices, as well as the propagators.

Next, consider the consequences of this separation for $\delta\Gamma(\phi)/\delta\phi(x)$. As a result of the observations made in the preceding paragraph, we can represent $[\delta\Gamma(\phi)/\delta\phi(x)]_{\text{bubble}}$ as in Fig. 7, and $[\delta\Gamma(\phi)/\delta\phi(x)]_{\text{nonbubble}}$ as in Fig. 8. Now refer back to the exact functional equation for $\delta\Gamma(\phi)/\delta\phi(x)$, given by Eq. (2.11), and represented graphically by Fig. 2. The loop expansion for the term proportional to $\phi(x)[\delta^2 Z/\delta j(x)\,\delta j(x)]$ is shown graphically in Fig. 9, and for the term proportional to $[\delta^3 Z/\delta j(x)\,\delta j(x)\,\delta j(x)]$ is illustrated by Fig. 10. From this analysis, it is obvious that we have the separation

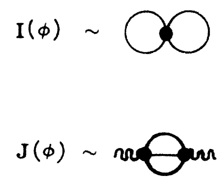

FIG. 3. Graphical representation of the two terms comprising the two-loop functional. The wavy line represents the explicit ϕ dependence of the indicated vertex. The solid line represents a free propagator in the presence of an external field.

$$\phi(x)\left[\frac{\delta^2 Z}{\delta j(x)\,\delta j(x)}\right] \sim \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{bubble}} + \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{nonbubble}} \tag{2.44}$$

and

$$\left[\frac{\delta^3 Z}{\delta j(x)\,\delta j(x)\,\delta j(x)}\right] \sim \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{nonbubble}} \text{ only}, \tag{2.45}$$

where ~ means "contributes to."

It is instructive to compare Figs. 7 and 8 with Figs. 9 and 10. From this comparison it is seen that the separation given by Eqs. (2.44) and (2.45) may be refined further. That is,

$$\phi(x)\left[\frac{\delta^2 Z}{\delta j(x)\,\delta j(x)}\right] \sim \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{bubble}} + \left\{\frac{\delta G}{\delta\phi} - \left[\frac{\delta\Gamma}{\delta\phi}\right]_{\text{nonbubble}}\right\} \tag{2.46}$$

and

$$\left[\frac{\delta^3 Z}{\delta j(x)\,\delta j(x)\,\delta j(x)}\right] \sim \left\{\frac{\delta}{\delta\phi}(\lambda\phi) - \left[\frac{\delta\Gamma}{\delta\phi}\right]_{\text{nonbubble}}\right\}. \tag{2.47}$$

The somewhat obtuse notation of (2.46) and (2.47) indicates that $[\delta\Gamma(\phi)/\delta\phi(x)]_{\text{nonbubble}}$ is obtained from $[\Gamma(\phi)]_{\text{nonbubble}}$ in two separate and unique ways: (1) from the differentiation of a propagator, as in Eq. (2.46), and (2) from the differentiation of a three-point vertex, as in (2.47). The division of $[\delta\Gamma/\delta\phi]_{\text{nonbubble}}$ given by (2.46) and (2.47) is

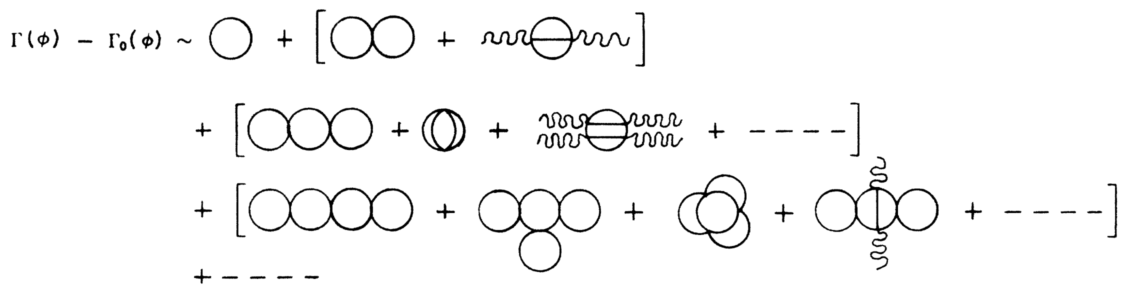

FIG. 4. Graphical representation of the loop expansion for $\Gamma(\phi)$, with conventions as in Fig. 3.

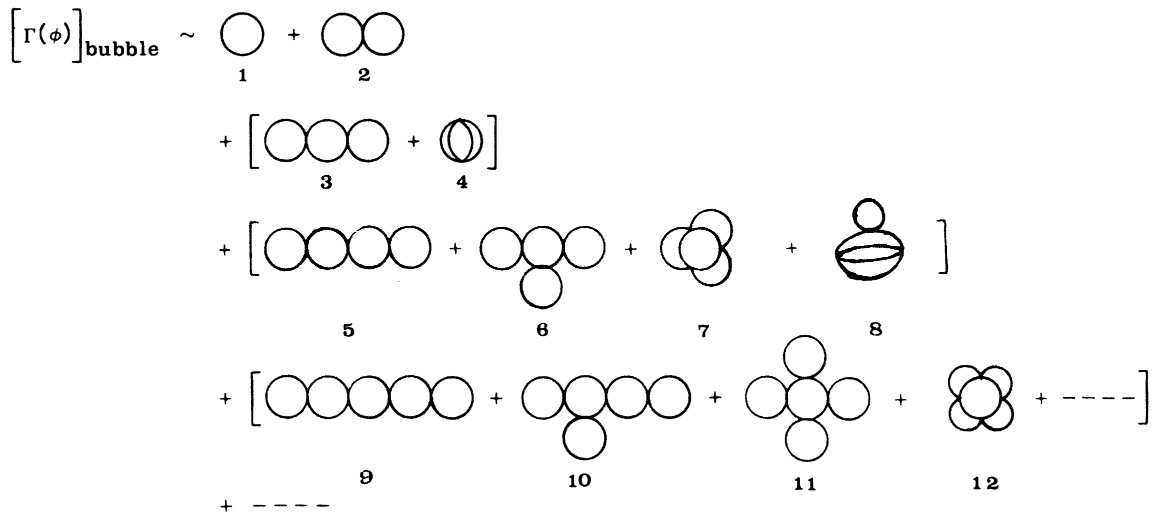

FIG. 5. Graphical representation of the loop expansion for $\Gamma(\phi)_{\text{bubble}}$, with conventions as in Fig. 3.

unique. All of the points discussed in this section are illustrated in detail by the two-loop functional discussed in Sec. II D.

Our next objective is to formulate a set of coupled functional equations which generate $[\Gamma(\phi)]_{\text{bubble}}$ in the large-N limit, based on the above discussion. We find it easier to present the final result, and then present the arguments leading to our conclusion. The functional differential equations which which we propose for large N are

$$\left[\frac{\delta\Gamma(\phi)}{\delta\phi_a(x)}\right]_{\text{bubble}}=\frac{\delta\Gamma_0(\phi)}{\delta\phi_a(x)}+\frac{i\lambda}{3!}\left\{2\phi_b(x)\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(x)}\right]_{\text{bubble}}+\phi_a(x)\left[\frac{\delta^2 Z(j)}{\delta j_b(x)\,\delta j_b(x)}\right]_{\text{bubble}}\right\}, \qquad (2.48)$$

with

$$\left[\frac{\delta^2\Gamma(\phi)}{\delta\phi_a(x)\,\delta\phi_b(y)}\right]_{\text{bubble}}=-\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(y)}\right]^{-1}_{\text{bubble}}$$

$$=-G^{-1}{}_{ab}(x,y)+\frac{i\lambda}{3!}\left\{2\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(x)}\right]_{\text{bubble}}+\delta_{ab}\left[\frac{\delta^2 Z(j)}{\delta j_c(x)\,\delta j_c(x)}\right]_{\text{bubble}}\right\}\delta^4(x-y), \qquad (2.49)$$

and

$$\frac{\delta}{\delta\phi_e(y)}\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\,\delta j_b(x)}\right]_{\text{bubble}}=\text{nonbubble}, \qquad (2.50)$$

where $G_{ab}(x,y)$ is defined in Eq. (2.21), and $\Gamma_0(\phi)$ is as defined in (2.16) and (2.18).

That Eq. (2.48) is a necessary condition is obvious; $[\delta^3 Z/\delta j(x)\delta j(x)\delta j(x)]$ must be dropped from (2.11) because of (2.47). However, because of (2.46) it is also obvious that this is not sufficient. Additional conditions, Eqs. (2.49) and (2.50), must be specified. To understand these, consider $[\delta/\delta\phi_b(y)][\delta\Gamma(\phi)/\delta\phi_a(x)]_{\text{bubble}}$ as given

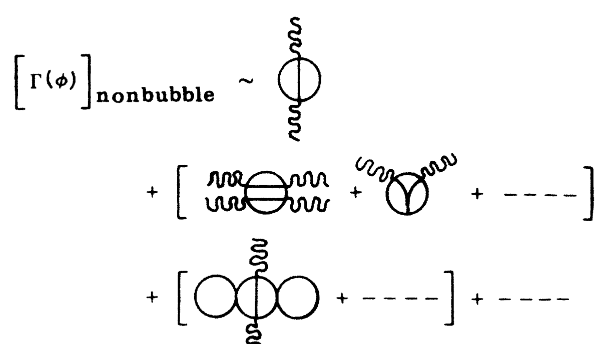

FIG. 6. Graphical representation of the loop expansion for $\Gamma(\phi)_{\text{nonbubble}}$, with conventions as in Fig. 3.

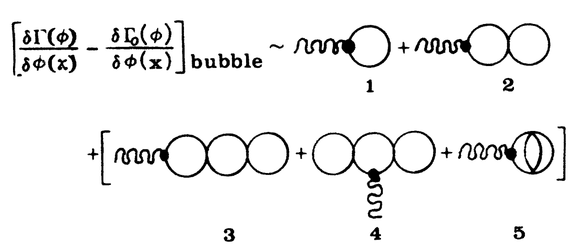

FIG. 7. Graphical representation of the loop expansion for $[\delta\Gamma/\delta\phi]_{\text{bubble}}$, with conventions as in Fig. 3.

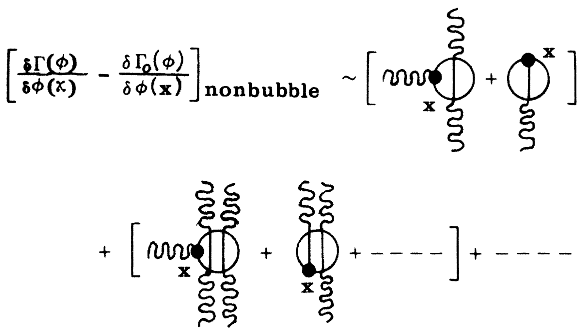

FIG. 8. Graphical representation of the loop expansion for $[\delta\Gamma/\delta\phi]_{\text{nonbubble}}$, with conventions as in Fig. 3.

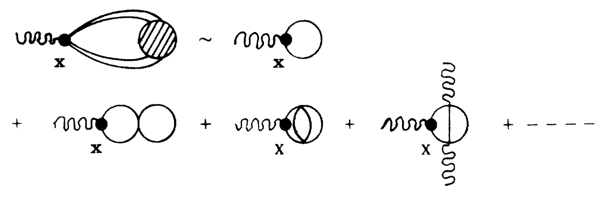

FIG. 9. Graphical representation of the loop expansion for $\phi(x)[\delta^2 Z(j)/\delta j(x)\delta j(x)]$, with conventions as in Figs. 1 and 3.

by the right-hand side of (2.48). There are two types of terms generated, those given by (2.49) and those given by (2.50). Since $[\delta^2 Z/\delta j(x)\delta j(x)]_{\text{bubble}}$ depends on ϕ only through the propagator, it is obvious that

$$\phi(x)\frac{\delta}{\delta\phi(y)}\left[\frac{\delta^2 Z}{\delta j(x)\delta j(x)}\right]_{\text{bubble}}$$

is quadratic in ϕ, *aside* from the propagator dependence on ϕ, since

$$\frac{\delta G}{\delta\phi}=-G\frac{\delta G^{-1}}{\delta\phi}G\,.$$

Thus, the terms coming from (2.50) generate contributions to $[\delta^2 Z/\delta j\delta j]$ which are quadratic in ϕ (*aside* from dependence originating in G), and hence generate contributions to $\delta\Gamma/\delta\phi(x)$ which are *cubic* in ϕ (aside from any propagator dependence on ϕ). The conclusion is that (2.50) generates *only* nonbubble graphs. With this identification, one obtains (2.49) from (2.48) by functional differentiation. This analysis is exemplified in detail in Sec. II D, and by Eqs. (2.33)–(2.38) in particular, where the two types of terms are clearly separated.

Now consider $\lambda\phi^4$ theory with O(N) invariance in the large-N limit. Since

$$\operatorname{tr}G_{ab}=G_{bb}\sim NG\,,$$

it is clear that for large N, $[\Gamma(\phi)]_{\text{bubble}}$ dominates[10] $[\Gamma(\phi)]_{\text{nonbubble}}$ by at least a factor of N in each fixed order in the loop expansion, i.e., for *n fixed* loops. Thus, for N large, we may replace Eq. (2.50) by

$$\frac{\delta}{\delta\phi_e(y)}\left[\frac{\delta^2 Z(j)}{\delta j_a(x)\delta j_b(x)}\right]_{\text{bubble}}=O\left(\frac{1}{N}\right)\quad(N\to\infty)\,. \tag{2.51}$$

We now deal with a subtle issue in order to complete the justification of (2.48)–(2.51) as the correct equations in the large-N limit. One notes that $\Gamma(\phi)_{\text{bubble}}$ can be usefully divided into *two* classes of graphs: (1) those graphs which form tree diagrams of bubbles, i.e., the branches are made up of linear chains of bubbles, and (2) those graphs in which trees formed of bubbles interact to form at least one closed loop of bubbles. Let us call the graphs of class 1 *bubble tree* graphs, and those of class 2 *bubble loop* graphs. These two classes of graphs are easily distinguished. For example, in Fig. 5, graphs 4, 7, 8, and 12 are bubble loop graphs, and the others are bubble tree graphs. These particular bubble loop graphs can be obtained by forming a single bubble loop from a bubble tree of 2, 3, 3, and 4 bubbles, respectively.

Now consider N large, with a fixed number of closed loops in the conventional sense. It is easy to see that the bubble tree graphs dominate the bubble loop graphs (with the same number of *conventional* closed loops) by at least a factor of N. Specific examples are found in Fig. 5, where graph 3 is proportional to N^3, while graph 4 is proportional to N^2, and graphs 5 and 6 are proportional to N^4, while graphs 7 and 8 are proportional to N^3. Hence the bubble loop graphs are subdominant in the large-N limit, and can also be neglected to leading order in N.

Similarly $[\delta\Gamma(\phi)/\delta\phi(x)]_{\text{bubble}}$ can be divided into the same two classes. For example, graph 3 of Fig. 5 leads to graphs 3 and 4 of Fig. 7, and graph 4 of Fig. 5 generates graph 5 of Fig. 7. (We have not drawn topologically equivalent graphs

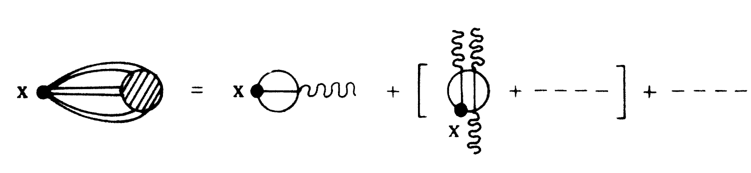

FIG. 10. Graphical representation for $[\delta^3 Z(j)/\delta j(x)\delta j(x)\delta j(x)]$, with conventions as in Figs. 1 and 3.

separately.) Our next task is to translate this classification into analytical terms.

The analysis we seek is a refinement of Eq. (2.46) and the discussion following. To do so we examine a specific example, graph 5 of Fig. 7. This graph is obtained by the contribution of (2.46) to (2.11), but with $\delta^2 Z/\delta j(x)\delta j(y)$ constructed from the first graph of Fig. 6. Further examples show that we may write, in a schematic way (where ~ means "contributes to"),

$$\phi(x)\left[\frac{\delta^2 Z}{\delta j(x)\delta j(x)}\right]_{\text{nonbubble}} \sim \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{bubble loops}} + \left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\text{nonbubble}}, \qquad (2.52)$$

which indicates that terms on the left-hand side of (2.52) generate the right-hand side of (2.52). Most importantly, the *only* way of generating graph 5 of Fig. 7 is by means of this particular term. Thus, if we dropped

$$\phi(x)\left[\frac{\delta^2 Z}{\delta j(x)\delta j(x)}\right]_{\text{nonbubble}}$$

from the right-hand side of (2.11), we would not generate the bubble loops.

Putting these various arguments together, we arrive at Eqs. (2.48)–(2.51) as representing the large-N limit of the theory. In particular, we expect this set of equations to generate the bubble tree graphs. Although we are aware that we have not provided a satisfactory mathematical proof, our conclusion is supported by the results presented in Sec. III and Appendix B. There it is clear, for the special case of a constant external field, that (2.48) and (2.49) do indeed generate the bubble tree graphs when our solution for the effective potential is reexpressed in terms of the perturbative loop expansion. Naturally the reader who desires a proof will require more rigorous analysis; however, our results do indeed seem to characterize the large-N limit of $\lambda\phi^4$ theory.

III. THE EFFECTIVE POTENTIAL

A. Bubble approximation

Although Eqs. (2.48)–(2.51) provide a closed set of equations for the generating functional $[\Gamma(\phi)]_{\text{bubble}}$, the actual solution of these equations is hampered by a lack of knowledge of the free-particle Green's function, $G_{ab}(x,y)$, in the presence of an *arbitrary* external field $\phi_a(x)$. However, no such difficulty exists if one restricts the external field to a *constant* field ϕ_a, as one can compute the appropriate free-particle Green's function in both coordinate space and momentum space for this special case. In particular, for a constant field ϕ_a, one has the coordinate-space representations

$$G^{-1}{}_{ab}(x,y;\phi) = [(\Box_x + \mu^2)\delta_{ab} + \delta_{ab}\tfrac{1}{6}\lambda\phi^2 + \tfrac{1}{3}\lambda\phi_a\phi_b]\,\delta^4(x-y) \qquad (3.1)$$

and

$$\begin{aligned}\langle x|G_{ab}|y\rangle &= G_{ab}(x,y;\phi)\\ &= \left\langle x\left|\left[\frac{1}{\Box_x+\mu^2+\frac{1}{2}\lambda\phi^2}\left(\frac{\phi_a\phi_b}{\phi^2}\right)+\frac{1}{\Box_x+\mu^2+\frac{1}{2}\lambda\phi^2}\left(\delta_{ab}-\frac{\phi_a\phi_b}{\phi^2}\right)\right]\right|y\right\rangle\\ &= G(x,y;\mu^2+\tfrac{1}{2}\lambda\phi^2)\frac{\phi_a\phi_b}{\phi^2} + G(x,y;\mu^2+\tfrac{1}{6}\lambda\phi^2)\left(\delta_{ab}-\frac{\phi_a\phi_b}{\phi^2}\right). \end{aligned} \qquad (3.2)$$

As a consequence of the limitation to constant external fields, the functional differential equations for $\delta\Gamma(\phi)/\delta\phi(x)$ reduce to an *ordinary* equation for $\partial V(\phi)/\partial\phi_a$, where $V(\phi)$ is the effective potential for the theory. Since

$$\Gamma(\phi) = -V(\phi)\int d^4x, \text{ for } \phi = \text{constant}, \qquad (3.3)$$

the system of equations (2.48)–(2.51) for the bubble graphs reduces to the following coupled equations for $[V(\phi)]_{\text{bubble}}$ (with the subscript "bubble" omitted):

$$\frac{\partial V(\phi)}{\partial\phi_a} - \frac{\partial V_0(\phi)}{\partial\phi_a} = -\frac{i\lambda}{3!}[2\phi_b\tilde{G}_{ab}(x,x;\phi) + \phi_a\tilde{G}_{bb}(x,x;\phi)], \qquad (3.4)$$

where

$$-\tilde{G}^{-1}{}_{ab}(x,y;\phi) = -G^{-1}{}_{ab}(x,y;\phi) + \frac{i\lambda}{3!}[2\tilde{G}_{ab}(x,x;\phi) + \delta_{ab}\tilde{G}_{cc}(x,x;\phi)]\delta^4(x-y) \qquad (3.5)$$

and

$$\frac{\partial}{\partial\phi_c}\tilde{G}_{ab}(x,x;\phi) = \text{nonbubble} \qquad (3.6a)$$

$$\simeq 0 \text{ for } N \text{ large}. \qquad (3.6b)$$

We have defined

$$\tilde{G}_{ab}(x,y;\phi) = \left[\frac{\delta^2 Z(j)}{\delta j_a(x)\delta j_b(y)}\right]_{\phi=\text{const}} \qquad (3.7)$$

for the exact Green's function in the presence of a constant field.

B. Large-N limit

The coupled system of equations may be simplified in the large-N limit by noting that

$$\phi_b G_{ab}(x,y;\phi) = \phi_a G(x,y;\mu^2 + \tfrac{1}{2}\lambda\phi^2)$$

and

$$G_{bb}(x,y;\phi) = G(x,y;\mu^2 + \tfrac{1}{2}\lambda\phi^2) + (N-1)G(x,y;\mu^2 + \tfrac{1}{6}\lambda\phi^2). \tag{3.8}$$

Owing to the dominance of the terms proportional to δ_{ab} in the propagator, we may replace Eqs. (3.4) and (3.5) by

$$\frac{\partial V(\phi)}{\partial\phi_a} - \frac{\partial V_0(\phi)}{\partial\phi_a} = -\tfrac{1}{6}i\lambda\phi_a \tilde{G}_{bb}(x,x;\phi) \tag{3.9}$$

and

$$-\tilde{G}^{-1}{}_{ab}(x,y;\phi) = -(\Box_x + \mu^2 + \tfrac{1}{6}\lambda\phi^2)\delta_{ab}\delta^4(x-y) + \tfrac{1}{6}i\lambda\delta_{ab}\tilde{G}_{cc}(x,x;\phi)\delta^4(x-y) \tag{3.10}$$

in the large-N limit. We can also write the exact propagator in the presence of a constant field as

$$\tilde{G}_{ab}(x,y;\phi) = \tilde{G}_1(x,y;\mu^2,\lambda\phi^2)\frac{\phi_a\phi_b}{\phi^2} + \tilde{G}_2(x,y;\mu^2,\lambda\phi^2)\left(\delta_{ab} - \frac{\phi_a\phi_b}{\phi^2}\right). \tag{3.11}$$

It is convenient to define

$$G(x,y;m^2) = \left\langle x \left| \frac{1}{\Box_x + m^2} \right| y \right\rangle \tag{3.12}$$

and

$$\tilde{G}(x,y;\mu^2,\lambda\phi^2) = \tilde{G}_2(x,y;\mu^2,\lambda\phi^2), \tag{3.13}$$

so that

$$G^{-1}(x,y;m^2) = (\Box_x + m^2)\,\delta^4(x-y). \tag{3.14}$$

The coupled set of equations to be solved in the large-N limit are

$$\frac{\partial V(\phi)}{\partial\phi_a} - \frac{\partial V_0(\phi)}{\partial\phi_a} = -\tfrac{1}{6}iN\lambda\phi_a\tilde{G}(y,y;\mu^2,\lambda\phi^2), \tag{3.15}$$

$$\tilde{G}^{-1}(x,y;\mu^2,\lambda\phi^2) = G^{-1}(x,y;\mu^2 + \tfrac{1}{6}\lambda\phi^2) - \tfrac{1}{6}iN\lambda\tilde{G}(y,y;\mu^2,\lambda\phi^2)\delta^4(x-y), \tag{3.16}$$

and

$$\frac{\partial\tilde{G}(x,x;\mu^2,\lambda\phi^2)}{\partial\phi_a} \cong O\left(\frac{1}{N}\right) \quad \text{(for large } N\text{)}. \tag{3.17}$$

[There is also a dependence of $\tilde{G}$ on $N\lambda$ which has not been explicitly indicated in (3.11) and (3.13) as no confusion will result.]

This completes the formulation of the equations for $\partial V/\partial\phi$ valid for the large-N limit. Remarkably, these equations can be solved to give an exact solution for $V(\phi)$ valid for large N, and not dependent on a perturbation expansion in λ.

C. The $1/N$ expansion

In this section we attempt to describe a systematic $1/N$ expansion[11] which should clarify the sense in which our results may be described as exact in the large-N limit. In identifying the leading terms for N large, we considered all the vacuum graphs with n loops (n *fixed*), and selected the dominant vacuum graphs for N large, which turned out to be *all* bubble graphs with n loops. At this stage, the set of all bubble graphs with n loops is summed over n to yield a system of equations which characterize nonperturbatively this set of graphs.

In particular, for $n = 2$ *loops*, the *complete* contribution of the bubble graph gives

$$[V(\phi)]_{\text{bubble}:n=2} = (N\lambda)\left[Nf_{21}(\lambda\phi^2) + f_{22}(\lambda\phi^2) + \frac{1}{N}f_{23}(\lambda\phi^2)\right], \tag{3.18}$$

while the nonbubble graphs with 2 loops give

$$[V(\phi)]_{\text{nonbubble}:n=2} = (\lambda^2\phi^2)[N\bar{g}_{21}(\lambda\phi^2) + \bar{g}_{22}(\lambda\phi^2)] = (N\lambda)\left[g_{21}(\lambda\phi^2) + \frac{1}{N}g_{22}(\lambda\phi^2)\right]. \tag{3.19}$$

For $n = 3$ *loops*, one has for the bubble graphs

$$[V(\phi)]_{\text{bubble}:n=3} = (N\lambda)^2\left[Nf_{31}(\lambda\phi^2) + f_{32}(\lambda\phi^2) + \frac{1}{N}f_{33}(\lambda\phi^2) + \frac{1}{N^2}f_{34}(\lambda\phi^2)\right], \tag{3.20}$$

while the nonbubble graphs contribute

$$[V(\phi)]_{\text{nonbubble}:n=3} = (N\lambda)^2\left[g_{31}(\lambda\phi^2) + \frac{1}{N}g_{32}(\lambda\phi^2) + \frac{1}{N^2}g_{33}(\lambda\phi^2)\right]. \tag{3.21}$$

The general case with $n+1$ *loops* gives

$$[V(\phi)]_{\text{bubble}:n+1} = (N\lambda)^n\left[Nf_{n1}(\lambda\phi^2) + f_{n2}(\lambda\phi^2) + \frac{1}{N}f_{n3}(\lambda\phi^2) + \cdots + \frac{1}{N^n}f_{n,n+2}(\lambda\phi^2)\right] \tag{3.22}$$

and

$$[V(\phi)]_{\text{nonbubble}:n+1} = (N\lambda)^n\left[g_{n1}(\lambda\phi^2) + \frac{1}{N}g_{n2}(\lambda\phi^2) + \cdots + \frac{1}{N^n}g_{n,n+1}(\lambda\phi^2)\right]. \tag{3.23}$$

In the above, the $f_{ij}(\lambda\phi^2)$ and $g_{ij}(\lambda\phi^2)$ are (in principle) calculable functions of $\lambda\phi^2$. The form of Eqs. (3.18)–(3.23) has its origin in the structure of n-loop vacuum diagrams, built from

(1) free propagators which depend on $\lambda\phi^2$,

(2) four-point vertices which depend on λ.

(3) pairs of three-point vertices which give $(\lambda\phi)^2$, and

(4) traces over internal indices which give a finite polynomial in N, which *only* appears as overall multiplicative factors in various Feynman integrals.

(This is the only way in which dependence on N may arise in an individual graph.) The reader can then easily verify the form of Eqs. (3.18)–(3.23) by examining specific vacuum diagrams.

Note that in our version of the large-N limit, the term $(N\lambda)^2 g_{31}(\lambda\phi^2)$ from $n=3$ is dropped, while the term $N^2\lambda f_{21}(\lambda\phi^2)$ from $n=2$ is retained. This emphasizes the nature of our large-N limit, which does not compare terms with a different number of closed loops. We now add (3.22) and (3.23) and sum the result over n. This gives an expression for the effective potential, which accounts for *all* graphs, and hence describes $V(\phi)$ *completely* in terms of a $1/N$ expansion. The result is

$$V(\phi^2) = V_0(\phi^2) + \sum_{n=1}^{\infty}(N\lambda)^n\left[NV_{n1}(\lambda\phi^2) + V_{n2}(\lambda\phi^2) + \cdots + \frac{1}{N^n}V_{n,n+2}(\lambda\phi^2)\right] \tag{3.24}$$

$$= V_0(\phi^2) + NV_1(\lambda\phi^2; N\lambda) + V_2(\lambda\phi^2; N\lambda) + \frac{1}{N}V_3(\lambda\phi^2; N\lambda) + \cdots, \tag{3.25}$$

where (3.25) is obtained from (3.24) by interchanging the sum in the square brackets with the sum over n. It is clear that our large-N limit is obtained from (3.25) by keeping $N\lambda$ and $\lambda\phi^2$ fixed, and letting N get large, with the result

$$V(\phi^2) - V_0(\phi^2) \sim NV_1(\lambda\phi^2; N\lambda). \tag{3.26}$$

Our method calculates $V_1(\lambda\phi^2, N\lambda)$ *exactly*, and in this sense gives the limiting behavior of $\lambda\phi^2$ theory for N large. [As yet we have not learned how to calculate the correction terms to (3.26).]

D. Solution for the effective potential

We find it easier to solve Eqs. (3.15–(3.17) in momentum space. To this end we define

$$\langle x|G(m^2)|y\rangle = \int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}G(p^2;m^2), \tag{3.27}$$

$$\langle x|\tilde{G}(\mu^2,\lambda\phi^2)|y\rangle = \int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\tilde{G}(p^2;\mu^2+\tfrac{1}{6}\lambda\phi^2), \tag{3.28}$$

$$\int\frac{d^4p}{(2\pi)^4}G(p^2,m^2) = B(m^2), \tag{3.29}$$

and

$$\int\frac{d^4p}{(2\pi)^4}\tilde{G}(p^2,\mu^2+\tfrac{1}{6}\lambda\phi^2) = \tilde{B}(\mu^2+\tfrac{1}{6}\lambda\phi^2). \tag{3.30}$$

As is evident from (3.15) and (3.16) the dependence of $\tilde{G}$ on the mass variables *for large N* occurs only in the combination $\mu^2+\frac{1}{6}\lambda\phi^2$. [If one is not working in the large-N limit, $\tilde{G}$ depends on both $G(p^2,\mu^2+\frac{1}{2}\lambda\phi^2)$ and $G(p^2,\mu^2+\frac{1}{6}\lambda\phi^2)$. Thus for arbitrary N, $\tilde{G}$ depends on μ^2 and $\lambda\phi^2$ separately.] We shall see very shortly that $\tilde{G}(p^2,m^2)$ depends on a cutoff; however, we do not indicate this variable explicitly among the arguments of $\tilde{G}$.

Using the above definition, Eq. (3.16) (for the *unrenormalized* Green's functions) becomes

$$\tilde{G}(p^2;m^2)^{-1} = G(p^2,m^2)^{-1} - \tfrac{1}{6}iN\lambda\tilde{B}(m^2), \tag{3.31}$$

where of course $m^2=\mu^2+\frac{1}{6}\lambda\phi^2$ is of particular interest. It follows directly from (3.31) that

$$\tilde{G}(p^2;m^2) = G(p^2;m^2) + \tfrac{1}{6}iN\lambda\tilde{G}(p^2;m^2)G(p^2;m^2)\tilde{B}(m^2) \tag{3.32}$$

and

$$\tilde{G}(p^2;m^2) = \frac{G(p^2;m^2)}{1-\frac{1}{6}iN\lambda G(p^2;m^2)\tilde{B}(m^2)}. \tag{3.33}$$

We can find $\tilde{B}(m^2)$ implicitly in terms of the free propagator alone. Integrate (3.32) over p^2 to obtain

$$\tilde{B}(m^2) = B(m^2) + \tfrac{1}{6}iN\lambda\tilde{B}(m^2)\int\frac{d^4p}{(2\pi)^4}\tilde{G}(p^2;m^2)G(p^2;m^2). \tag{3.34}$$

Multiply (3.33) by $G(p^2;m^2)$ and insert the result in (3.34) giving

$$\tilde{B}(m^2) = B(m^2) + \tfrac{1}{6}iN\lambda\tilde{B}(m^2) \int \frac{d^4p}{(2\pi)^4} \frac{1}{G(p^2;m^2)^{-1}[G(p^2;m^2)^{-1} - \frac{1}{6}iN\lambda\tilde{B}(m^2)]} . \tag{3.35}$$

Since

$$G(p^2;m^2) = \frac{-1}{p^2 - m^2 + i\epsilon}, \tag{3.36}$$

$$B(m^2) = \frac{i}{16\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2 + m^2}{m^2}\right)\right], \tag{3.37}$$

where Λ^2 is a cutoff. Thus,

$$\tilde{B}(m^2) = B(m^2) + \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{p^2 - m^2} - \frac{1}{p^2 - m^2 + \frac{1}{6}iN\lambda\tilde{B}(m^2)}\right].$$

Performing the integration and using (3.29) and (3.30), finally

$$\tilde{B}(m^2) = B(m^2 - \tfrac{1}{6}iN\lambda\tilde{B}(m^2)), \tag{3.38}$$

or more explicitly

$$\tilde{B}(m^2) = \frac{i}{16\pi^2}\left\{\Lambda^2 - [m^2 - \tfrac{1}{6}iN\lambda\tilde{B}(m^2)] \times \ln\left[\frac{\Lambda^2 + m^2 - \frac{1}{6}iN\lambda\tilde{B}(m^2)}{m^2 - \frac{1}{6}iN\lambda\tilde{B}(m^2)}\right]\right\}. \tag{3.39}$$

Equation (3.39) determines $\tilde{B}(m^2)$ as a solution of a transcendental equation. The effective potential is then determined by

$$\frac{\partial V(\phi^2)}{\partial \phi_a} = (\mu^2 + \tfrac{1}{6}\lambda\phi^2)\phi_a - \tfrac{1}{6}iN\lambda\phi_a\tilde{B}(\mu^2 + \tfrac{1}{6}\lambda\phi^2), \tag{3.40}$$

with $\tilde{B}(m^2)$ computed from Eq. (3.39). Since $\tilde{B}(m^2)$ is in principle known, one can solve (3.40) to find $V(\phi^2)$, by numerical integration if necessary.

There are two issues which must be faced before the solution for the effective potential can be considered satisfactory. They are the following: (1) The functions we are discussing are given in terms of the *unrenormalized* parameters of theory, and (2) a detailed numerical study of Eqs. (3.39) and (3.40) is not completely trivial, and it may obscure the qualitative features of the theory. Fortunately, both of these objections are easily met: (1) The coupled set of equations are renormalized in Secs. IV A and IV B, and (2) the qualitative behavior of the renormalized equations can be studied without appeal to detailed numerical solutions.

IV. RENORMALIZATION

A. Mass and wave-function renormalization

It is obvious from Eq. (3.32) that there is no wave-function renormalization in the large-N limit. The mass renormalization is also trivial. Define

$$\overline{\mu}^2 = \mu^2 + \delta\mu^2 = \mu^2 - \tfrac{1}{6}iN\lambda\tilde{B}(\mu^2) \tag{4.1}$$

to be finite. (Note that $\overline{\mu}^2$ is the physical mass only if $\langle\tilde{\phi}\rangle = 0$.) One obtains

$$\frac{\partial V(\phi^2)}{\partial \phi_a} = (\overline{\mu}^2 + \tfrac{1}{6}\lambda\phi^2)\phi_a - \tfrac{1}{6}iN\lambda\phi_a\overline{B}(\overline{\mu}^2;\tfrac{1}{6}\lambda\phi^2), \tag{4.2}$$

where we have defined

$$\overline{B}(\overline{\mu}^2;\tfrac{1}{6}\lambda\phi^2) = \tilde{B}(\mu^2 + \tfrac{1}{6}\lambda\phi^2) - \tilde{B}(\mu^2), \tag{4.3}$$

so that

$$\overline{B}(\overline{\mu}^2;0) = 0 . \tag{4.4}$$

From the definition (4.3), and from (3.39)

$$\overline{B}(\overline{\mu}^2;y) = \frac{i}{16\pi^2}\overline{\mu}^2 \ln\left(\frac{\Lambda^2 + \overline{\mu}^2}{\overline{\mu}^2}\right) - \frac{i}{16\pi^2}[\overline{\mu}^2 + y - \tfrac{1}{6}iN\lambda\overline{B}(\overline{\mu}^2;y)] \times \ln\left[\frac{\Lambda^2 + \overline{\mu}^2 + y - \frac{1}{6}iN\lambda\overline{B}(\overline{\mu}^2;y)}{\overline{\mu}^2 + y - \frac{1}{6}iN\lambda\overline{B}(\overline{\mu}^2;y)}\right]. \tag{4.5}$$

Thus the mass renormalization has removed the quadratic divergence, and the renormalization conditions

$$\tilde{G}(0;\mu^2)^{-1} = \overline{\mu}^2, \qquad \frac{\partial}{\partial p^2}\tilde{G}(p^2;\mu^2)^{-1}\Big|_{p^2=0} = -1 \tag{4.6}$$

are satisfied for the two-point function.

B. Coupling renormalization

Following Coleman and Weinberg[7] we specify the renormalized coupling constant by choosing the value of $\partial^4 V/\partial\phi^4$ at a fixed value of ϕ_a. For the case $\overline{\mu}^2 \neq 0$, one may choose this to be $\phi_a = 0$, which then fixes the 1PI four-point Green's function when all momenta vanish. When $\overline{\mu}^2 = 0$, this prescription cannot be adopted because of infrared divergences, so that we fix the coupling constant by specifying the value of $\partial^4 V/\partial\phi^4$ at $\phi^2 = M^2$. From Eq. (4.2) it is straightforward to compute

$$\frac{\partial^4 V}{\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d}=\left\{\tfrac{1}{3}\lambda(\delta_{ab}\delta_{cd}+\delta_{bc}\delta_{ad}+\delta_{ac}\delta_{bd})\left[1-\frac{iN\lambda}{6}\frac{\partial\overline{B}}{\partial y}(\overline{\mu}^2;y)\right]\right.$$
$$\left.-iN\tfrac{1}{54}\lambda^3(\delta_{ab}\phi_c\phi_d+\text{permutations})\frac{\partial^2\overline{B}(\overline{\mu}^2;y)}{\partial y^2}-iN\tfrac{1}{162}\lambda^4\phi_a\phi_b\phi_c\phi_d\frac{\partial^3\overline{B}(\overline{\mu}^2;y)}{\partial y^3}\right\}_{y=\frac{1}{6}\lambda\phi^2}, \quad (4.7)$$

which provides the basis of the discussion of the coupling renormalization. Because of the technical problem induced by the infrared divergence, we discuss the cases $\overline{\mu}^2\neq 0$ and $\overline{\mu}^2=0$ separately.

(1) $\overline{\mu}^2\neq 0$. We require

$$\left.\frac{\partial^4 V}{\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d}\right|_{\phi=0}=\tfrac{1}{3}\overline{\lambda}(\delta_{ab}\delta_{cd}+\delta_{bc}\delta_{ad}+\delta_{ac}\delta_{bd}), \quad (4.8)$$

which implies, from Eqs. (4.5) and (4.7), that

$$\overline{\lambda}=\lambda\left[1-\frac{iN\lambda}{6}\frac{\partial\overline{B}(\overline{\mu}^2;y)}{\partial y}\right]_{y=0}. \quad (4.9)$$

Using (4.9) with (4.2) we have

$$\frac{\partial V}{\partial\phi_a}=\overline{\mu}^2\phi_a+\tfrac{1}{6}\lambda\phi^2\phi_a\left[1-\frac{iN\lambda}{6}\frac{\partial\overline{B}(\overline{\mu}^2;0)}{\partial y}\right]-\tfrac{1}{6}iN\lambda\phi_a\left[\overline{B}(\overline{\mu}^2;\tfrac{1}{6}\lambda\phi^2)-\frac{\lambda\phi^2}{6}\frac{\partial\overline{B}(\overline{\mu}^2;0)}{\partial y}\right] \quad (4.10)$$

or

$$\frac{\partial V}{\partial\phi_a}=(\overline{\mu}^2+\tfrac{1}{6}\overline{\lambda}\phi^2)\phi_a-\tfrac{1}{6}iN\overline{\lambda}\phi_a B_R(\overline{\mu}^2;\tfrac{1}{6}\overline{\lambda}\phi^2), \quad (4.11)$$

where we have defined the *renormalized* quantity

$$\overline{\lambda}B_R(\overline{\mu}^2;\tfrac{1}{6}\overline{\lambda}\phi^2)=\lambda\left[\overline{B}(\overline{\mu}^2;\tfrac{1}{6}\lambda\phi^2)-\frac{\lambda\phi^2}{6}\frac{\partial\overline{B}(\overline{\mu}^2;0)}{\partial y}\right]. \quad (4.12)$$

It is clear from (4.4) and (4.12) that

$$B_R(\overline{\mu}^2;0)=0$$

and

$$\left.\frac{\partial B_R(\overline{\mu}^2;y)}{\partial y}\right|_{y=0}=0 \quad (4.13)$$

by construction. Equation (4.13) implies for ϕ^2 small that

$$B_R(\overline{\mu}^2;y)\underset{y\to 0}{=}O(y^2). \quad (4.14)$$

We prove in Appendix A that $B_R(\overline{\mu}^2;y)$ is finite and independent of cutoff. As a result of that discussion, one finds that

$$\left(1+\frac{N\overline{\lambda}}{96\pi^2}\right)B_R(\overline{\mu}^2;y)=-\frac{i}{16\pi^2}y-\frac{i}{16\pi^2}[\overline{\mu}^2+y-\tfrac{1}{6}iN\overline{\lambda}B_R(\overline{\mu}^2;y)]\ln\left[\frac{\overline{\mu}^2}{\overline{\mu}^2+y-\frac{1}{6}iN\overline{\lambda}B_R(\overline{\mu}^2;y)}\right]. \quad (4.15)$$

Equations (4.11) and (4.15) specify $V(\phi^2)$ in terms of renormalized quantities only. This result is *exact* in the large-N limit (with $\overline{\lambda}\phi^2$ and $N\overline{\lambda}$ held fixed). Since $B_R(\overline{\mu}^2;y)$ can be evaluated (numerically) from (4.15), $V(\phi^2)$ is completely known (numerically) in terms of the finite arbitrary parameters of the theory. As we shall see in Sec. V, we may analyze the qualitative behavior of the effective potential without recourse to detailed numerical analysis.

Of course, the system (4.11) and (4.15) could be solved by successive approximations, which would reconstruct the renormalized loop expansion in the large-N limit. Obviously this is considerably less interesting than our nonperturbative solution to these equations.

For illustrative purposes, and for use in Sec. V we compute $B_R(\overline{\mu}^2;y)$ for small y. Making use of (4.14) one finds from (4.15) that

$$B_R(\overline{\mu}^2;y)\underset{y\to 0}{\sim}\frac{i}{\overline{\mu}^2}\frac{y^2}{(32\pi^2)}+O(y^3). \quad (4.16)$$

Combining this with (4.11), we have for the large-N limit of the potential

$$V(\phi^2)\underset{\overline{\lambda}\phi^2\to 0}{\sim}\tfrac{1}{2}\overline{\mu}^2\phi^2+\frac{\overline{\lambda}}{4!}(\phi^2)^2+\frac{N(\overline{\lambda}\phi^2)^3}{432\overline{\mu}^2(96\pi^2)}+O(\phi^8), \quad (4.17)$$

which has the form dictated by Eq. (3.26).

(2) $\overline{\mu}^2=0$. The presence of infrared divergences becomes evident if one computes the derivatives of $B_R(\overline{\mu}^2;y)$ from (4.15) and takes the limit $\overline{\mu}^2\to 0$. Thus, instead of (4.8) we choose

$$\left.\frac{\partial^4 V}{\partial\phi_a\partial\phi_b\partial\phi_c\partial\phi_d}\right|_{\phi^2=M^2}=\tfrac{1}{3}\lambda_M(\delta_{ab}\delta_{dc}+\delta_{bc}\delta_{ad}+\delta_{ac}\delta_{bd}), \tag{4.18}$$

where the left-hand side of (4.18) is to be evaluated by averaging over the sphere $\phi^2=M^2$, M^2 is an arbitrary mass, and λ_M is the renormalized coupling constant appropriate to this convention. From (4.7) we find that

$$\lambda_M=\lambda-\tfrac{1}{6}iN\lambda^2\left[\frac{\partial\overline{B}(0;y)}{\partial y}+\frac{4y}{N}\frac{\partial^2\overline{B}(0;y)}{\partial y^2}+\frac{4y^2}{N(N+2)}\frac{\partial^3\overline{B}(0;y)}{\partial y^3}\right]_{y=\frac{1}{6}\lambda M^2}. \tag{4.19}$$

Proceeding in a manner similar to that of (4.10)–(4.12) we obtain from (4.18) and (4.19)

$$\frac{\partial V(\phi^2)}{\partial\phi_a}=\tfrac{1}{6}\lambda_M\phi_a[\phi^2-iNB_M(\tfrac{1}{6}\lambda_M\phi^2)], \tag{4.20}$$

where we have defined

$$\lambda_M B_M(\tfrac{1}{6}\lambda_M\phi^2)=\lambda\left\{\overline{B}(0;\tfrac{1}{6}\lambda\phi^2)-\tfrac{1}{6}\lambda\phi^2\left[\frac{\partial\overline{B}(0;y)}{\partial y}+\frac{4y}{N}\frac{\partial^2\overline{B}(0;y)}{\partial y^2}+\frac{4y^2}{N(N+2)}\frac{\partial^3\overline{B}(0;y)}{\partial y^3}\right]_{y=\frac{1}{6}\lambda M^2}\right\}. \tag{4.21}$$

We demonstrate in Appendix A that $B_M(\frac{1}{6}\lambda_M\phi^2)$ is finite and independent of cutoff. It is further shown there that

$$B_M(\tfrac{1}{6}\lambda_M\phi^2)=[\phi^2-iNB_M(\tfrac{1}{6}\lambda_M\phi^2)]\left\{\frac{B_M(\frac{1}{6}\lambda_M M^2)}{[M^2-iNB_M(\frac{1}{6}\lambda_M M^2)]}+\frac{i\lambda_M}{96\pi^2}\ln\left[\frac{\phi^2-iNB_M(\frac{1}{6}\lambda_M\phi^2)}{M^2-iNB_M(\frac{1}{6}\lambda_M M^2)}\right]\right\}. \tag{4.22}$$

Equations (4.20) and (4.22) determine the effective potential, given M^2 and λ_M. Equation (4.22) is a homogeneous equation in that it does not specify $B_M(\frac{1}{6}\lambda_M M^2)$, i.e., it is *form-invariant* for any value of M^2, which is a direct consequence of the homogeneous Callan-Symanzik equation for a massless theory.[1,2,7]

It might appear from (4.22) that $B_M(\frac{1}{6}\lambda_M M^2)$ is an undetermined constant, independent of λ_M and M^2, but this is not the case. The renormalization condition (4.18), combined with (4.20), requires that

$$\left[N(N+2)\frac{\partial B_M(\frac{1}{6}\lambda_M\phi^2)}{\partial\phi^2}+4(N+2)\frac{M^2\partial^2 B_M(\frac{1}{6}\lambda_M\phi^2)}{\partial(\phi^2)^2}+4M^4\frac{\partial^3 B_M(\frac{1}{6}\lambda_M\phi^2)}{\partial(\phi^2)^3}\right]_{\phi^2=M^2}=0, \tag{4.23}$$

which relates $B_M(\frac{1}{6}\lambda_M M^2)$ to λ_M and M^2. We found it extremely tedious and unenlightening to evaluate (4.23) using (4.22). However, we wish to compute (4.23) in the one-loop approximation to illustrate that (4.23) is nontrivial. The one-loop approximation can be obtained by neglecting $B_M(\frac{1}{6}\lambda_M\phi^2)$ in the right-hand side of (4.22). The result is

$$B_M(\tfrac{1}{6}\lambda_M\phi^2)_{1\text{-loop}}=\frac{i\lambda_M}{96\pi^2}\phi^2[\ln(\phi^2/M^2)+D], \tag{4.24}$$

where

$$B_M(\tfrac{1}{6}\lambda_M M^2)_{1\text{-loop}}=\left(\frac{i\lambda_M}{96\pi^2}M^2\right)D. \tag{4.25}$$

Applying (4.23) to this approximation, one finds that

$$B_M(\tfrac{1}{6}\lambda_M\phi^2)_{1\text{-loop}}=\frac{i\lambda_M}{96\pi^2}\phi^2\left[\ln(\phi^2/M^2)-\frac{(N^2+6N+4)}{N(N+2)}\right], \tag{4.26}$$

which demonstrates that $B_M(\frac{1}{6}\lambda_M M^2)$ is not an independent constant. Combining (4.26) with (4.20), one obtains

$$V(\phi^2)_{1\text{-loop}}=\frac{\lambda_M\phi^4}{4!}\left\{1+\frac{N\lambda_M}{96\pi^2}\left[\ln\left(\frac{\phi^2}{M^2}\right)-\frac{(3N^2+14N+8)}{2N(N+2)}\right]\right\}, \tag{4.27}$$

which agrees with other determinations of $V(\phi^2)_{1\text{-loop}}$ obtained by direct application of the loop expansion.[5] [One may compare (4.27) with Eq. (3.10) of Coleman and Weinberg[7] for $N=1$, by replacing $\frac{1}{6}\lambda\phi^2$ by $\frac{1}{2}\lambda\phi^2$ in the free-particle propagator, and the subsequent development. This comparison with $N=1$ is possible at the one-loop level, since the *only* one-loop vacuum graph is a bubble graph.]

V. STABILITY OF THE VACUUM

Since we have available a nonperturbative solution for the effective potential, we may examine the stability of the vacuum and possible spontaneous symmetry breakdown, without recourse to the loop expansion. The stable vacuum is found by locating the absolute minimum of the potential, for which a necessary condition is $\partial V/\partial\phi_a=0$. One must then establish whether this extremum is in fact the absolute minimum. If this occurs for $\phi_a\neq 0$, then one has a broken symmetry, and the true ground state will not have the full O(N) symmetry of the Lagrangian. Since we must deal with a number of technical details, we examine the cases $\overline{\mu}^2\neq 0$ and $\overline{\mu}^2=0$ separately.

A. $\overline{\mu}^2\neq 0$

One solution to $\partial V(\phi^2)/\partial\phi_a=0$, with $\partial V/\partial\phi_a$ given by (4.11), is obviously $\phi_a=0$. If there is a solution with $\phi_a\neq 0$, it requires

$$[\overline{\mu}^2+\tfrac{1}{6}\overline{\lambda}\phi^2-\tfrac{1}{6}iN\overline{\lambda}B_R(\overline{\mu}^2;\tfrac{1}{6}\overline{\lambda}\phi^2)]=0\,. \tag{5.1}$$

However, from Eq. (4.15), this in turn requires

$$\left(1+\frac{N\overline{\lambda}}{96\pi^2}\right)iB_R(\overline{\mu}^2;\tfrac{1}{6}\overline{\lambda}\phi^2)\overset{?}{=}\frac{\overline{\lambda}}{96\pi^2}\phi^2 \text{ for } \phi^2>0\,. \tag{5.2}$$

To continue the analysis, we present the following intermediate result:

$$iB_R(\overline{\mu}^2;y)=0 \text{ for } y=0 \textit{ only}\,. \tag{5.3}$$

Proof. $iB_R(\overline{\mu}^2;y_0)=0$ requires from (4.15)

$$(\overline{\mu}^2+y_0)\ln\left(\frac{\overline{\mu}^2+y_0}{\overline{\mu}^2}\right)=y_0\,. \tag{5.4}$$

The only solution to (5.4) is $y_0=0$. Thus, $iB_R(\overline{\mu}^2;y)$ only has a zero at $y=0$.

We note that if $N\overline{\lambda}>-96\pi^2$, Eq. (5.2) can only be satisfied for $iB_R(\overline{\mu}^2;y)$ real and positive (negative) for y positive (negative). The behavior of $iB_R(\overline{\mu}^2;y)$ may be determined in the neighborhood of the origin from (4.16) and (4.17), where it is stated that

$$iB_R(\overline{\mu}^2;y)\underset{y\to 0}{\sim}\frac{-y^2}{\overline{\mu}^2(32\pi^2)}+O(y^3) \tag{5.5}$$

and

$$V(\phi^2)\underset{\overline{\lambda}\phi^2\to 0}{\sim}\tfrac{1}{2}\overline{\mu}^2\phi^2+\frac{\overline{\lambda}}{4!}(\phi^2)^2+\frac{N(\overline{\lambda}\phi^2)^3}{432\overline{\mu}^2(96\pi^2)}+\cdots\,. \tag{4.17}$$

Thus, for y sufficiently small, the sign of $iB_R(\overline{\mu}^2;y)$ is determined by (5.5). Since $iB_R(\overline{\mu}^2;y)$ has no zeros other than $y=0$, it cannot change sign; this fact may be combined with (5.2) to test for a possible extremum. If (5.1) is satisfied, then (5.1) and (5.2) combined imply that

$$6\left(\frac{\overline{\mu}^2}{\overline{\lambda}}\right)\frac{(N\overline{\lambda}+96\pi^2)}{96\pi^2}\overset{?}{=}\phi_0^{\,2}>0\,, \tag{5.6}$$

where the (possible) extremum of $V(\phi^2)$ is denoted as $\phi_0^{\,2}$. The origin is a minimum of $V(\phi^2)$ for $\overline{\mu}^2>0$ and a maximum for $\overline{\mu}^2<0$. From (4.15) and (5.1) we observe that $iB_R(\overline{\mu}^2;y)$ changes from real to complex (or the converse) as ϕ^2 passes through this extremum. This observation makes it possible to discuss the stability of the vacuum without detailed numerical solutions.

Consider first $\overline{\mu}^2>0$. From (4.11) and (4.15) this means that $V(\phi^2)$ is real in the neighborhood of the origin and becomes *complex* for $\phi^2>\phi_0^{\,2}$, which means that the vacuum is *unstable* for all $\phi^2\geqslant\phi_0^{\,2}$. Thus, if $V(\phi^2)$ has an extremum at $\phi_0^{\,2}\neq 0$ for $\overline{\mu}^2\neq 0$, it is expected to be a maximum. Thus the only minimum occurs at $\phi^2=0$, and the potential will continue to decrease for all $\phi^2>\phi_0^{\,2}$. Therefore, to obtain a stable vacuum one must exclude the possible maximum at $\phi^2=\phi_0^{\,2}$, which limits $0<1+(N\overline{\lambda})^{-1}96\pi^2$.[13] If $N\overline{\lambda}$ is outside this range, $\phi^2=0$ is no longer a minimum of the potential. The various possibilities that occur for $\overline{\mu}^2>0$ are summarized in Table I.

Now consider $\overline{\mu}^2<0$,[12] in which case the origin is a maximum of $V(\phi^2)$, so that $\phi_a=0$ cannot be a stable vacuum. This is consistent with (3.32), which for $\phi^2=0$ becomes

$$\tilde{G}(p^2;\mu^2)^{-1}=\overline{\mu}^2-p^2\,. \tag{5.7}$$

If $\overline{\mu}^2<0$, $\overline{\mu}^2$ is not the physical mass, $\phi_a=0$ cannot be the ground state, (5.7) is not the *physical* two-point function, so that $\langle\tilde{\phi}\rangle\neq 0$ *must* occur. The potential will have a minimum if (5.6) is satisfied which will be an absolute minimum of the effective potential, if $\partial V(\phi^2)/\partial\phi_a>0$ for all $\phi^2>\phi_0^{\,2}$. If (5.6) is not satisfied, there will be no minimum, which limits $N\overline{\lambda}$ to the range $0<1+(N\overline{\lambda})^{-1}96\pi^2$.[13] The various possibilities that occur for $\overline{\mu}^2<0$ (see Ref. 12) are summarized in Table II. (See note

TABLE I. Qualitative behavior of the effective potential for $\overline{\mu}^2>0$. Columns 1-3 are determined by Eqs. (5.5), (5.6), and (4.11), respectively. (See also Ref. 13 and note added in proof.)

$N\overline{\lambda}$	$\frac{\partial V}{\partial\phi}=0:\phi^2\neq 0$	Stable vacuum
$0<1+(N\overline{\lambda})^{-1}96\pi^2$	No	Local minimum ($\phi_a=0$)
$-96\pi^2<N\overline{\lambda}<0$	Yes (maximum)	None ($\phi_a=0$ relative minimum)

TABLE II. Same as Table I, but for $\bar{\mu}^2 < 0$. (See also Refs. 12 and 13 and note added in proof.)

$N\bar{\lambda}$	$\frac{\partial V}{\partial \phi} = 0 : \phi^2 \neq 0$	Stable vacuum
$0 < 1 + (N\bar{\lambda})^{-1} 96\pi^2$	Yes (minimum)	Local mimimum ($\phi^2 \neq 0$)
$-96\pi^2 < N\bar{\lambda} < 0$	No	None

added in proof.)

Let us fix $\bar{\mu}^2 < 0$ and $0 < 1 + (N\bar{\lambda})^{-1} 96\pi^2$, so that there is a minimum of the potential at $\phi_0{}^2 \neq 0$. Define

$$\langle \bar{\phi}_a \rangle = \eta_a \tag{5.8}$$

at the minimum. We can calculate the physical propagator at zero momentum from (4.11), which gives[12]

$$\left.\frac{\partial^2 V(\phi^2)}{\partial \phi_a \partial \phi_b}\right|_{\phi_a = \eta_a} = \eta_a \eta_b \tfrac{1}{3}\bar{\lambda}\left[1 - \frac{iN\bar{\lambda}}{6}\frac{\partial B_R(\bar{\mu}^2; y)}{\partial y}\right]_{y = \frac{1}{6}\bar{\lambda}\phi_0^2}. \tag{5.9}$$

From (4.15)

$$\frac{\partial B_R(\bar{\mu}^2; y)}{\partial y} = \frac{\frac{i}{16\pi^2}\ln\left[\frac{\bar{\mu}^2 + y - \frac{1}{6} iN\bar{\lambda} B_R(\bar{\mu}^2; y)}{\bar{\mu}^2}\right]}{1 - \frac{N\bar{\lambda}}{96\pi^2}\left\{\ln\left[\frac{\bar{\mu}^2 + y - \frac{1}{6} iN\bar{\lambda} B_R(\bar{\mu}^2; y)}{\bar{\mu}^2}\right]\right\}} \tag{5.10}$$

$$\xrightarrow[y \to \frac{1}{6}\bar{\lambda}\phi_0^2]{} -\frac{6i}{N\bar{\lambda}}. \tag{5.11}$$

Hence

$$\left.\frac{\partial^2 V(\phi^2)}{\partial \phi_a \partial \phi_b}\right|_{\phi_0^2} = 0, \tag{5.12}$$

to leading order in the $1/N$ expansion. This correctly predicts that the inverse propagators vanish at zero momentum. However, Eqs. (3.31)–(3.37) are not accurate enough to determine the mass of the "σ mesons" in the event of broken symmetry.[14]

B. $\bar{\mu}^2 = 0$

We now discuss the possibility of spontaneous symmetry breakdown for massless $\lambda\phi^4$ theory with O(N) symmetry. If spontaneous symmetry breakdown is to occur, the right-hand side of (4.20) must vanish, which requires

$$[\phi_0{}^2 - iNB_M(\tfrac{1}{6}\lambda_M \phi_0{}^2)] \stackrel{?}{=} 0. \tag{5.13}$$

Inserting (5.13) into (4.22) implies

$$B_M(\tfrac{1}{6}\lambda_M \phi_0{}^2) = 0, \tag{5.14}$$

which when combined with (5.13) means that

$$\phi_0{}^2 = 0. \tag{5.15}$$

Therefore,

$$\frac{\partial V(\phi^2)}{\partial \phi_a} = 0 \quad \text{for } \phi^2 = 0 \text{ only}. \tag{5.16}$$

Since

$$[\phi^2 - iNB_M(\tfrac{1}{6}\lambda_M \phi^2)] \neq 0 \quad \text{for } \phi^2 \neq 0, \tag{5.17}$$

there are no other extrema. Stability requires $\partial V(\phi^2)/\partial \phi_a$ remain real and positive for all ϕ^2. Therefore it must also be real and positive at any other conveniently chosen value of ϕ^2, M^2 say, as can be verified by examination of (4.20) and (4.22). Thus, stability of the vacuum is achieved if

$$\lambda_M[M^2 - iNB_M(\tfrac{1}{6}\lambda_M M^2)] \equiv K \tag{5.18}$$

is real and positive, in which case $V(\phi^2)$ will have an absolute minimum at $\phi = 0$. If K is not real and positive, there will be no stable vacuum.

If $\phi^2 = 0$ is a stable vacuum, the physical mesons *remain* massless, as there is no spontaneous symmetry breakdown. However, see note added in proof.

VI. RENORMALIZATION GROUP

Our discussion of the massless theory required the introduction of a mass M whose only function was to define the renormalized coupling constant λ_M, and to set the scale of the renormalized field, but which is otherwise arbitrary. This arbitrariness is reflected in the form invariance of Eq. (4.22), which should be contrasted with (4.15) for the massive theory. In the massless theory a small change in M can be compensated by a change in λ_M and an appropriate rescaling of the field. In this section we study the consequences of this rescaling.

Following Coleman and Weinberg,[7] the generating functional for massless ϕ^4 theory satisfies a homogeneous Callan-Symanzik equation,[7]

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma(\lambda)\int d^4x\, \phi_a(x)\frac{\delta}{\delta \phi_a(x)}\right]\Gamma(\phi^2) = 0, \tag{6.1}$$

for an appropriate β and γ. (In *this* section we write λ_M as λ, since no confusion will result.) Dimensional analysis indicates that β and γ only depend on λ. The functional $\Gamma(\phi^2)$ can be expanded in terms of 1PI Green's functions to give the more familiar equation[2]

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma\right]\Gamma^{(n)}(x_1,\ldots,x_n)=0. \tag{6.2}$$

We can apply (6.1) directly to our work by considering the special case $\phi(x)=$constant, leading to a partial differential equation for the effective potential[7]

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\gamma\phi_b\frac{\partial}{\partial\phi_b}\right]V(\phi^2)=0. \tag{6.3}$$

In our formulation of the problem, it is more convenient to work with $\partial V/\partial\phi_a$, so that

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+\gamma\left(1+\phi_b\frac{\partial}{\partial\phi_b}\right)\right]\frac{\partial V(\phi^2)}{\partial\phi_a}=0. \tag{6.4}$$

We have shown in Sec. III A that the wave-function renormalization is finite in the large-N limit, which implies[7] that $\gamma(\lambda)=0$ in this limit, so that (6.4) becomes

$$\left[2M^2\frac{\partial}{\partial M^2}+\beta(\lambda)\frac{\partial}{\partial\lambda}\right]\frac{\partial V(\phi^2)}{\partial\phi_a}=0 \quad \text{for } N \text{ large.} \tag{6.5}$$

Since $\partial V/\partial\phi_a$ is known nonperturbatively from Eqs. (4.20) and (4.22), we may reverse the usual arguments and *compute* $\beta(\lambda)$ from the known behavior of $\partial V/\partial\phi_a$ in the large-N limit. On dimensional grounds, we may write

$$B_M(\tfrac{1}{6}\lambda\phi^2)=i\phi^2F(\phi^2/M^2,\lambda), \tag{6.6}$$

where F is a dimensionless quantity dependent on ϕ^2/M^2 and λ. Then

$$\frac{\partial V}{\partial\phi_a}=\tfrac{1}{6}\lambda\phi_a\phi^2[1+NF(\phi^2/M^2,\lambda)] \tag{6.7}$$

and

$$F(z,\lambda)=[1+NF(z,\lambda)]\left\{\frac{F(1,\lambda)}{1+NF(1,\lambda)}+\frac{\lambda}{96\pi^2}\ln\left[\frac{z(1+NF(z,\lambda))}{1+NF(1,\lambda)}\right]\right\}. \tag{6.8}$$

Substituting into (6.5) we find that

$$\beta(\lambda)=\frac{2N\lambda z(\partial/\partial z)F(z,\lambda)}{\{1+NF(z,\lambda)+N\lambda[\partial F(z,\lambda)/\partial\lambda]\}}. \tag{6.9}$$

The *apparent* dependence of the right-hand side of (6.9) on z is in fact not present. (If there were such a dependence, we would have an inconsistency.) It is straightforward to compute $\partial F(z,\lambda)/\partial z$ and $\partial F(z,\lambda)/\partial\lambda$ from (6.8). We then find

$$\beta(\lambda)=\frac{(N\lambda^2/48\pi^2)[1+NF(1,\lambda)]}{\{1-(N\lambda/96\pi^2)[1+NF(1,\lambda)]+N\lambda[(1+NF(1,\lambda))^{-1}-N\lambda/96\pi^2]\partial F(1,\lambda)/\partial\lambda\}}, \tag{6.10}$$

which is only dependent on λ, as it should be.

Since $F(1,\lambda)$ vanishes for $\lambda\to0$, we have

$$\beta(\lambda)\underset{\lambda\to0}{\longrightarrow}\frac{N\lambda^2}{48\pi^2}+\cdots, \tag{6.11}$$

as expected. The possibility of a nontrivial zero of $\beta(\lambda)$ for some value of $\lambda\neq0$ can be studied in the large-N limit by an analysis of (6.10) without recourse to perturbation theory. The function $\beta(\lambda)$ will vanish if the numerator of (6.10) vanishes. However, since $\partial V/\partial\phi_a\neq0$ except for $\phi^2=0$ (as discussed in Sec. V), we see that (6.7) implies that

$$[1+NF(z,\lambda)]\neq0 \text{ for any } z. \tag{6.12}$$

In particular, we may choose M^2 to be at the minimum, if there is one, fixing $z=1$, implying

$$[1+NF(1,\lambda)]\neq0, \tag{6.13}$$

so that the numerator of $\beta(\lambda)$ never vanishes. If the denominator of (6.10) were infinite for some value of λ, then $\beta(\lambda)$ could vanish at this point. A preliminary (but not exhaustive) study seems to show that this is unlikely.

We argued that $\beta(\lambda)$ most likely did not have a nontrivial zero, since we proved that the numerator of (6.10) did not vanish. Further, we related this property to the nonvanishing of Eq. (6.7) for $\phi^2\neq0$. *If* the numerator had vanished, it would have correlated two features of the theory[8]:

(1) $\beta(\lambda)=0$

and (6.14)

(2) $\left.\dfrac{\partial V(\phi^2)}{\partial\phi_a}\right|_{\phi^2=M^2}=0$

at *this* value of λ. This shows that the spontaneous symmetry breakdown of the massless theory has as a necessary condition the existence of a nontrivial Gell-Mann and Low eigenvalue. We believe that this condition for spontaneous symmetry breaking may be a general feature of *infrared stable*, *massless* field theories which have no *fundamental* parameters with dimension of length, and only a single coupling constant. (The model of Coleman and Weinberg,[5] with two coupling constants, does not seem to have this property.) In quantum electrodynamics this correlates a possible Gell-Mann–Low eigenvalue with $\langle\bar{\psi}\psi\rangle \neq 0$, and the breakdown of γ_5 invariance.

VII. CONCLUSIONS

We have shown that it is possible to compute the effective potential *exactly* in $\lambda\phi^4$ theory, with O(N) symmetry, in the limit of N large. As a consequence, we were able to analyze the stability of the vacuum and the possible spontaneous symmetry breakdown of the theory. We found that the existence of a local minimum in the large-N limit required the renormalized coupling constant to satisfy $0 < 1 + (N\lambda)^{-1}96\pi^2$. (It is not clear whether these are true bounds, or only an indication of the breakdown of the $1/N$ expansion.[13]) We were also able to argue that $\beta(\lambda)$, the function that appears in the Callan-Symanzik equation in the massless theory, is unlikely to have a nontrivial zero. Furthermore, we showed that the existence of a nontrivial zero of $\beta(\lambda)$ in the massless theory is a necessary condition for spontaneous symmetry breakdown in the massless case.

In Appendix B we showed how to truncate our exact system of equations (for large N) so as to generate the leading logarithm approximation.[6] In fact the leading-logarithm approximation generates a *spurious* singularity in ϕ^2 space. However, the approximation is not expected to be valid in the region of the singularity.

At the moment the study of the large-N limit in particle physics is largely academic. (It does have applications to statistical mechanics.) Its virtue is that it presents a method which avoids a perturbation expansion in the coupling constant in favor of an expansion in $1/N$. This allows one to analyze many issues of principle which are not accessible in the usual perturbation expansions. These methods might be useful in physically interesting theories if the corrections to the large-N limit are small.

Note added. One may combine Eqs. (4.11) and (4.15) to obtain a single equation describing the effective potential. It is convenient to define

$$\mathfrak{M}^2 = 2\frac{\partial V}{\partial\phi^2}\,, \tag{7.1}$$

whereupon one obtains

$$\mathfrak{M}^2 = \bar{\mu}^2 + \tfrac{1}{6}\bar{\lambda}\phi^2\left(1 + \frac{N\bar{\lambda}}{96\pi^2}\right)^{-1} + \frac{N\bar{\lambda}}{(N\bar{\lambda} + 96\pi^2)}\mathfrak{M}^2 \ln\left(\frac{\mathfrak{M}^2}{\bar{\mu}^2}\right). \tag{7.2}$$

This equation achieves a more compact form if one rescales the classical field by

$$\phi^2 \to \frac{N}{96\pi^2}\,\phi^2 \tag{7.3}$$

and defines

$$g = \frac{N\bar{\lambda}}{(96\pi^2 + N\bar{\lambda})}\,, \tag{7.4}$$

so that[14]

$$\mathfrak{M}^2 = \bar{\mu}^2 + \tfrac{1}{6}g\phi^2 + g\mathfrak{M}^2 \ln\left(\frac{\mathfrak{M}^2}{\bar{\mu}^2}\right). \tag{7.5}$$

This is the "gap equation" of the theory. Equation (7.5) provides a convenient equation for studying symmetry breaking. Spontaneous symmetry breaking thus requires

$$\mathfrak{M}^2 = 0$$

and

$$\frac{-6\bar{\mu}^2}{g} = {\phi_0}^2, \tag{7.6}$$

which is *identical* to Eq. (5.6).

The rescaling (7.4) makes it clear that $N\bar{\lambda} = -96\pi^2$ is a singular point of the theory, and the analysis of Sec. V shows that $0 > g > -\infty$ is unphysical. Unfortunately, Eq. (7.5), taken by itself, tends to obscure the existence of bounds for $N\bar{\lambda}$. One aspect of this property may be recovered by expanding (7.5) in the neighborhood of $\phi^2 = 0$, where

$$\mathfrak{M}^2 \underset{\phi^2\to 0}{\sim} \bar{\mu}^2 + \frac{1}{6}\,\frac{g}{1-g}\,\phi^2 + \frac{1}{72\bar{\mu}^2}\left(\frac{g}{1-g}\right)^3\phi^4 + O(\phi^6), \tag{7.7}$$

which exhibits the singular nature of $g = 1$ ($N\bar{\lambda} = \infty$). This phenomenon can also be observed in (4.17), which is the analog of (7.7) *before* the rescaling (7.3).

The many-field limit is not new. In statistical mechanics the large-N limit gives rise to the spherical model,[15] invented by M. Kac. In this limit, the Hartree approximation becomes exact.[16] The study of the large-N limit has also been advocated by other workers.[17]

Note added in proof.

From (7.1) and (7.2) one easily finds that

$$\frac{\partial V(\phi^2)}{\partial\phi^2} \underset{\phi^2\to\infty}{\sim} -\frac{\phi^2}{12}\,\frac{96\pi^2}{N}\,(\ln\phi^2)^{-1}\,, \tag{7.8}$$

which is identical to (B8) obtained from the renormalization group. Accordingly, the effective potential in the large-N limit has no lower bound for all choices of the free parameters. This difficulty is also signaled by the presence of a tachyon state in the model.[14]

A more detailed study of the gap equation (in collaboration with L. Abbot) shows that if a local minimum of the potential occurs at some $\phi^2>0$, one always finds

$$\mathrm{Re}\,\frac{\partial V}{\partial\phi^2}=0\,,\qquad \mathrm{Re}\,\frac{\partial^2 V}{\partial(\phi^2)^2}<0\,, \tag{7.9}$$

and

$$\mathrm{Im}\,\frac{\partial V}{\partial\phi^2}\neq 0$$

at a *larger* value of ϕ^2, which is consistent with the behavior reported in Sec. IV and Eq. (7.8).

It has been conjectured[14] that $V(\phi^2)$ may have a minimum for very large ϕ^2, outside the domain of validity of the large-N expansion, which in fact represents the true vacuum of the theory. If so, the analysis of Sec. VI suggests that $\beta(\lambda)=0$ is required for this to occur.

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APPENDIX A: RENORMALIZATION

In this appendix we show that the quantities $B_R(\bar\mu^2;y)$ and $B_M(y)$, defined by Eqs. (4.12) and (4.21), respectively, are finite and independent of cutoff. [In what follows we suppress the $\bar\mu^2$ dependence of $B_R(\bar\mu^2;y)$ to simplify our notation.] From Eqs. (4.5) and (4.12) we can write

$$(\bar\lambda/\lambda)B_R(z)=\frac{i}{16\pi^2}\,\bar\mu^2\ln\left(\frac{\Lambda^2+\bar\mu^2}{\bar\mu^2}\right)-(\lambda/\bar\lambda)z\left[\frac{\partial\bar B(y)}{\partial y}\right]_{y=0}-\frac{i}{16\pi^2}\left[\bar\mu^2+z-\tfrac{1}{6}iN\bar\lambda B_R(z)\right]\ln\left[\frac{\Lambda^2+\bar\mu^2+z-\frac{1}{6}iN\bar\lambda B_R(z)}{\bar\mu^2+z-\frac{1}{6}iN\bar\lambda B_R(z)}\right], \tag{A1}$$

where $z=\frac{1}{6}\bar\lambda\phi^2$, and $\bar\lambda(\lambda)$ is the renormalized (unrenormalized) coupling constant. From (4.5) one finds that

$$\left[\frac{\partial\bar B(y)}{\partial y}\right]_{y=0}=\frac{i}{16\pi^2}\left(\frac{\bar\lambda}{\lambda}\right)\left[\frac{\Lambda^2}{\Lambda^2+\bar\mu^2}-\ln\left(\frac{\Lambda^2+\bar\mu^2}{\bar\mu^2}\right)\right]. \tag{A2}$$

Taking the limit $\Lambda^2\to\infty$ in (A1) and noting that $B_R(z)/\Lambda^2\to 0$, since the quadratic divergence has been eliminated with the mass renormalization, one obtains from substitution of (A2) into (A1)

$$\bar\lambda B_R(z)\left\{\frac{1}{\lambda}+\frac{N}{96\pi^2}\ln\left[\frac{\Lambda^2}{\bar\mu^2+z-\frac{1}{6}iN\bar\lambda B_R(z)}\right]\right\}=-\frac{i}{16\pi^2}z-\frac{i}{16\pi^2}\left[\bar\mu^2+z\right]\ln\left[\frac{\bar\mu^2}{\bar\mu^2+z-\frac{1}{6}iN\bar\lambda B_R(z)}\right]. \tag{A3}$$

If $B_R(z)$ is indeed finite and independent of cutoff, then the right-hand side of (A3) is also finite, so that consistency demands that we show that the left-hand side of (A3) is also independent of Λ^2. However, from (4.9) and (A2), we relate the renormalized and the unrenormalized coupling constants by the expression

$$\frac{1}{\lambda}=\frac{1}{\bar\lambda}\left\{1+\frac{N\bar\lambda}{96\pi^2}\left[-\ln\left(\frac{\Lambda^2}{\bar\mu^2}\right)+1\right]\right\}. \tag{A4}$$

When (A4) is substituted into the left-hand side of (A3), all Λ^2 dependence is eliminated, and one arrives at Eq. (4.15) of the text.

The discussion of $B_M(z)$ is similar.

Combining Eq. (4.5) with (4.21) one obtains

$$\lambda_M B_M(z)=-\frac{i\lambda}{16\pi^2}\left[z-\tfrac{1}{6}iN\lambda_M B_M(z)\right]\ln\left[\frac{\Lambda^2}{z-\frac{1}{6}iN\lambda_M B_M(z)}\right]-\left(\frac{\lambda^2}{\lambda_M}\right)z\left[\frac{\partial\bar B(0,y)}{\partial y}+\frac{4y}{N}\frac{\partial^2\bar B(0,y)}{\partial y^2}+\frac{4y^2}{N(N+2)}\frac{\partial^3\bar B(0,y)}{\partial y^3}\right]_{y=\lambda_M{}^2/6}, \tag{A5}$$

with $z=\frac{1}{6}\lambda_M\phi^2$. From the renormalization condition (4.19), one may reexpress the second term on the right-hand side of (A5) in terms of $(\lambda_M-\lambda)$. Substituting (4.19) into (A5) yields

$$\lambda_M B_M(\tfrac{1}{6}\lambda_M\phi^2) = -\frac{i\lambda\lambda_M}{96\pi^2}[\phi^2 - iNB_M(\tfrac{1}{6}\lambda_M\phi^2)]\ln\left[\frac{6\Lambda^2/\lambda_M}{\phi^2 - iNB_M(\frac{1}{6}\lambda_M\phi^2)}\right] - \frac{i}{N}(\lambda_M-\lambda)\phi^2. \tag{A6}$$

Setting $\phi^2=M^2$ in (A6) gives

$$\frac{1}{\lambda} = \frac{1}{\lambda_M}\frac{M^2}{[M^2 - iNB_M(\frac{1}{6}\lambda_M M^2)]} - \frac{N}{96\pi^2}\ln\left[\frac{6\Lambda^2/\lambda_M}{M^2 - iNB_M(\frac{1}{6}\lambda_M M^2)}\right]. \tag{A7}$$

Eliminating λ from (A6) using (A7), one arrives at the final result given by Eq. (4.22) of the text. Hence $B_M(\frac{1}{6}\lambda_M\phi^2)$ can be expressed entirely in terms of renormalized quantities, and is finite.

APPENDIX B: LEADING-LOGARITHM APPROXIMATION

Several authors have computed the effective potential in $\lambda\phi^4$ theory and considered the *leading-logarithm* approximation to $V(\phi^2)$ in the large-N limit.[6] This approximation is obtained from the solution to the renormalization group Eqs. (6.3) with $\beta(\lambda)$ computed to lowest nontrivial order. Equivalently, one sums the *linear chain* of bubble graphs which contribute to the effective potential. The linear-chain (leading-logarithm) approximation only sums a subset of the bubble graphs which we are able to treat exactly. The purpose of this appendix is to discuss how our system of equations may be truncated to produce the leading-logarithm approximation.

In order to facilitate the comparison with the work of other authors, we discuss our system of equations *before* mass and coupling renormalization has been implemented (cf. Sec. III D). The question of renormalization is easily dealt with after our equations have been truncated, and we have the advantage that we need not treat the cases $\bar{\mu}^2\neq 0$ and $\bar{\mu}^2=0$ separately. [For purposes of detailed comparison, we attempt to make contact with the work of Jackiw,[6] his Sec. III C, especially his Eqs. (3.25)–(3.27), which we denote by (J3.25)–(J3.27).]

The effective potential in the large-N limit is given by Eqs. (3.29) and (3.40). It is convenient to rewrite (3.39) [or equivalently (3.35)] as

$$\tilde{B}(m^2) = B(m^2) + \tfrac{1}{6}iN\lambda\tilde{B}(m^2)\int\frac{d^4p}{(2\pi)^4}\frac{[G(p^2,m^2)]^2}{[1-\frac{1}{6}iN\lambda G(p^2,m^2)\tilde{B}(m^2)]}, \tag{B1}$$

where $m^2=\mu^2+\frac{1}{6}\lambda\phi^2$. Expand the denominator of (B1) in powers of $N\lambda G(p^2,m^2)\tilde{B}(m^2)$, so that

$$\tilde{B}(m^2) = B(m^2) + \tfrac{1}{6}iN\lambda\tilde{B}(m^2)\int\frac{d^4p}{(2\pi)^4}[G(p^2,m^2)]^2 + [\tfrac{1}{6}iN\lambda\tilde{B}(m)]^2\int\frac{d^4p}{(2\pi)^4}[G(p^2,m^2)]^3 + [\tfrac{1}{6}iN\lambda\tilde{B}(m)]^3\int\frac{d^4p}{(2\pi)^4}[G(p^2,m^2)]^4 + \cdots, \tag{B2}$$

which has the graphical representation shown in Fig. 11. The infinite series in (B2) still expresses the exact content of the large-N limit, as no approximations have as yet been made.

One must solve (3.39) or (B2) for $\tilde{B}(m^2)$, and substitute the result in (3.40) to obtain $\partial V/\partial\phi_a$. In the text we solve this exactly, in terms of the transcendental Eq. (3.39), which has a solution by numerical methods. An alternate (but less useful) approach is to solve (B2) approximately. Suppose that we *linearize* (B2) in that we only keep terms in (B2) which depend linearly on $\tilde{B}(m^2)$. Let us denote this approximate solution as $\tilde{B}_1(m^2)$. Then

$$\tilde{B}_1(m^2) = B(m^2)\left\{1-\tfrac{1}{6}iN\lambda\int\frac{d^4p}{(2\pi)^4}[G(p^2,m^2)]^2\right\}^{-1} \tag{B3a}$$

$$= \frac{i}{16\pi^2}\frac{g(m^2)}{[1-\lambda a_n f(m^2)]}, \tag{B3b}$$

where

$$B(m^2) = \frac{i}{16\pi^2}g(m^2) = \Lambda^2 - m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right), \tag{J3.26b}$$

$$f(m^2) = \frac{dg(m^2)}{dm^2} = \frac{\Lambda^2}{\Lambda^2+m^2} - \ln\left(\frac{\Lambda^2+m^2}{m^2}\right), \tag{J3.26a}$$

and $a_n=N/96\pi^2$.

Let us improve the solution for $\tilde{B}(m^2)$ by substituting (B3a) back into the right-hand side of (B2). There is now an infinite number of terms on the right-hand side of (B2) which depend on powers of $\tilde{B}_1(m^2)$. Now truncate this new infinite series at the quadratic term on the right, i.e., at $[\tilde{B}_1(m^2)]^2$, so that

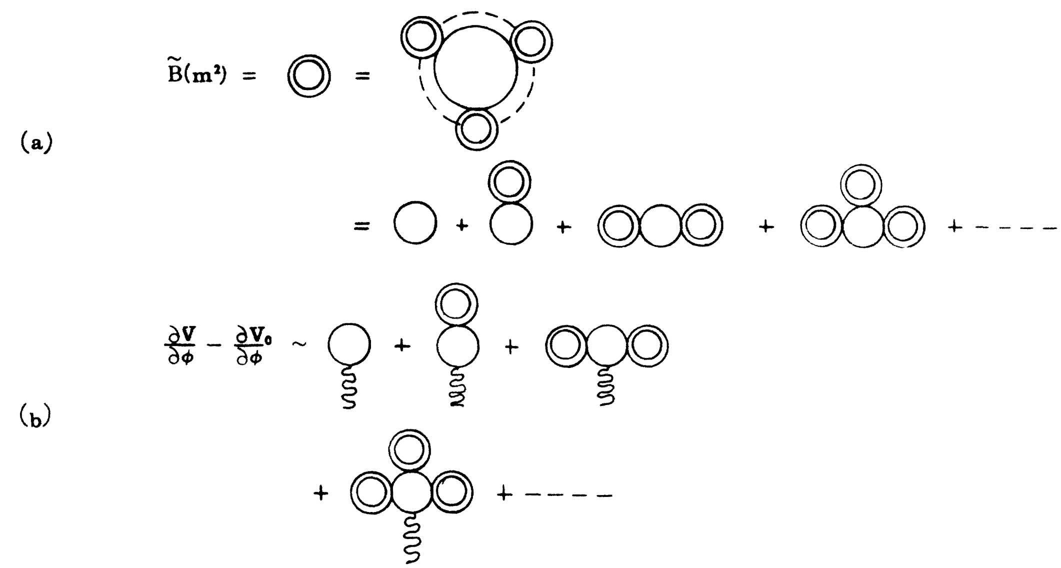

FIG. 11. (a) Graphical representation of Eqs. (3.35) and (B2) for $\tilde{B}(m^2)$, with conventions as in Figs. 1 and 3, but with ϕ a constant field. (b) Graphical representation of Eqs. (3.40) and (B2) for $\partial V/\partial\phi$, with conventions as in (a).

$$\tilde{B}_2(m^2) = B(m^2) + \tfrac{1}{6} iN\lambda B_1(m^2) \int \frac{d^4p}{(2\pi)^4} [G(p^2, m^2)]^2 + [\tfrac{1}{6} iN\lambda B_1(m^2)]^2 \int \frac{d^4p}{(2\pi)^4} [G(p^2, m^2)]^3. \tag{B4}$$

Carrying out the integrations one obtains

$$\tilde{B}_2(m^2) = g(m^2) + \tfrac{1}{6} iN\lambda \left[\frac{i}{16\pi^2} g(m^2)\right] \frac{[-i/16\pi^2 f(m^2)]}{[1-\lambda a_n f(m^2)]} + \left(\frac{iN\lambda}{6}\right)^2 \left\{\frac{i}{16\pi^2} \frac{g(m^2)}{[1-\lambda a_n f(m^2)]}\right\}^2 \left[\frac{i}{16\pi^2} \frac{\partial f(m^2)}{\partial m^2}\right] \tag{B5a}$$

$$= \frac{i}{16\pi^2} \left\{ g(m^2)\left[\frac{1}{1-\lambda a_n f(m^2)}\right] + \lambda^2 a_n^{\,2} \frac{[g(m^2)]^2}{[1-\lambda a_n f(m^2)]^2} \frac{\partial f(m^2)}{\partial m^2} \right\}. \tag{B5b}$$

The approximate solution to the effective potential (corresponding to the leading-logarithm approximation) is

$$\frac{\partial V(\phi^2)}{\partial \phi_a} - \frac{\partial V_0(\phi^2)}{\partial \phi_a} \simeq [m^2 + \lambda a_n \tilde{B}_2(m^2)]. \tag{B6}$$

We see above that fairly drastic approximations must be made to reduce our exact equations to (B5) and (B6).

Jackiw[6] computes $V(\phi^2) - V_0(\phi^2)$ in his Eq. (3.25) for the *massless* theory, in the leading-logarithm or linear-chain approximation, for N large. From his results we have calculated $[\partial V(\phi^2)/\partial\phi_a] - \partial V_0(\phi^2)/\partial\phi_a$, which after rearranging terms coincides exactly with our (B5) and (B6). Further Jackiw[6] and Coleman and Weinberg[6] find for the renormalized effective potential in the leading-logarithm approximation when N gets large,

$$V(\phi^2) \simeq \frac{1}{4!} \bar{\lambda}\phi^4 \frac{1}{1-\bar{\lambda} a_n \ln\phi^2}. \tag{B7}$$

Equation (B6) has a spurious singularity in ϕ^2 space. Of course the leading-logarithm approximation is a rather poor method for computing the effective potential in the region where the singularity occurs. For ϕ^2 very large

$$V(\phi^2) \underset{\phi^2\to\infty}{\sim} \frac{-\phi^4}{4! a_n \ln\phi^2}, \tag{B8}$$

which is independent of $\bar{\lambda}$ and monotonically decreasing. This behavior is identical to that given by (7.8).

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[1]There are an enormous number of papers relevant to this topic. For a recent review see S. Coleman, 1973 Erice lectures (unpublished).

[2]C. G. Callan, Phys. Rev. D 2, 1541 (1973); K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

[3]The idea of generating functionals for proper vertices originates with J. Schwinger, Proc. Natl. Acad. Sci. USA 37, 452 (1951); 37, 455 (1951). The path-integral

formulation of quantum theory is given by Feynman, in R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965). For reviews, see Ref. 1; H. M. Fried, *Functional Methods and Models in Quantum Field Theory* (MIT Press, Cambridge, Mass., 1972); B. Zumino, in *Lectures on Elementary Particles and Quantum Field Theory*, 1970 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1970).

[4]The explicit construction of the generating functional for the one-particle irreducible vertices is given by G. Jona-Lasinio [Nuovo Cimento 34, 1790 (1964)] and C. DeDominicis and P. C. Martin [J. Math. Phys. 5, 14 (1964)].

[5]The loop expansion for the 1PI Green's functions is discussed by B. C. DeWitt [*Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965), unpublished work, and private communication], S. Coleman and E. Weinberg [Phys. Rev. D 7, 1888 (1973)], B. W. Lee and J. Zinn-Justin [*ibid.* 5, 3121 (1972)], and R. Jackiw [*ibid.* 9, 1686 (1974)].

[6]The leading-logarithm approximation to the effective potential is obtained from renormalization-group methods by Coleman and Weinberg, Ref. 5, and from summation of diagrams by Jackiw, Ref. 5.

[7]Coleman and Weinberg, Ref. 5.

[8]We understand that D. Gross and A. Neveu [Phys. Rev. D (to be published)] have studies massless $(\bar{\psi}\psi)^2$ theory in the large-N limit in two dimensions. Their result may be relevant to this point.

[9]Although this is probably known to many people, we first learned it from T. F. Wong and G. S. Guralnik [Phys. Rev. D 3, 3028 (1971)], who used it to construct the 2-loop contribution to the propagator in the σ model.

[10]We learned this from S. Coleman and R. Jackiw (private communication).

[11]We wish to thank H.-S. Tsao for his aid in clarifying our ideas about the N^{-1} expansion.

[12]Equation (4.15) is not quite right for $\bar{\mu}^2 < 0$, since the denominator of the free propagator can vanish even for Euclidean momentum. The $i\epsilon$ prescription must be kept, and the potential develops an imaginary part. We have not presented the modifications required, since the reader will have no difficulty in incorporating this feature into our formalism, guided by Appendix B2 of Coleman and Weinberg, Ref. 5.

[13]T. T. Wu (private communication) suggests that these may not be genuine bounds. Rather, it is possible that the N^{-1} expansion is no longer valid for $0 > N\bar{\lambda} > -96\pi^2$. We have not been able to resolve the issue.

[14]An analysis of symmetry breaking is presented in Sec. V. A more detailed study is given by S. Coleman, R. Jackiw, and H. D. Politzer [Phys. Rev. D (to be published)]. The gap equation has been derived by L. Dolan and R. Jackiw [Phys. Rev. D 9, 3320 (1974)] in the context of finite-temperature problems. R. Jackiw (private communication) has also derived the gap equation in the zero-temperature case.

[15]For a review and references, see S.-k. Ma, Rev. Mod. Phys. 45, 589 (1973). A very economical derivation of the large-N limit by functional methods without the cumbersome analysis of Feynman graphs is given by H. J. Schnitzer, this issue, Phys. Rev. D 10, 2042 (1974).

[16]P. C. Martin (private communication).

[17]K. G. Wilson, Phys. Rev. D 7, 2911 (1973); G.'t Hooft, Nucl. Phys. (to be published); G. P. Canning, Bohr Institute report (unpublished).