[5]S. Coleman, Commun. Math. Phys. 31, 265 (1973).
[6]D. Boulware, Ann. Phys. (N.Y.) 56, 140 (1970).
[7]See, for example, E. S. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).
[8]G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972); C. G. Bollini, J. Giambiagi, and A. Gonzales Domingues, Nuovo Cimento 31, 550 (1964).

PHYSICAL REVIEW D VOLUME 10, NUMBER 6 15 SEPTEMBER 1974

Relativistic center-of-mass variables for composite systems with arbitrary internal interactions*

R. A. Krajcik
Physics Department, Case Western Reserve University, Cleveland, Ohio 44106 and Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544†

L. L. Foldy
Physics Department, Case Western Reserve University, Cleveland, Ohio 44106
(Received 26 October 1973; revised manuscript received 6 May 1974)

We show how a class of relativistic center-of-mass (c.m.) variables for a composite system with arbitrary internal interactions can be constructed to any order in $1/c^2$ by means of a nonsingular unitary transformation which arises from a study of the Lie algebra of the Poincaré group. The class of c.m. variables so constructed subsumes the c.m. variables previously obtained by means of the singular Gartenhaus-Schwartz transformation. We explicitly determine the c.m. variables to order $1/c^2$, and as an example consider both internal electromagnetic (EM) interactions, where simplifications are pointed out with regard to a previous study, and external EM interactions, where the complete form of the "correction" term to the Foldy-Wouthuysen EM interaction Hamiltonian is given. The results of some earlier studies of relativistic corrections to phenomenological potentials are also shown to be included in our results, and separability is briefly discussed.

I. INTRODUCTION

With the possible exception of those physical systems which are currently described as "elementary particles," most physical systems—molecules, atoms, nuclei, . . . —can profitably be considered to be composite systems formed from "more elementary particles." In this paper, we will consider only those composite systems for which the creation or destruction of the constituent particles, whether virtual or real, plays a minor role in the phenomenon under study. The composite system can then be regarded as composed of a fixed number of relatively stable particles. Dynamical variables for the composite system can then be introduced, and related to the Lie algebra of the invariance group through the expression of its infinitesimal generators in terms of these, where the otherwise arbitrary internal interactions of the composite system are now expressed as direct interactions. The composite system can then be studied in terms of the Lie algebra of the Galilean or Poincaré group.

The infinitesimal generators of the invariance group can be expressed in terms of the dynamical variables of the composite system in either of two ways. The first, and most familiar, is to express the generators in terms of the constituent particle variables[1] by which we will mean the set of variables $\{\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu\}$, where $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ are the position, momentum, and spin of the μth particle of the composite system. The second is to express the generators in terms of a (not necessarily unique) set of center-of-mass and internal variables which we shall collectively refer to as c.m. variables. These variables are defined in terms of the constituent particles such that the infinitesimal generators take a specific form in which they are related to the center-of-mass variables in the same way as for an irreducible representation. This form of the generators shall be referred to as the "single particle" form. What is interesting, and what we will be most concerned with, is that the above dual procedure of expressing the infinitesimal generators in terms of dynamical variables may, with advantage, be inverted. Instead of expressing the infinitesimal generators in terms of a given set of c.m. variables such that the Lie algebra of the invariance group is satisfied, the Lie algebra of the invariance group can be used to construct a class of c.m. variables which are defined in terms of the constituent particle variables. In this way, relativistic c.m. variables can be constructed.

The advantage of c.m. variables is that they have the effect of describing a composite system as if it were in fact a composite system; that is, as if it were a "single particle" whose behavior as a whole is separated from the details of its internal structure.[2] The description of composite systems in terms of c.m. variables has been particularly useful during the past few years in the study of soft-photon scattering from a bound system of particles. The study had been instigated by the question of an explicit demonstration of the low-energy theorem (LET) for Compton scattering[3] and the Drell-Hearn-Gerasimov (DHG) sum rule[4] for composite systems,[5] and an explicit treatment of c.m. variables, especially to order $1/c^2$, has been applied to loosely bound systems by a number of authors.[6-9] More recently, the incorporation of some special cases of internal interactions into composite systems to order $1/c^2$ has been considered.[10-12]

The purpose of this paper will now be set down. First, as the contents of the original articles concerned with the validity of the LET for Compton scattering and the DHG sum rule for composite systems[6-9] have been elaborated upon and extended in scope, confusion has arisen with regard to the following point: The "correction" to the Foldy-Wouthuysen (FW) electromagnetic (EM) interaction Hamiltonian as originally discussed[6-9] and rederived in later publications[11,12] *is not the complete "correction" term to order $1/c^2$*. It is important to realize that in the series of publications concerned with the validity of the LET for Compton scattering and the DHG sum rule, *only those terms which could contribute to the theorem and the rule were ever considered.* Furthermore, we have been informed that an oversight with respect to this point has recently occurred in the literature.[13] To preclude further confusion in the literature, the complete expression for the "correction" to the FW EM interaction Hamiltonian has been included in Sec. III of this paper.

Second, we develop a general procedure for systematically constructing a class of c.m. variables satisfying the conditions referred to above for a composite system with *arbitrary internal interactions* to any order in $1/c^2$. The procedure makes use of the Lie algebra of the Poincaré group, and leads to the definition of a unitary transformation alluded to in a previous publication.[8] The resulting c.m. variables are more general than those constructed by means of the Gartenhaus-Schwartz (GS) transformation,[14] and in contradistinction involves a nonsingular transformation. The relationship between the GS transformation and our procedure is discussed, and explicit results are given to order $1/c^2$.

Third, we wish to point out that some results previously derived by others with regard to c.m. variables in the presence of internal EM interactions[10] may in fact be had by inspection, and that the results of some earlier studies of relativistic corrections to phenomenological potentials[15,16] are included in our results as a special case. In this connection, separability[1] is briefly discussed.

II. DYNAMICAL VARIABLES

A. The Poincaré group

If the infinitesimal generators of the Poincaré group are denoted by $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$, and represent respectively the generator for infinitesimal space translations, space rotations, "velocity" transformations, and time translations, then the unitary transformation

$$e^{-i\vec{a}\cdot\vec{\mathcal{P}}-i\vec{\theta}\cdot\vec{\mathcal{J}}+i\vec{u}\cdot\vec{\mathcal{K}}+i\tau\mathcal{H}} \tag{2.1}$$

changes the state of the system in the active sense to a new one which is space-translated by $\vec{a}$, rotated by an angle θ about $\hat{\theta}=\vec{\theta}/\theta$, boosted by a "velocity" $u=c\tanh^{-1}(v/c)$ in the direction $\hat{v}=\vec{v}/v$, and advanced in time by τ, and the infinitesimal generators satisfy the Lie algebra

$$[\mathcal{P}_i,\mathcal{P}_j]=[\mathcal{P}_i,\mathcal{H}]=[\mathcal{J}_i,\mathcal{H}]=0\,, \tag{2.2a}$$

$$[\mathcal{J}_i,\mathcal{J}_j]=i\epsilon_{ijk}\mathcal{J}_k\,,\quad [\mathcal{J}_i,\mathcal{P}_j]=i\epsilon_{ijk}\mathcal{P}_k\,,\quad [\mathcal{J}_i,\mathcal{K}_j]=i\epsilon_{ijk}\mathcal{K}_k\,, \tag{2.2b}$$

$$[\mathcal{K}_i,\mathcal{P}_j]=i\delta_{ij}\mathcal{H}/c^2,\quad [\mathcal{K}_i,\mathcal{K}_j]=-i\epsilon_{ijk}\mathcal{J}_k/c^2\,,\quad [\mathcal{K}_i,\mathcal{H}]=i\mathcal{P}_i\,, \tag{2.2c}$$

where the units are such that $\hbar=1$. In terms of the constituent particle variables $\{\vec{r}_\mu,\vec{p}_\mu,\vec{s}_\mu\}$, where $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ represent the position, momentum, and spin of the μth particle of the composite system, and in the presence of arbitrary internal interactions, the infinitesimal generators of the Poincaré group may be expressed as follows[1]:

$$\vec{\mathcal{P}}=\sum_\mu \vec{p}_\mu\,, \tag{2.3a}$$

$$\vec{\mathcal{J}}=\sum_\mu(\vec{r}_\mu\times\vec{p}_\mu+\vec{s}_\mu), \tag{2.3b}$$

$$\vec{\mathcal{K}}=\sum_\mu\vec{K}_\mu+\beta\vec{V}\,,\quad \vec{K}_\mu=\frac{1}{2c^2}(\vec{r}_\mu H_\mu+H_\mu\vec{r}_\mu)-\frac{\vec{s}_\mu\times\vec{p}_\mu}{H_\mu+m_\mu c^2}-t\vec{p}_\mu\,, \tag{2.3c}$$

$$\mathcal{H}=\sum_\mu H_\mu+\beta U,\quad H_\mu=(p_\mu{}^2c^2+m_\mu{}^2c^4)^{1/2}\,, \tag{2.3d}$$

where the constituent particle variables are taken to satisfy the familiar free-particle commutation relations

$$[r_\mu^i, r_\nu^j]=[p_\mu^i, p_\nu^j]=[s_\mu^i, r_\nu^j]=[s_\mu^i, p_\nu^j]=0 , \tag{2.4a}$$

$$[r_\mu^i, p_\nu^j]=i\delta_{\mu\nu}\delta_{ij} , \tag{2.4b}$$

$$[s_\mu^i, s_\nu^j]=i\delta_{\mu\nu}\epsilon_{ijk}s_\nu^k , \tag{2.4c}$$

U is an internal interaction potential, $\vec{V}$ is the corresponding "interaction boost," and β is simply a parameter, here introduced to distinguish the contribution due to the presence of internal interaction, which allows us to "turn off" the internal interaction by letting $\beta\to 0$. Note that $\vec{\mathcal{P}}$ and $\vec{\mathcal{J}}$ retain their free-particle forms, so that from Eq. (2.2c) $\vec{\mathcal{K}}$ must change if $\mathcal{H}$ does. Furthermore, because the commutation relations of the constituent particle variables remain unchanged in the presence of internal interaction, the Lie algebra restricts the form of U and $\vec{V}$. Equation (2.2a) implies that U is a rotationally and translationally invariant function,

$$[U,\vec{\mathcal{P}}]=[U,\vec{\mathcal{J}}]=0 , \tag{2.5a}$$

Eq. (2.2b) that $\vec{V}$ is a vector under rotations,

$$[\mathcal{J}_i, V_j]=i\epsilon_{ijk}V_k , \tag{2.5b}$$

and Eq. (2.2c) that U and $\vec{V}$ are related by

$$[V_i, \mathcal{P}_j]=i\delta_{ij}U/c^2 , \tag{2.5c}$$

and $\vec{V}$ satisfies

$$\left[V_i, \sum_\mu K_\mu^j\right]-(i\leftrightarrow j)+\beta[V_i, V_j]=0 , \tag{2.5d}$$

where $(i\leftrightarrow j)$ means interchange i and j of the previous term. These are the only independent conditions on U and $\vec{V}$. The question of independent commutation relations of the Poincaré group is discussed in Sec. II C.

The infinitesimal generators of the Poincaré group can also be expressed in "single particle" form, by which we will mean the following[1,2]:

$$\vec{\mathcal{P}}=\vec{P} , \tag{2.6a}$$

$$\vec{\mathcal{J}}=\vec{R}\times\vec{P}+\vec{S} , \tag{2.6b}$$

$$\vec{\mathcal{K}}=\frac{1}{2c^2}(\vec{R}\mathcal{H}+\mathcal{H}\vec{R})-\frac{\vec{S}\times\vec{P}}{\mathcal{H}+h(\beta)}-t\vec{P} , \tag{2.6c}$$

$$\mathcal{H}=[P^2c^2+h^2(\beta)]^{1/2} . \tag{2.6d}$$

These expressions serve to define $\vec{R}$, $\vec{P}$, $\vec{S}$, and h, the center-of-mass position, momentum, spin (internal angular momentum), and mass (internal Hamiltonian) of the composite system in terms of the infinitesimal generators of the Poincaré group. Or, given Eq. (2.3), the above expressions serve to define $\vec{R}$, $\vec{P}$, $\vec{S}$, and h in terms of the constituent particle variables $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$. In terms of the infinitesimal generators of the Poincaré group, the Lie algebra implies the following commutation relations among $\vec{R}$, $\vec{P}$, $\vec{S}$, and h, and conversely:

$$[R_i, R_j]=[P_i, P_j]=[S_i, R_j]=[S_i, P_j]=0 , \quad [h, R_j]=[h, P_j]=[h, S_j]=0 , \tag{2.7a}$$

$$[R_i, P_j]=i\delta_{ij} , \tag{2.7b}$$

$$[S_i, S_j]=i\epsilon_{ijk}S_k . \tag{2.7c}$$

In Sec. II C, it will be shown how the remaining $3N-2$ independent (3-component, internal) variables of an N-particle composite system can be constructed in the presence of arbitrary internal interactions to any order in $1/c^2$.

B. The Galilean group

The Lie algebra of the Galilean group can be obtained from the Lie algebra of the Poincaré group by contracting the latter group with respect to speed and space.[17] This may be accomplished by allowing the speed of light c to increase without bound wherever it occurs explicitly in the Lie algebra of the Poincaré group as given by Eq. (2.2). Care must be taken, however, with respect to the generator $\mathcal{H}$. In the contraction, it is assumed that $\mathcal{H}$ is expandable in powers of $1/c^2$, where the leading term is the total rest energy of the system. With this understanding, the Lie algebra of the Galilean group follows directly from Eq. (2.2).

$$[\mathcal{P}_i, \mathcal{P}_j]=[\mathcal{P}_i, \mathcal{H}]=[\mathcal{J}_i, \mathcal{H}]=0 , \tag{2.8a}$$

$$[\mathcal{J}_i, \mathcal{J}_j]=i\epsilon_{ijk}\mathcal{J}_k, \quad [\mathcal{J}_i, \mathcal{P}_j]=i\epsilon_{ijk}\mathcal{P}_k , \quad [\mathcal{J}_i, \mathcal{K}_j]=i\epsilon_{ijk}\mathcal{K}_k , \tag{2.8b}$$

$$[\mathcal{K}_i, \mathcal{P}_j]=i\delta_{ij}M, \quad [\mathcal{K}_i, \mathcal{K}_j]=0 , \quad [\mathcal{K}_i, \mathcal{H}]=i\mathcal{P}_i , \tag{2.8c}$$

where $M=\sum_\mu m_\mu$, and m_μ is rest mass of the μth particle of the composite system. Although the Lie algebra of the Galilean group is the limiting case of that of the Poincaré group in the sense described above, it is significantly different in structure. Note in particular that the Lie algebra of the Galilean group does not close on itself, but rather contains an element which is neither zero nor a homogeneous, linear function of the infinitesimal generators of the group. The presence of the constant M in the Lie algebra of the Galilean group implies additional complications not associated with the Poincaré group, most notable of which is the Bargmann superselection rule which forbids the creation or destruction of matter.[18] In spite of the additional complications which arise, the presence of the constant M is necessary if the Lie algebra is to correspond to physical particles,[19] so that not only does the Lie algebra that would have resulted had $\mathcal{H}$ not been expanded in powers of $1/c^2$ have a nontrivial central extension, that is replacing zero with a constant times the identity operator, but it is precisely this extension which corresponds to physical particles.

If the generator $\vec{\mathcal{K}}$ is also assumed to be expandable in powers of $1/c^2$, then, in terms of the constituent particle variables, the infinitesimal generators of the Galilean group can be obtained by contracting those of the Poincaré group with respect to the speed of light c. From Eq. (2.3),

$$\vec{\mathcal{P}}=\sum_\mu \vec{p}_\mu, \tag{2.9a}$$

$$\vec{\mathcal{J}}=\sum_\mu(\vec{r}_\mu \times \vec{p}_\mu+\vec{s}_\mu), \tag{2.9b}$$

$$\vec{\mathcal{K}}=\sum_\mu(m_\mu\vec{r}_\mu - t\vec{p}_\mu)+\beta\vec{V}^{(0)}, \tag{2.9c}$$

$$\mathcal{H}=\sum_\mu\left(m_\mu c^2+\frac{\vec{p}_\mu^{\,2}}{2m_\mu}\right)+\beta U^{(0)}, \tag{2.9d}$$

where the superscript indicates orders in $1/c^2$. The "interaction boost" $\vec{V}^{(0)}$ may be set equal to zero without loss of generality by a procedure discussed in Ref. 1, and for expediency this choice will be made in this paper. The conditions on $U^{(0)}$ are then that it be a rotationally and translationally invariant function such that $[\sum_\mu m_\mu \vec{r}_\mu, U^{(0)}]=0$.

By analogy with classical mechanics, we introduce a set of nonrelativistic c.m. variables through the following definitions[1]:

$$M=\sum_\mu m_\mu, \quad \vec{P}=\sum_\mu \vec{p}_\mu, \quad \vec{R}=\sum_\mu \frac{m_\mu \vec{r}_\mu}{M}, \tag{2.10a}$$

$$\vec{r}_\mu=\vec{\rho}_\mu+\vec{R}, \quad \vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}, \quad \vec{s}_\mu=\vec{\sigma}_\mu, \tag{2.10b}$$

$$\vec{S}=\sum_\mu(\vec{\rho}_\mu\times\vec{\pi}_\mu+\vec{\sigma}_\mu), \tag{2.10c}$$

where $\vec{R}$ and $\vec{P}$ are the c.m. position and momentum, $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ are the internal position, momentum, and spin associated with particle μ, $\vec{S}$ is the total internal spin of the composite system, and the $\vec{\rho}_\mu$ and $\vec{\pi}_\mu$ satisfy the following subsidiary conditions:

$$\sum_\mu m_\mu \vec{\rho}_\mu=0, \tag{2.11a}$$

$$\sum_\mu \vec{\pi}_\mu=0. \tag{2.11b}$$

The c.m. variables so introduced have rather special properties. In terms of these c.m. variables, the infinitesimal generators of the Galilean group assume a "single particle" form. From Eq. (2.9),

$$\vec{\mathcal{P}}=\vec{P}, \tag{2.12a}$$

$$\vec{\mathcal{J}}=\vec{R}\times\vec{P}+\vec{S}, \tag{2.12b}$$

$$\vec{\mathcal{K}}=M\vec{R}-t\vec{P}, \tag{2.12c}$$

$$\mathcal{H}=h(\beta)+\frac{\vec{P}^2}{2M}, \tag{2.12d}$$

where h, a rotationally and translationally invariant function of internal c.m. variables only, is the internal Hamiltonian of the composite system

$$h(\beta)=Mc^2+h^{(0)}(\beta), \qquad h^{(0)}(\beta)=\sum_\mu\frac{\vec{\pi}_\mu^{\,2}}{2m_\mu}+\beta U^{(0)}. \tag{2.12e}$$

Furthermore, the commutation relations satisfied by these c.m. variables are consistent with those given by Eq. (2.7) in the case of the Poincaré group

$$[R_i,\{\vec{\rho}_\mu,\vec{\pi}_\mu,\vec{\sigma}_\mu\}]=[P_i,\{\vec{\rho}_\mu,\vec{\pi}_\mu,\vec{\sigma}_\mu\}]=0, \tag{2.13a}$$

$$[R_i,R_j]=[P_i,P_j]=[\rho_\mu^i,\rho_\nu^j]=[\pi_\mu^i,\pi_\nu^j]=0, \tag{2.13b}$$

$$[S_i,R_j]=[S_i,P_j]=[h,R_j]=[h,P_j]=[h,S_j]=0, \tag{2.13c}$$

$$[R_i,P_j]=i\delta_{ij}, \quad [\rho_\mu^i,\pi_\nu^j]=i(\delta_{\mu\nu}-m_\nu/M)\delta_{ij}, \tag{2.13d}$$

$$[S_i,S_j]=i\epsilon_{ijk}S_k, \quad [S_i,\rho_\mu^j]=i\epsilon_{ijk}\rho_\mu^k, \quad [S_i,\pi_\mu^j]=i\epsilon_{ijk}\pi_\mu^k, \quad [S_i,\sigma_\mu^j]=i\epsilon_{ijk}\sigma_\mu^k, \quad [\sigma_\mu^i,\sigma_\nu^j]=i\delta_{\mu\nu}\epsilon_{ijk}\sigma_\mu^k. \tag{2.13e}$$

It follows that the infinitesimal generators as defined in Eqs. (2.12) in terms of these c.m. variables must satisfy the Lie algebra of the Galilean group.

C. Relativistic c.m. variables

Using the Lie algebra of the Poincaré group, we will show how one can construct relativistic c.m. variables to any order in $1/c^2$ for a composite system with arbitrary internal interactions. Our criteria for constructing these variables will be the following. Expressed in terms of the relativistic c.m. variables, we require that (1) the infinitesimal generators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ satisfy the Lie algebra of the Poincaré group as given by Eq. (2.2), (2) the infinitesimal generators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ have the single-particle form as given by Eq. (2.6), and (3) the infinitesimal generators $\vec{\mathcal{P}}$ and $\vec{\mathcal{J}}$ retain their nonrelativistic forms even in the presence of arbitrary internal interactions as given by Eqs. (2.12a), (2.12b), and (2.10c). The relativistic c.m. variables constructed on the basis of the above criteria are not unique. We will find that the above criteria and our assumption that the generators $\vec{\mathcal{K}}$ and $\mathcal{H}$ are expandable in powers of $1/c^2$ in the presence of internal interactions lead to a whole class of relativistic c.m. variables. (If $\vec{\mathcal{K}}$ and $\mathcal{H}$ are analytic functions of $1/c^2$ in the presence of internal interaction, i.e., expandable in powers of $1/c^2$, then there are no other solutions.) Furthermore, there may exist relativistic c.m. variables which satisfy our criteria, but the generators $\vec{\mathcal{K}}$ and $\mathcal{H}$ are not expandable in powers of $1/c^2$ in the presence of internal interactions. This class of relativistic c.m. variables is not considered in this paper.

We begin by observing that the nonrelativistic c.m. variables, as defined in the previous section, satisfy the criteria above with respect to the Galilean (rather than the Poincaré) group. With sufficient insight, one could presumably define a new set of c.m. variables to some higher order in $1/c^2$ which would satisfy the above criteria to the corresponding order with respect to the Poincaré group. One may approach the problem, however, in a more systematic way. On the basis of previous work,[1] one may ask whether this new set of c.m. variables, defined in terms of the constituent particle variables to some higher order in $1/c^2$, can be constructed from the nonrelativistic c.m. variables, defined in the last section, by means of a unitary transformation. The answer is yes, and provides insight and a means of constructing this class of relativistic c.m. variables.

Consider the Lie algebra of the Poincaré group as given in Eq. (2.2). Of the nine commutation relations between $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$, only six are independent:

$$[\mathcal{P}_i, \mathcal{P}_j]=0 , \tag{2.14a}$$

$$[\mathcal{J}_i, \mathcal{J}_j]=i\epsilon_{ijk}\mathcal{J}_k , \quad [\mathcal{J}_i, \mathcal{P}_j]=i\epsilon_{ijk}\mathcal{P}_k , \quad [\mathcal{J}_i, \mathcal{K}_j]=i\epsilon_{ijk}\mathcal{K}_k , \tag{2.14b}$$

$$[\mathcal{K}_i, \mathcal{P}_j]=i\delta_{ij}\mathcal{H}/c^2 , \quad [\mathcal{K}_i, \mathcal{K}_j]=-i\epsilon_{ijk}\mathcal{J}_k/c^2 . \tag{2.14c}$$

As shown in Appendix A, the remaining three commutators involving $\mathcal{H}$ can be derived from five others. Of those commutators above, the last two, Eqs. (2.14c), are the most important. The first of these shows that of the four generators, $\mathcal{H}$ plays a minor role with respect to the independent commutation relations above, in the sense that it is embodied in $\vec{\mathcal{K}}$ and "projected out" by $\vec{\mathcal{P}}$. This will imply that the relativistic c.m. variables constructed by means of a unitary transformation will satisfy our criteria with respect to the generator $\mathcal{H}$ if it satisfies it with respect to the generator $\vec{\mathcal{K}}$. From the second commutator, an explicit manifestation of the Wigner rotation so closely associated with relativistic kinematics, we will show that the proposed unitary transformation exists for all orders in $1/c^2$.

We now make one other observation. Consider the infinitesimal generators of the Poincaré group as given in Eqs. (2.3). Let us write any one of these as $G_\alpha(\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu)$, $\alpha=1, 2, \ldots, 10$. If we substitute in each of these for the operators $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ in terms of the *Galilean c.m. variables* $\vec{R}$, $\vec{P}$, $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ as given in Eqs. (2.10b), we now have for the generators

$$G_\alpha(\vec{r}_\mu=\vec{\rho}_\mu+\vec{R},\ \vec{p}_\mu=\vec{\pi}_\mu+m_\mu\vec{P}/M,\ \vec{s}_\mu=\vec{\sigma}_\mu) ,$$

which, by virtue of the commutation relations (2.13) and subsidiary conditions (2.11) satisfied by these variables, will continue to satisfy the Lie algebra of the Poincaré group. In terms of these variables, however, G_α will not have the single-particle form of Eqs. (2.6). Our objective will be to bring them to this form by means of a unitary transformation.

Suppose there exists a Hermitian operator $\phi=\phi(\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu)$ such that

$$G_\alpha(\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu) = e^{i\phi}G_\alpha\left(\vec{r}_\mu=\vec{\rho}_\mu+\vec{R}, \vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}, \vec{s}_\mu=\vec{\sigma}_\mu\right)e^{-i\phi} , \tag{2.15a}$$

where G_α is any one of the infinitesimal generators of the Poincaré group expressed in single-particle form, as given by Eq. (2.6), on the left-hand side of Eq. (2.15a), and in terms of constituent particle variables, as given by Eq. (2.3), on the right-hand side, and $\vec{R}$, $\vec{P}$, $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ satisfy Eqs. (2.13) and (2.11). The essence of Eq. (2.15a) is the statement that if $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ are replaced by the *relativistic* c.m. variables defined by the equations

$$\vec{r}_\mu=e^{i\phi}(\vec{\rho}_\mu+\vec{R})e^{-i\phi} , \tag{2.15b}$$

$$\vec{p}_\mu=e^{i\phi}\left(\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}\right)e^{-i\phi} , \tag{2.15c}$$

$$\vec{s}_\mu=e^{i\phi}\vec{\sigma}_\mu e^{-i\phi} , \tag{2.15d}$$

in Eq. (2.3), then the infinitesimal generators of the Poincaré group $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ would satisfy condition (2) of the criteria set down at the beginning of this subsection, or

$$G(\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu)=G(\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu) , \tag{2.15e}$$

where $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ are given by Eqs. (2.15b)–(2.15d). Because the transformation is unitary, the commutation relations of $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$

among themselves are still given by Eq. (2.4), since $\vec{R}$, $\vec{P}$, $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ satisfy Eqs. (2.13) and (2.11). This implies that $\vec{\mathcal{P}}$, $\vec{\mathfrak{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ satisfy the Lie algebra of the Poincaré group, so that condition 1 is satisfied.

The interpretation of ϕ as it occurs in Eqs. (2.15a)–(2.15c), is to be as follows. The function ϕ defines the possible *relationships* between objects *whose abstract meaning remains the same,* $\{\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu\}$ and $\{\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu\}$, so that $\vec{p}_\mu$ and $\vec{\pi}_\mu$, for example, always represent the momentum and internal momentum of the μth particle, respectively. The *relationship* between the two, however, is determined by ϕ. Assuming $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ to be *given,* the $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ can be *defined* in terms of these by determining ϕ, performing the unitary transformation, and then inverting the results. The form of ϕ will now be determined.

From condition (3), ϕ must satisfy the equations

$$\vec{\mathcal{P}} = e^{i\phi}\vec{\mathcal{P}}e^{-i\phi}, \tag{2.16a}$$

$$\vec{\mathfrak{J}} = e^{i\phi}\vec{\mathfrak{J}}e^{-i\phi}, \tag{2.16b}$$

so that $\phi = \phi(\vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu)$ is a rotationally and translationally invariant function. Next consider the generator $\vec{\mathcal{K}}$. From Eq. (2.15a) we have

$$\vec{\mathcal{K}}(\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu) = e^{i\phi}\vec{\mathcal{K}}\left(\vec{r}_\mu = \vec{\rho}_\mu + \vec{R},\ \vec{p}_\mu = \vec{\pi}_\mu + \frac{m_\mu}{M}\vec{P},\ \vec{s}_\mu = \vec{\sigma}_\mu\right)e^{-i\phi}. \tag{2.16c}$$

Because the generator $\vec{\mathcal{K}}$, in particular, satisfies the Lie algebra of the Poincaré group [condition (1)], and ϕ commutes with the generator $\vec{\mathcal{P}}$ [condition (3)], the first commutation relation of Eq. (2.14c) implies that the generator $\mathcal{H}$ satisfies Eq. (2.15a) if $\vec{\mathcal{K}}$ does, so that Eq. (2.16c) is sufficient to determine ϕ. For convenience, Eq. (2.16c) will be rewritten as follows, but it will mean the same thing:

$$\vec{\kappa} = e^{i\phi}\vec{\mathfrak{k}}e^{-i\phi}. \tag{2.16d}$$

We will now show that Eq. (2.16d) can always be solved for ϕ, provided that ϕ can be expanded in a power series in $1/c^2$. This is an important result. It shows that a unitary, nonsingular equivalent of the Gartenhaus-Schwartz transformation[14] exists, and can, in principle, be determined to any order in $1/c^2$ for arbitrary internal interactions of a composite system. The proof is as follows, and makes use of a technique found useful in molecular physics.[20]

Let $\xi = e^{-i\phi}$; then with Eq. (2.16d) written as

$$\xi\vec{\kappa} = \vec{\mathfrak{k}}\xi \tag{2.16e}$$

ξ, $\vec{\kappa}$, and $\vec{\mathfrak{k}}$ may be expanded in powers of $1/c^2$,

$$\xi = I + \sum_{n=1}\frac{\xi^{(n)}}{c^{2n}}, \tag{2.17a}$$

$$\vec{\kappa} = \vec{\kappa}^{(0)} + \sum_{n=1}\frac{\vec{\kappa}^{(n)}}{c^{2n}}, \tag{2.17b}$$

$$\vec{\mathfrak{k}} = \vec{\mathfrak{k}}^{(0)} + \sum_{n=1}\frac{\vec{\mathfrak{k}}^{(n)}}{c^{2n}}, \tag{2.17c}$$

and substituted into Eq. (2.16e). Collecting terms of like order in $1/c^2$,

$$\vec{\kappa}^{(0)} = \vec{\mathfrak{k}}^{(0)}, \tag{2.18a}$$

$$\vec{\kappa}^{(1)} = \vec{\mathfrak{k}}^{(1)} + [\vec{\kappa}^{(0)}, \xi^{(1)}], \tag{2.18b}$$

$$\vec{\kappa}^{(2)} = \vec{\mathfrak{k}}^{(2)} + [\vec{\kappa}^{(0)}, \xi^{(2)}] + \vec{\mathfrak{k}}^{(1)}\xi^{(1)} - \xi^{(1)}\vec{\kappa}^{(1)}, \tag{2.18c}$$

$$\cdots,$$

$$\vec{\kappa}^{(n)} = \vec{\mathfrak{k}}^{(n)} + [\vec{\kappa}^{(0)}, \xi^{(n)}] + \sum_{\alpha=1}^{n-1}(\vec{\mathfrak{k}}^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\vec{\kappa}^{(n-\alpha)}), \tag{2.18d}$$

$$\cdots.$$

Note that Eq. (2.18a) corresponds to the Galilean case discussed in the previous subsection. Furthermore, from Eq. (2.12c), $\vec{\kappa}^{(0)} = M\vec{R} - t\vec{P}$, which together with Eq. (2.16a) implies that $[\vec{\kappa}^{(0)}, \xi^{(n)}]$ is just the $\vec{P}$ gradient of $\xi^{(n)}$ times iM, or

$$M[\vec{R}, \xi^{(n)}] = \vec{\kappa}^{(n)} - \vec{\mathfrak{k}}^{(n)} - \sum_{\alpha=1}^{n-1}(\vec{\mathfrak{k}}^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\vec{\kappa}^{(n-\alpha)}). \tag{2.19a}$$

This equation has the solution[21]

$$\xi^{(n)} = \frac{-i}{M}\int_0^{\vec{P}} d\vec{P}\cdot\left[\vec{\kappa}^{(n)} - \vec{\mathfrak{k}}^{(n)} - \sum_{\alpha=1}^{n-1}(\vec{\mathfrak{k}}^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\vec{\kappa}^{(n-\alpha)})\right] - i\Pi^{(n)}, \tag{2.19b}$$

where the $\Pi^{(n)}$ are arbitrary functions of internal c.m. variables only, provided the $\vec{P}$ curl of the right-hand side of Eq. (2.19a) vanishes. This is equivalent to the statement

$$\left[\kappa_i^{(0)}, \kappa_j^{(n)} - k_j^{(n)} - \sum_{\alpha=1}^{n-1}(k_j^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\kappa_j^{(n-\alpha)})\right] - (i \leftrightarrow j) = 0. \tag{2.20}$$

The demonstration that Eq. (2.20) is satisfied for all n is given in Appendix B. It requires only that the second commutation relation of Eq. (2.14c) be satisfied, or more generally, that condition (1) be satisfied.

Because the $\phi^{(n)}$ are related to the $\xi^{(n)}$ in an increasingly complicated but direct fashion,

$$\sum_{n=1} \frac{\xi^{(n)}}{c^{2n}} = \sum_{n=1} \frac{(-i)^n}{n!}\left[\sum_{m=1} \frac{\phi^{(m)}}{c^{2m}}\right]^n \; ; \tag{2.21}$$

that is,

$$\xi^{(1)} = -i\phi^{(1)} \, , \tag{2.21a}$$

$$\xi^{(2)} = -i\phi^{(2)} + \frac{i^2}{2!}[\phi^{(1)}]^2 \, , \tag{2.21b}$$

$$\xi^{(3)} = -i\phi^{(3)} + \frac{i^2}{2!}\{\phi^{(1)}, \phi^{(2)}\} - \frac{i^3}{3!}[\phi^{(1)}]^3 \, , \tag{2.21c}$$

$$\cdots ,$$

where $\{\;,\;\}$ is an anticommutator, or conversely,

$$\phi^{(1)} = i\xi^{(1)} \, , \tag{2.22a}$$

$$\phi^{(2)} = i\xi^{(2)} - \frac{i}{2!}[\xi^{(1)}]^2 \, , \tag{2.22b}$$

$$\phi^{(3)} = i\xi^{(3)} - \frac{i}{2!}\{\xi^{(1)}, \xi^{(2)}\} + \frac{i}{2!}[\xi^{(1)}]^3 - \frac{i}{3!}[\xi^{(1)}]^3 \, , \tag{2.22c}$$

$$\cdots ,$$

the $\phi^{(n)}$ are also shown to exist for all n, and the resultant ξ is unitary. The $\phi^{(n)}$ could equally well be determined, once their existence had been established, by expanding Eq. (2.16d) directly in $1/c^2$.

It is well to keep in mind at this point that the Lie algebra of the infinitesimal generators $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ which led to the results above is only satisfied, with respect to the constituent particle variables, if the conditions upon U and $\vec{V}$ as given by Eq. (2.5) are satisfied. In particular, it will be expedient to solve Eq. (2.5c) for

$$\vec{V} = \frac{1}{2c^2}(\vec{R}U + U\vec{R}) + \vec{W} \, , \tag{2.23a}$$

where $\vec{W}$ is a translationally invariant vector function,

$$[\mathcal{P}_i, W_j] = 0 \, , \tag{2.23b}$$

$$[\mathcal{J}_i, W_j] = i\epsilon_{ijk} W_k \, , \tag{2.23c}$$

which is related to U through Eq. (2.5d).

It is also important to note that the class of c.m. variables defined by ξ (or ϕ) is parametrized by the two functions Π and $\beta\vec{V}$. It is by virtue of the arbitrary functions $\Pi^{(n)}$ that the c.m. variables generated by this procedure will be more general than those generated by the Gartenhaus-Schwartz transformation.[14] In the next section, $\phi^{(1)}$ and the corresponding c.m. variables are explicitly constructed and a comparison is made.

D. c.m. variables to order $1/c^2$ and comparison

We begin with Eq. (2.18b), Eq. (2.18a) being the Galilean case. Its solution for $\xi^{(1)}$, or from Eq. (2.21a) $\phi^{(1)}$, is given by Eq. (2.19b),

$$-i\phi^{(1)} = \xi^{(1)} = \frac{-i}{M}\int_0^{\vec{P}} d\vec{P}\cdot(\vec{\kappa}^{(1)} - \vec{K}^{(1)}) - i\Pi^{(1)} \, . \tag{2.24}$$

Explicitly expressing $\vec{\kappa}^{(1)}$ and $\vec{K}^{(1)}$ in c.m. variables according to the convention of Eq. (2.16c) immediately yields $\phi^{(1)}$. From Eqs. (2.6c) and (2.3c), respectively,

$$\vec{\mathcal{K}}(\vec{R}, \vec{P}, \vec{\rho}_\mu, \vec{\pi}_\mu, \vec{\sigma}_\mu) = \frac{1}{2c^2}\left\{\vec{R}\left[Mc^2 + h^{(0)}(\beta) + \frac{\vec{P}^2}{2M} + \cdots\right] + \text{H.c.}\right\} - \frac{\vec{S}\times\vec{P}}{(2Mc^2 + \cdots)} - t\vec{P} \, ,$$

so that

$$\vec{\kappa}^{(1)} = \frac{1}{2}\left\{\vec{R}\left[h^{(0)}(\beta) + \frac{\vec{P}^2}{2M}\right] + \text{H.c.}\right\} - \frac{\vec{S}\times\vec{P}}{2M} \, , \tag{2.25a}$$

where $h^{(0)}$ is given by Eq. (2.12e), and

$$\vec{\mathcal{K}}\left(\vec{r}_\mu = \vec{\rho}_\mu + \vec{R},\; \vec{p}_\mu = \vec{\pi}_\mu + \frac{m_\mu}{M}\vec{P},\; \vec{s}_\mu = \vec{\sigma}_\mu\right) = \sum_\mu \left\{\frac{1}{2c^2}\left[(\vec{\rho}_\mu + \vec{R})\left(m_\mu c^2 + \frac{[\vec{\pi}_\mu + (m_\mu/M)\vec{P}]^2}{2m_\mu} + \cdots\right) + \text{H.c.}\right] - \frac{\vec{\sigma}_\mu \times [\vec{\pi}_\mu + (m_\mu/M)\vec{P}]}{(2m_\mu c^2 + \cdots)} - t\left(\vec{\pi}_\mu + \frac{m_\mu}{M}\vec{P}\right)\right\} + \frac{\beta\vec{V}^{(1)}}{c^2} + \cdots ,$$

so that

$$\vec{\mathbf{K}}^{(1)}=\sum_\mu\left\{\frac{1}{2}\left[\left(\vec{\rho}_\mu\frac{\vec{\pi}_\mu{}^2}{2m_\mu}+\frac{\vec{\rho}_\mu\vec{\pi}_\mu\cdot\vec{\mathbf{P}}}{M}+\frac{\vec{\mathbf{R}}\vec{\pi}_\mu{}^2}{2m_\mu}+\frac{m_\mu}{M}\frac{\vec{\mathbf{R}}\vec{\mathbf{P}}^2}{2M}\right)+\text{H.c.}\right]-\frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu}{2m_\mu}-\frac{\vec{\sigma}_\mu\times\vec{\mathbf{P}}}{2M}\right\}+\beta\vec{\mathbf{V}}^{(1)}\ , \tag{2.25b}$$

where

$$\vec{\mathbf{V}}^{(1)}=\vec{\mathbf{R}}U^{(0)}+\vec{\mathbf{W}}^{(1)}, \tag{2.25c}$$

and Eq. (2.11) has been used. Consequently, with the aid of the Eq. (2.25c),

$$\phi^{(1)}=\frac{1}{M}\int_0^{\vec{\mathbf{P}}}d\vec{\mathbf{P}}\cdot\left\{-\beta\vec{\mathbf{W}}^{(1)}-\tfrac{1}{2}\sum_\mu\left[\left(\vec{\rho}_\mu\frac{\vec{\pi}_\mu{}^2}{2m_\mu}+\frac{\vec{\rho}_\mu\cdot\vec{\mathbf{P}}\vec{\pi}_\mu}{2M}+\frac{\vec{\rho}_\mu\vec{\pi}_\mu\cdot\vec{\mathbf{P}}}{2M}\right)+\text{H.c.}\right]+\sum_\mu\frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu}{2m_\mu}\right\}+\Pi^{(1)}. \tag{2.26a}$$

This may be easily integrated to yield the final result:

$$\phi^{(1)}=-\tfrac{1}{2}\sum_\mu\left(\frac{\vec{\rho}_\mu\cdot\vec{\mathbf{P}}\vec{\pi}_\mu\cdot\vec{\mathbf{P}}}{2M^2}+\text{H.c.}\right)-\tfrac{1}{2}\sum_\mu\left(\frac{\vec{\rho}_\mu\cdot\vec{\mathbf{P}}\vec{\pi}_\mu{}^2}{2m_\mu M}+\text{H.c.}\right)+\sum_\mu\frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu\cdot\vec{\mathbf{P}}}{2m_\mu M}+\omega_\pi^{(1)}-\frac{\beta}{M}\int_0^{\vec{\mathbf{P}}}d\vec{\mathbf{P}}\cdot\vec{\mathbf{W}}^{(1)}+\beta\zeta_\pi^{(1)}\ , \tag{2.26b}$$

$$\Pi^{(1)}=\omega_\pi^{(1)}+\beta\zeta_\pi^{(1)}\ ,$$

where $\Pi^{(1)}$ is an arbitrary function of internal c.m. variables only. Other than the function $\Pi^{(1)}$, which indicates an arbitrariness in the choice of the c.m. variables even in the absence of internal interactions, $\beta\to 0$, and $\vec{\mathbf{W}}^{(1)}$, which contains the contribution to the c.m. variables specifically due to internal interactions, $\phi^{(1)}$ is the Hermitian operator discussed earlier with respect to the LET for Compton scattering and the DHG sum rule.[8]

The relationship between the constituent particle variables and the c.m. variables to order $1/c^2$ is easily determined from Eqs. (2.15b) through (2.15d). The results are

$$\begin{aligned}\vec{\mathbf{r}}_\mu=\vec{\rho}_\mu+\vec{\mathbf{R}}&-\frac{1}{2c^2}\left[\vec{\rho}_\mu\cdot\frac{\vec{\mathbf{P}}}{M}\left(\frac{\vec{\pi}_\mu}{m_\mu}+\frac{\vec{\mathbf{P}}}{2M}\right)+\text{H.c.}\right]-\frac{1}{2c^2}\sum_\nu\left(\frac{\vec{\pi}_\nu{}^2\vec{\rho}_\nu}{2m_\nu M}+\text{H.c.}\right)\\&+\sum_\nu\frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu)\times\vec{\mathbf{P}}}{2M^2c^2}-\frac{\vec{\sigma}_\mu\times\vec{\mathbf{P}}}{2m_\mu Mc^2}+\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu}{2m_\nu Mc^2}+\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{\mathbf{P}}}{2M^2c^2}-\frac{\beta}{Mc^2}\vec{\mathbf{W}}^{(1)}\\&+\frac{i}{c^2}\left[\omega_\pi^{(1)}-\frac{\beta}{M}\int_0^{\vec{\mathbf{P}}}d\vec{\mathbf{P}}\cdot\vec{\mathbf{W}}^{(1)}+\beta\zeta_\pi^{(1)},\vec{\rho}_\mu\right],\end{aligned} \tag{2.27a}$$

$$\vec{\mathbf{p}}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{\mathbf{P}}+\left(\frac{\vec{\pi}_\mu{}^2}{2m_\mu}-\frac{m_\mu}{M}\sum_\nu\frac{\vec{\pi}_\nu{}^2}{2m_\nu}+\frac{\vec{\pi}_\mu\cdot\vec{\mathbf{P}}}{2M}\right)\frac{\vec{\mathbf{P}}}{Mc^2}+\frac{i}{c^2}\left[\omega_\pi^{(1)}-\frac{\beta}{M}\int_0^{\vec{\mathbf{P}}}d\vec{\mathbf{P}}\cdot\vec{\mathbf{W}}^{(1)}+\beta\zeta_\pi^{(1)},\vec{\pi}_\mu\right], \tag{2.27b}$$

$$\vec{\mathbf{s}}_\mu=\vec{\sigma}_\mu-\frac{\vec{\sigma}_\mu\times(\vec{\pi}_\mu\times\vec{\mathbf{P}})}{2m_\mu Mc^2}+\frac{i}{c^2}\left[\omega_\pi^{(1)}-\frac{\beta}{M}\int_0^{\vec{\mathbf{P}}}d\vec{\mathbf{P}}\cdot\vec{\mathbf{W}}^{(1)}+\beta\zeta_\pi^{(1)},\vec{\sigma}_\mu\right]. \tag{2.27c}$$

The corresponding expressions for $\vec{\mathbf{R}}$, $\vec{\mathbf{P}}$, $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$ in terms of $\vec{\mathbf{r}}_\mu$, $\vec{\mathbf{p}}_\mu$, and $\vec{\mathbf{s}}_\mu$ result from inverting Eq. (2.27) while making use of Eq. (2.11). Note in particular that

$$\vec{\mathcal{P}}=\sum_\mu\vec{\mathbf{p}}_\mu=\vec{\mathbf{P}}\ , \tag{2.28a}$$

$$\vec{\mathfrak{J}}=\sum_\mu(\vec{\mathbf{r}}_\mu\times\vec{\mathbf{p}}_\mu+\vec{\mathbf{s}}_\mu)=\vec{\mathbf{R}}\times\vec{\mathbf{P}}+\vec{\mathbf{S}}\ ,$$

$$\vec{\mathbf{S}}=\sum_\mu(\vec{\rho}_\mu\times\vec{\pi}_\mu+\vec{\sigma}_\mu)\ , \tag{2.28b}$$

where $[\phi^{(1)},\vec{\mathfrak{J}}]=0$ has been used so that condition (3) is explicitly satisfied. Condition (2) is also easily shown to be satisfied:

$$\begin{aligned}\vec{\mathcal{K}}&=\sum_\mu\left\{\frac{1}{2c^2}\left[\vec{\mathbf{r}}_\mu\left(m_\mu c^2+\frac{\vec{\mathbf{p}}_\mu{}^2}{2m_\mu}\right)+\text{H.c.}\right]-\frac{\vec{\mathbf{s}}_\mu\times\vec{\mathbf{p}}_\mu}{2m_\mu c^2}-t\vec{\mathbf{p}}_\mu\right\}+\frac{\beta\vec{\mathbf{V}}^{(1)}}{c^2}\\&=\frac{1}{2c^2}\left[\vec{\mathbf{R}}\left(h(\beta)+\frac{\vec{\mathbf{P}}^2}{2M}\right)+\text{H.c.}\right]-\frac{\vec{\mathbf{S}}\times\vec{\mathbf{P}}}{2Mc^2}-t\vec{\mathbf{P}}\ ,\end{aligned}$$

$$h(\beta)=Mc^2+h^{(0)}(\beta)\ ,\quad h^{(0)}(\beta)=\sum_\mu\frac{\vec{\pi}_\mu{}^2}{2m_\mu}+\beta U^{(0)}\ , \tag{2.28c}$$

while the corresponding result for $\mathcal{K}$ follows from the first commutator of Eq. (2.14c). The relationship between the $h^{(n)}$ and $U^{(n)}$ also follows from the first commutator of Eq. (2.14c) together with Eq. (2.16c):

$$[P^2c^2+h^2(\beta)]^{1/2}=e^{i\phi}\left\{\sum_\mu\left[\left(\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}\right)^2c^2+m_\mu{}^2c^4\right]^{1/2}+\beta U\left(\vec{r}_\mu=\vec{\rho}_\mu+\vec{R},\ \vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P},\ \vec{s}_\mu=\vec{\sigma}_\mu\right)\right\}e^{-i\phi}, \tag{2.28d}$$

or, equivalently,

$$h(\beta)=Mc^2+\sum_{n=0}\frac{h^{(n)}(\beta)}{c^{2n}}, \tag{2.29a}$$

$$h^{(0)}(\beta)=\sum_\mu\frac{\vec{\pi}_\mu{}^2}{2m_\mu}+\beta U^{(0)}, \tag{2.29b}$$

$$h^{(1)}(\beta)=-\sum_\mu\frac{\vec{\pi}_\mu{}^4}{8m_\mu{}^3}-\sum_\mu\frac{\vec{\pi}_\mu\cdot\vec{P}}{m_\mu M}\left(\frac{\vec{\pi}_\mu{}^2}{2m_\mu}+\frac{\vec{\pi}_\mu\cdot\vec{P}}{2M}\right)+\beta\frac{\vec{P}^2}{2M^2}U^{(0)}+\beta U^{(1)}+i[\phi^{(1)},h^{(0)}(\beta)], \tag{2.29c}$$

$$\ldots,$$

where it must be remembered that

$$U^{(n)}=U^{(n)}\left(\vec{r}_\mu=\vec{\rho}_\mu+\vec{R},\ \vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P},\ \vec{s}_\mu=\vec{\sigma}_\mu\right)$$

in Eq. (2.29). The operators $\phi^{(n)}$ adjust the relationship between $\{\vec{r}_\mu,\vec{p}_\mu,\vec{s}_\mu\}$ and $\{\vec{R},\vec{P},\vec{\rho}_\mu,\vec{\pi}_\mu,\vec{\sigma}_\mu\}$ so that it is relativistically correct to order $1/c^{2n}$. Alternately, Eq. (2.29) can be inverted with respect to $h^{(n)}$ and $U^{(n)}$, and serve to define the $U^{(n)}$. Since the transformation $\exp[i\phi]$ is unitary, and $\{\vec{R},\vec{P},\vec{\rho}_\mu,\vec{\pi}_\mu,\vec{\sigma}_\mu\}$ satisfies Eqs. (2.13) and (2.11), $\{\vec{r}_\mu,\vec{p}_\mu,\vec{\sigma}_\mu\}$ satisfies Eq. (2.4), so that condition (1) is satisfied.

Comparing the c.m. variables as defined by Eq. (2.27) to order $1/c^2$ with those generated by the GS transformation by Osborn,[14] we note that in the absence of internal interactions, $\beta\to 0$, the c.m. variables of Osborn[14] correspond to the particular solution $\Pi^{(1)}=0$. In short, to order $1/c^2$, they are precisely given by Eq. (2.27) with $\omega_\pi^{(1)}=0$ and $\beta=0$, so that they are subsumed by our results. Since the c.m. variables of Bakamjian and Thomas[2] are the same as those obtained by Osborn,[14] our remarks apply to them also.

A comparison of our results with those resulting from the GS transformation[14] in the presence of internal interaction will be less direct. It will be accomplished by first deriving these variables to order $1/c^2$, and then comparing the results directly with Eq. (2.27) above. Here we are referring to the general GS transformation in which the generators $\vec{\mathcal{K}}$ and $\mathcal{H}$ explicitly depend upon $\vec{V}$ and U, respectively, and $\vec{V}$ and U are arbitrary functions of the dynamical variables except for the restrictions imposed by the Lie algebra.

The GS transformation is a singular transformation which maps the operator $\vec{P}$ into zero. It exists for all operators which commute with $\vec{P}$. The c.m. variables are found by transforming $\vec{r}_\mu-\vec{r}_\nu$, $\vec{p}_\mu$, and $\vec{s}_\mu$; $\vec{R}$ and $\vec{P}$ are defined by the infinitesimal generators of the Poincaré group. The transformation of $\vec{p}_\mu$ will be given as an example; the results for $\vec{r}_\mu-\vec{r}_\nu$ and $\vec{s}_\mu$ are similar. Let $\vec{p}_\mu(\alpha)$ be defined by

$$\vec{p}_\mu(\alpha)=\exp[i\tfrac{1}{2}\alpha(\vec{R}\cdot\vec{P}+\text{H.c.})]\times\vec{p}_\mu\exp[-i\tfrac{1}{2}\alpha(\vec{R}\cdot\vec{P}+\text{H.c.})], \tag{2.30a}$$

$$\tfrac{1}{2}(\vec{R}\cdot\vec{P}+\text{H.c.})=\frac{1}{2}\left(\vec{\mathcal{K}}\cdot\frac{\vec{\mathcal{P}}}{\mathcal{H}}+\text{H.c.}\right), \tag{2.30b}$$

where from Eq. (2.3) and to order $1/c^2$,

$$\tfrac{1}{2}(\vec{R}\cdot\vec{P}+\text{H.c.})=\sum_\mu\frac{m_\mu}{2M}(\vec{r}_\mu\cdot\vec{P}+\text{H.c.})+\frac{1}{Mc^2}\sum_\mu\left[\frac{1}{2}\left(\frac{\vec{p}_\mu{}^2}{2m_\mu}\vec{r}_\mu\cdot\vec{P}+\text{H.c.}\right)-\frac{\vec{s}_\mu\times\vec{p}_\mu\cdot\vec{P}}{2m_\mu}-\frac{m_\mu}{2M}\left(\vec{r}_\mu\cdot\vec{P}\sum_\nu\frac{\vec{p}_\nu{}^2}{2m_\nu}+\text{H.c.}\right)\right]+\frac{\beta}{Mc^2}\vec{W}^{(1)}\cdot\vec{P}, \tag{2.30c}$$

where Eqs. (2.23) have been used. The result of a straightforward calculation is

$$\vec{p}_\mu(\alpha)=\vec{p}_\mu-(1-e^{-\alpha})\vec{P}\left(\frac{m_\mu}{M}+\frac{\vec{\pi}_\mu{}^2}{2m_\mu Mc^2}-\frac{m_\mu}{M^2c^2}\sum_\nu\frac{\vec{\pi}_\nu{}^2}{2m_\nu}\right)-\frac{1}{Mc^2}(1-e^{-2\alpha})\frac{\vec{\pi}_\mu\cdot\vec{P}\vec{P}}{2M}+\frac{i\beta}{Mc^2}\sum_{n=1}(1-e^{-n\alpha})[A_n(\vec{P}),\vec{\pi}_\mu], \tag{2.30d}$$

where it has been assumed that

$$\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}=\sum_{n=1} A_n(\vec{P}) , \tag{2.31}$$

and $A_n(\vec{P})$ is zero, or contains n factors of $\vec{P}$. This permits the following replacement, to order $1/c^2$:

$$\frac{i\alpha\beta}{Mc^2}[\vec{W}^{(1)}\cdot\vec{P},\vec{p}_\mu]$$

$$=\frac{\alpha\beta}{Mc^2}\left[\left[\tfrac{1}{2}(\vec{R}\cdot\vec{P}+\text{H.c.}), \int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)}\right], \vec{\pi}_\mu\right]$$

$$=\frac{i\beta}{Mc^2}\sum_{n=1} \alpha n[A_n(\vec{P}),\vec{\pi}_\mu] , \tag{2.32}$$

which contributes to Eq. (2.30d) as indicated above. The limit of $\vec{p}_\mu(\alpha)$ as $\alpha\to\infty$ is defined to be the internal momentum variable $\vec{\pi}_\mu$. Taking the limit as $\alpha\to\infty$ of Eq. (2.30d) and inverting the result yields

$$\vec{p}_\mu=\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}+\left(\frac{\vec{\pi}_\mu^{\,2}}{2m_\mu}-\frac{m_\mu}{M}\sum_\nu\frac{\vec{\pi}_\nu^{\,2}}{2m_\nu}+\frac{\vec{\pi}_\mu\cdot\vec{P}}{2M}\right)\frac{\vec{P}}{Mc^2}$$
$$-\frac{i\beta}{Mc^2}\left[\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)},\vec{\pi}_\mu\right] , \tag{2.33}$$

which is to be compared to Eq. (2.27b). In this scheme, the commutation relations of the c.m. variables follow directly from Eq. (2.7), and GS transforming Eq. (2.4) with $\vec{r}_\mu-\vec{r}_\nu$ replacing $\vec{r}_\mu$. Choosing $\sum_\mu m_\mu\vec{\rho}_\mu=0$ yields Eq. (2.13), while $\sum_\mu\vec{\pi}_\mu=0$ results from GS transforming $\vec{P}=\sum_\mu\vec{p}_\mu$. Note that in the presence of internal interactions the GS transformation again corresponds to the particular solution $\Pi^{(1)}=0$, so that they are also subsumed by our results.

In the next section, we consider both internal and external electromagnetic interactions as instructive examples.

III. ELECTROMAGNETIC INTERACTIONS

A. Internal EM interactions

Other than the restriction given in the Introduction, the internal interactions considered thus far in this paper have been completely arbitrary. In this section, we wish to consider the old and familiar problem of two particles interacting electromagnetically, and treat this as a composite system with internal interactions. In particular, we wish to single out for consideration a more recent treatment of this problem by Close and Osborn[10] in order to consider some of their results as an example, and to illustrate an important point.

Following the quasi-field-theoretic approach of Close and Osborn,[10] one finds that, in terms of the constituent particle variables, the Hamiltonian for two particles interacting electromagnetically is, to order $1/c^2$,

$$\mathcal{H}=H+\beta U , \tag{3.1a}$$

$$H^{(0)}=\frac{\vec{p}_1^{\,2}}{2m_1}+\frac{\vec{p}_2^{\,2}}{2m_2} , \tag{3.1b}$$

$$H^{(1)}=-\frac{\vec{p}_1^{\,4}}{8m_1^3}-\frac{\vec{p}_2^{\,4}}{8m_2^3} , \tag{3.1c}$$

$$U^{(0)}=\frac{\epsilon_1\epsilon_2}{4\pi|\vec{r}_1-\vec{r}_2|} , \tag{3.1d}$$

$$U^{(1)}=-\frac{\epsilon_1\epsilon_2}{16\pi m_1m_2}\left[\vec{p}_1\frac{1}{|\vec{r}_1-\vec{r}_2|}\cdot\vec{p}_2+\text{H.c.}\right]-\frac{\epsilon_1\epsilon_2}{16\pi m_1m_2}\left[\vec{p}_1\cdot(\vec{r}_1-\vec{r}_2)\frac{1}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)\cdot\vec{p}_2+\text{H.c.}\right]$$
$$-\frac{\epsilon_1\mu_2}{2\pi m_1}\frac{\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_1}{|\vec{r}_1-\vec{r}_2|^3}+\frac{\epsilon_2\mu_1}{2\pi m_2}\frac{\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_2}{|\vec{r}_1-\vec{r}_2|^3}-\tfrac{8}{3}\mu_1\mu_2\vec{s}_1\cdot\vec{s}_2\delta^3(\vec{r}_1-\vec{r}_2)$$
$$+\frac{\mu_1\mu_2}{\pi}\left[\frac{\vec{s}_1\cdot\vec{s}_2}{|\vec{r}_1-\vec{r}_2|^3}-\frac{3\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^5}\right]+\left[\frac{\epsilon_1\epsilon_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)-\frac{\epsilon_2\mu_1}{2m_1}-\frac{\epsilon_1\mu_2}{2m_2}\right]\delta^3(\vec{r}_1-\vec{r}_2)$$
$$-\frac{\epsilon_2}{4\pi m_1}\left(2\mu_1-\frac{\epsilon_1}{2m_1}\right)\frac{\vec{s}_1\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_1}{|\vec{r}_1-\vec{r}_2|^3}+\frac{\epsilon_1}{4\pi m_2}\left(2\mu_2-\frac{\epsilon_2}{2m_2}\right)\frac{\vec{s}_2\cdot(\vec{r}_1-\vec{r}_2)\times\vec{p}_2}{|\vec{r}_1-\vec{r}_2|^3} . \tag{3.1e}$$

As one of their results, Close and Osborn[10] then derive the corresponding expression for the Hamiltonian in terms of the c.m. variables. This may be misleading.

The point is the following. Although not explicitly stated, the c.m. variables chosen by Close and Osborn[10] are those generated by the GS transformation in the presence of internal interactions. An advantage of these c.m. variables is that the form of the Hamiltonian follows directly from Eq. (3.1) by inspection, not calculation. In fact, one has

$$h(\beta)=\lim_{\alpha\to\infty}\exp(i\tfrac{1}{2}\alpha\{\vec{R}\cdot\vec{P}+\text{H.c}\})[H+\beta U]$$
$$\times\exp(-i\tfrac{1}{2}\alpha\{\vec{R}\cdot\vec{P}+\text{H.c.}\}) , \tag{3.2a}$$

where $\vec{R}$ is defined in terms of $\{\vec{r}_\mu,\vec{p}_\mu,\vec{s}_\mu\}$ by Eqs. (2.6) and (2.3), so that

$$h^{(0)}(\beta)=H^{(0)}(\infty)+\beta U^{(0)}(\infty)\,, \tag{3.2b}$$

$$h^{(1)}(\beta)=H^{(1)}(\infty)+\beta U^{(1)}(\infty)\,, \tag{3.2c}$$

$$\cdots\,,$$

$$h^{(n)}(\beta)=H^{(n)}(\infty)+\beta U^{(n)}(\infty)\,, \tag{3.2n}$$

where

$$U^{(n)}(\infty)=U^{(n)}(\vec{r}_\mu-\vec{r}_\nu\to\vec{\rho}_\mu-\vec{\rho}_\nu,\ \vec{p}_\mu\to\vec{\pi}_\mu,\ \vec{s}_\mu\to\vec{\sigma}_\mu)\,, \tag{3.2m}$$

and similarly for $H^{(n)}(\infty)$. Consequently, one can look at Eq. (3.1) and know what the form of the Hamiltonian is in terms of these c.m. variables. For comparison, we make the replacement $\vec{\pi}_1=\hat{p}$, $\vec{\pi}_2=-\hat{p}$, and $\vec{\rho}_1-\vec{\rho}_2=\hat{q}$. The result is

$$\mathcal{H}=[P^2c^2+h^2(\beta)]^{1/2}\,, \tag{3.3a}$$

$$h^{(0)}(\beta)=\frac{\hat{p}^2}{2m_1}+\frac{\hat{p}^2}{2m_2}+\beta\frac{\epsilon_1\epsilon_2}{4\pi|\hat{q}|}\,, \tag{3.3b}$$

$$\begin{aligned}h^{(1)}(\beta)=-\left(\frac{\hat{p}^4}{8m_1^3}+\frac{\hat{p}^4}{8m_2^3}\right)+\beta\Bigg\{&\frac{\epsilon_1\epsilon_2}{16\pi m_1m_2}\left(\hat{p}\frac{1}{|\hat{q}|}\cdot\hat{p}+\text{H.c.}\right)+\frac{\epsilon_1\epsilon_2}{16\pi m_1m_2}\left(\hat{p}\cdot\hat{q}\frac{1}{|\hat{q}|^3}\hat{q}\cdot\hat{p}+\text{H.c.}\right)\\&-\frac{M}{2\pi m_1m_2}(\epsilon_1\mu_2\vec{\sigma}_2+\epsilon_2\mu_1\vec{\sigma}_1)\cdot\frac{\hat{q}\times\hat{p}}{|\hat{q}|^3}+\frac{\epsilon_1\epsilon_2}{8\pi}\left(\frac{\vec{\sigma}_1}{m_1^2}+\frac{\vec{\sigma}_2}{m_2^2}\right)\cdot\frac{\hat{q}\times\hat{p}}{|\hat{q}|^3}\\&-\tfrac{8}{3}\mu_1\mu_2\vec{\sigma}_1\cdot\vec{\sigma}_2\delta^3(\hat{q})+\frac{\mu_1\mu_2}{\pi}\left(\frac{\vec{\sigma}_1\cdot\vec{\sigma}_2}{|\hat{q}|^3}-\frac{3\vec{\sigma}_1\cdot\hat{q}\vec{\sigma}_2\cdot\hat{q}}{|\hat{q}|^5}\right)\\&+\left[\frac{\epsilon_1\epsilon_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)-\frac{\epsilon_2\mu_1}{2m_1}-\frac{\epsilon_1\mu_2}{2m_2}\right]\delta^3(\hat{q})\Bigg\}\,,\end{aligned} \tag{3.3c}$$

which is to be compared with Eq. (5.11) of Close and Osborn.[10] Here $\vec{\sigma}_1$ and $\vec{\sigma}_2$ denote internal spin operators, that is, $\frac{1}{2}\hbar$ times the Pauli spin operators used by Close and Osborn. Since the form of h is known in terms of these c.m. variables, and the form of $\vec{S}$ follows from GS transforming $\vec{J}-\vec{R}\times\vec{P}$, the form of the remaining infinitesimal generators of the Poincaré group $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, and $\vec{\mathcal{K}}$ is also known from Eq. (2.6). So, one knows by inspection the detailed form of the generators expressed in terms of these c.m. variables before the relationship between these c.m. variables and the constituent particle variables has been explicitly determined, that is, before the nonsingular GS transformation, by which these c.m. variables are defined, has actually been performed in the presence of internal interactions.

This is quite different from the program we have outlined. In our procedure, the central question with regard to c.m. variables, constituent particle variables, and relativistic invariance of a composite system has been what relativistic invariance implies about the relationship between the two sets of variables. We have found that a whole class of c.m. variables, defined with respect to the constituent particle variables, is permitted by the Lie algebra of the Poincaré group, that is, is consistent with relativistic invariance of a composite system, and that the detailed form of the infinitesimal generators then follows accordingly. In particular, to continue our example, $h^{(0)}$ and $h^{(1)}$ would be given by Eqs. (2.29b) and (2.29c). The conceptual difference between the GS transformation and our procedure may be used to advantage, however. If $\Pi^{(1)}=0$, the form of $h^{(0)}$ and $h^{(1)}$ is known in terms of the c.m. variables from the notion of the GS transformation as discussed above and given by Eqs. (3.3b) and (3.3c) (before $\phi^{(1)}$ has been determined), while the effect of the internal EM interaction upon the relationship between the two sets of variables is particularly simple to determine by means of $\phi^{(1)}$. Given the choice for $\vec{V}^{(1)}$ used by Close and Osborn[22] [Eq. (6.1)],

$$\vec{V}^{(1)}=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2)U^{(0)}\,, \tag{3.4a}$$

we have

$$\vec{W}^{(1)}=\{\tfrac{1}{2}(\vec{r}_1+\vec{r}_2)-\vec{R}\}U^{(0)}\,, \tag{3.4b}$$

$$-\frac{\beta}{M}\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}^{(1)}=-\frac{\beta}{2M}(\vec{\rho}_1+\vec{\rho}_2)\cdot\vec{P}\frac{\epsilon_1\epsilon_2}{4\pi|\vec{\rho}_1-\vec{\rho}_2|}\,, \tag{3.4c}$$

where Eq. (2.25c) has been used. Or, in terms of $\hat{q}$, using Eq. (2.2a), the above is equivalent to

$$-\frac{\beta}{M}\int_0^{\vec{P}}d\vec{P}\cdot\vec{W}^{(1)}=-\frac{\beta}{2M^2}(m_2-m_1)\hat{q}\cdot\vec{P}\frac{\epsilon_1\epsilon_2}{4\pi|\hat{q}|}\,. \tag{3.4d}$$

The operator $\phi^{(1)}$ is then given by Eq. (2.26b), and the c.m. variables are given by Eq. (2.27). Note that both $\hat{p}$ and $\vec{R}$ explicitly depend upon the internal interaction. Also, one may verify that $h^{(1)}$, as defined by Eq. (2.29c), is in fact equal to the expression given by Eq. (3.3c), with $\phi^{(1)}$ defined above.

In contrast, it is of interest to note that there always exists a $\vec{V}$ such that the resultant c.m. variables, defined in terms of $\{\vec{r}_\mu, \vec{p}_\mu, \vec{s}_\mu\}$, are independent of the internal interaction. In the example above, that $\vec{V}$ would be

$$\vec{V}^{(1)}=\frac{(m_1\vec{r}_1+m_2\vec{r}_2)}{M}\;\frac{\epsilon_1\epsilon_2}{4\pi|\vec{r}_1-\vec{r}_2|}=\vec{R}\,U^{(0)}\ , \qquad (3.5)$$

for which $\vec{W}^{(1)}$ would be zero, and $\phi^{(1)}$ would be independent of β. In the language of the GS transformation, for this particular $\vec{V}^{(1)}$, $\vec{R}$ as defined by the generators of the Poincaré group would be independent of the internal interaction, so that the GS transformation could not introduce the internal interaction into the definition of the internal c.m. variables. Such an example is contained in Sec. IV.

B. External EM interactions

We wish to reconsider the problem of the LET for Compton scattering and the DHG sum both as an example and in order to bring forward the complete form of the "correction" to the FW EM interaction Hamiltonian introduced in a previous publication.[8] Although the existence of additional terms has long been known,[23] the EM interaction Hamiltonian discussed previously[8] considered explicitly only such terms as would contribute to the LET for Compton scattering and the DHG sum rule. More recently, however, terms which would contribute to higher order in photon momentum have become of interest,[13] and require the complete form.

To consider the LET for Compton scattering and the DHG sum rule with regard to a composite system, the simplest choice was made both for the composite system and the c.m. variables: $\beta\to 0$, which corresponds to a "loosely bound" system of particles, and $\Pi^{(1)}=0$. External EM interaction was introduced in the usual way, that is, by the sum of the FW reduced EM interaction Hamiltonians of the constituent particles. To order $1/c^2$, the c.m. variables were generated by means of the unitary transformation $\exp[i\phi^{(1)}/c^2]$ for which the replacement $\vec{p}_\mu\to\vec{p}_\mu'\equiv\vec{p}_\mu-\epsilon_\mu\vec{A}(\vec{r}_\mu)$ had been made.[8] It was found that a "correction" term, $H_{\Delta\mathrm{FW}}$, was required in order to preserve relativistic invariance to order $1/c^2$, and hence satisfy the LET for Compton scattering and DHG sum rule. It arose from purely kinematic considerations, that is, for want of a set of c.m. variables which satisfied the criteria given at the beginning of the previous section. The complete form of $H_{\Delta\mathrm{FW}}$ is as follows:

$$H_{\Delta\mathrm{FW}}=-\sum_\mu \tfrac{1}{2}\epsilon_\mu[\vec{E}(\vec{r}_\mu)\cdot\vec{a}_\mu'+\mathrm{H.c.}]+\sum_\mu\frac{\epsilon_\mu}{4m_\mu Mc^2}\left(1-\frac{m_\mu}{M}\right)\vec{\rho}_\mu\cdot\vec{\nabla}_\mu[\vec{\nabla}_\mu\cdot\vec{E}(\vec{r}_\mu)]\ , \qquad (3.6a)$$

where

$$\vec{a}_\mu'=-\frac{1}{2}\left[\vec{\rho}_\mu\cdot\frac{\vec{P}'}{Mc^2}\left(\frac{\vec{\pi}_\mu'}{m_\mu}+\frac{\vec{P}'}{2M}\right)+\mathrm{H.c.}\right]$$
$$-\tfrac{1}{2}\sum_\nu\left(\frac{\vec{\pi}_\nu'^{\,2}\vec{\rho}_\nu}{2m_\nu Mc^2}+\mathrm{H.c.}\right)+\sum_\nu\frac{(\vec{\rho}_\nu\times\vec{\pi}_\nu')\times\vec{P}'}{2M^2c^2}-\frac{\vec{\sigma}_\mu\times\vec{P}'}{2m_\mu Mc^2}+\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{\pi}_\nu'}{2m_\nu Mc^2}+\sum_\nu\frac{\vec{\sigma}_\nu\times\vec{P}'}{2M^2c^2}\ , \qquad (3.6b)$$

$\vec{\pi}_\mu'\equiv\vec{\pi}_\mu(\vec{p}_\mu\to\vec{p}_\mu-\epsilon_\mu\vec{A}(\vec{r}_\mu))$, and similarly for $\vec{P}'$. In the previous publication,[8] the second sum of Eq. (3.6a) was considered to be part of the EM Hamiltonian which could not contribute to the LET for Compton scattering or the DHG sum rule in the same sense that the collection of Darwin and quadrupole-moment terms were considered to be part of $H_{\mathrm{FW}}^{(2)}$. Since the noncontributing part of the EM Hamiltonian could be dealt with cursorily, only those terms which could contribute to the LET for Compton scattering and the DHG sum rule were explicitly given.

Recall that $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ occurring in Eq. (3.6) are given by the nonrelativistic expressions in Eq. (2.10b), the correct relationship between the constituent particle variables and the c.m. variables being generated by the unitary transformation to order $1/c^2$. In essence, one may correctly calculate to any order in $1/c^2$ using the familiar nonrelativistic relationship between the constituent particle variables and the c.m. variables provided the unitary transformation defined by Eq. (2.15) is then used to account for the difference between the nonrelativistic relationship and the relativistically correct one. In explicit calculations such as the direct verification of the LET for Compton scattering and the DHG sum rule for composite systems, this can be a considerable advantage.

Finally, it should be emphasized that it is not necessary to verify the LET for Compton scattering with regard to composite systems by a direct calculation of the scattering amplitude.[6-9,11] All that is required of the composite system is a verification of the fundamental assumptions on which the LET for Compton scattering depends. In particular, with the assumption of an energy gap between the "ground state," that is, the state from which Compton scattering takes place, and the first excited state, if one can show that the matrix elements of the 4-current for the composite system are time-reversal invariant and transform properly under a Lorentz transformation to order $1/c^2$, then the 4-current is conserved to order $1/c^2$ and the LET for Compton scattering follows from an argument by Lapidus and Chou.[3] The demonstration that the matrix elements of the 4-current for the "loosely bound" system of particles considered above satisfy these conditions is given in Appendix C.

IV. CORRECTIONS TO PHENOMENOLOGICAL POTENTIALS

We wish to briefly consider a topic closely related to relativistic c.m. variables, that of relativistic corrections to phenomenological potentials, as these corrections are already contained in Eq. (2.29c), and higher-order equations in $1/c^2$. Inverting Eq. (2.29c) with respect to $h^{(1)}$ and $U^{(1)}$, we obtain the relativistic corrections to order $1/c^2$ of the phenomenological potential $U^{(0)}$, for an N-particle composite system:

$$\beta U^{(1)} = -\beta \frac{\vec{P}^2}{2M^2} U^{(0)} - i\beta\left[-\tfrac{1}{2}\sum_\mu\left(\frac{\vec{\rho}_\mu\cdot\vec{P}\,\vec{\pi}_\mu\cdot\vec{P}}{2M^2} + \text{H.c.}\right) - \tfrac{1}{2}\sum_\mu\left(\frac{\vec{\rho}_\mu\cdot\vec{P}\,\vec{\pi}_\mu{}^2}{2m_\mu M} + \text{H.c.}\right) + \sum_\mu \frac{\vec{\sigma}_\mu\times\vec{\pi}_\mu\cdot\vec{P}}{2m_\mu M},\ U^{(0)}\right]$$
$$+ i\beta\left[\frac{1}{M}\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)} - \zeta_\pi^{(1)},\ h^{(0)}(\beta)\right] - i\beta[\omega_\pi^{(1)},\ U^{(0)}] + h^{(1)}(\beta) - h^{(1)}(0)\,, \tag{4.1a}$$

where

$$\Pi^{(1)} = \omega_\pi^{(1)} + \beta\zeta_\pi^{(1)}\ ,$$

and

$$h^{(1)}(0) = -\sum_\mu \frac{\vec{\pi}_\mu{}^4}{8m_\mu{}^3} + i\left[\omega_\pi^{(1)}, \sum_\mu \frac{\vec{\pi}_\mu{}^2}{2m_\mu}\right]. \tag{4.1b}$$

The corrections to the phenomenological potential $U^{(0)}$ in Eq. (4.1) are to be compared with the results of Shirokov[15] and the closely related more recent results of Bhakar.[16] Implicit in Shirokov's paper and explicit in Bhakar's paper is the assumption that $\vec{W}^{(1)} = 0$, that is,

$$\vec{V}^{(1)} = \vec{R}\,U^{(0)}\ . \tag{4.2}$$

Neither considers terms which are functions of internal variables only. Consequently, Shirokov's expression for $U^{(1)}$ [Eq. (23)] is given by Eq. (4.1) with $N=2$,

$$\vec{W}^{(1)} = 0\ , \tag{4.3a}$$

and

$$h^{(1)}(\beta) - h^{(1)}(0) - i\beta[\omega_\pi^{(1)},\ U^{(0)}] - i\beta[\zeta_\pi^{(1)},\ h^{(0)}(\beta)] = 0\ . \tag{4.3b}$$

It is thereby included in our expression as a special case. Furthermore, the relativistic corrections given by Bhakar[16] are included in those given by Shirokov,[15] and so are too included in our expression for $U^{(1)}$. Unfortunately, there are typographical errors in the integral operator form of the relativistic corrections as given by both Shirokov [Eq. (22)] and Bhakar [Eq. (2.14)]. Corrections to the typographical errors are included with Refs. 15 and 16.

Several remarks are now in order. First, we note that Eq. (4.1a) represents nothing more than a particular rewriting of the first-order relationship between (i) U and $\vec{V}$ of one representation of the Poincaré generators [Eqs. (2.3)] and (ii) $h(\beta)$ of another equally valid representation [Eqs. (2.6)]. The essence of this first-order relationship is that, given $U^{(0)}$ [or $h^{(0)}(\beta)$], (a) any $U^{(1)}$ and $\vec{V}^{(1)}$ which satisfy the Lie algebra satisfy the first-order relationship with $h^{(1)}(\beta)$, and (b) assuming a translationally invariant vector function $\vec{W}^{(1)}$ [$=\vec{V}^{(1)} - \vec{R}U^{(0)}$], any $U^{(1)}$ which satisfies the first-order relationship with $h^{(1)}(\beta)$ satisfies the Lie algebra. The first statement determines $h^{(1)}(\beta)$ to within an "integration constant" [Eq. (2.29c)]; the second statement is what makes the relationship of interest in studying relativistic corrections to phenomenological potentials [Eq. (4.1a)].

Within the representation of the Poincaré generators in Eqs. (2.3), the forms of U and $\vec{V}$ are restricted by conditions imposed by the Lie algebra. One can assess the forms of U and $\vec{V}$ in Eqs. (2.3), however, on the basis of physical criteria other than that of relativity, and here is where the condition of separability must play a vital role.[1,24] Briefly, separability demands that any multiparticle system must have the property that for all divisions of the system into two subsystems of particles infinitely far removed from the other, the resultant subsystems must be completely independent of one another. This condition then limits the physically acceptable forms of U and $\vec{V}$. Furthermore, since ϕ depends upon the form of $\vec{V}$, one can easily see that any effect the condition of separability has upon $\vec{V}$ will in turn be reflected in the definition of the relativistic c.m. variables. In particular, beyond $N=2$, the simplest choice for $\vec{V}^{(1)}$, namely,

$$\vec{V}^{(1)}=\vec{R}\,U^{(0)}\ ,$$

is clearly not acceptable since it is not separable even in the case where $U^{(0)}$ is a sum of two-particle interactions such that $U^{(0)}$ is itself separable. For with this choice, if the system consisted of two subsystems I and II each with internal interactions but with no interactions between the two, then the transformations under a Lorentz boost of the state vector of each subsystem would depend on the location and dynamics of the other. Hence the general need for the function $\vec{W}^{(1)}$ is clear from this example.

One other example will be given to clarify the connection between separability, the first-order relationship between U, $\vec{V}$, and $h(\beta)$, and relativistic c.m. variables. Suppose one is given a representation of the Poincaré generators in Eqs. (2.3) up to order $1/c^2$ in which U and $\vec{V}$ are both separable and covariant and defined as follows[25]:

$$U^{(0)}=\tfrac{1}{2}\sum_{\mu\nu}u_{\mu\nu}^{(0)}\ , \tag{4.4a}$$

$$\vec{V}^{(1)}=\tfrac{1}{2}\sum_{\mu\nu}\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)}\ , \tag{4.4b}$$

$$\begin{aligned}U^{(1)}=\tfrac{1}{2}\sum_{\mu\nu}\Bigg\{&-\frac{P_{\mu\nu}{}^2}{2M_{\mu\nu}{}^2}u_{\mu\nu}^{(0)}+\frac{i(m_\nu-m_\mu)}{4m_\mu m_\nu M_{\mu\nu}}[(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})p_{\mu\nu}{}^2+p_{\mu\nu}{}^2(\vec{P}_{\mu\nu}\cdot\vec{r}_{\mu\nu}),u_{\mu\nu}^{(0)}]\\&+\frac{i}{4M_{\mu\nu}{}^2}[(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})+(\vec{P}_{\mu\nu}\cdot\vec{p}_{\mu\nu})(\vec{P}_{\mu\nu}\cdot\vec{r}_{\mu\nu}),u_{\mu\nu}^{(0)}]-\frac{i}{2M_{\mu\nu}}\left[\left(\frac{\vec{s}_\mu}{m_\mu}-\frac{\vec{s}_\nu}{m_\nu}\right)\times\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu},u_{\mu\nu}^{(0)}\right]\Bigg\}\\&+\frac{i}{2}\sum_{\mu\nu\sigma}\left\{\frac{(\vec{p}_\mu+\vec{p}_\nu+\vec{p}_\sigma)}{m_\mu+m_\nu+m_\sigma}\cdot[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]+[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]\cdot\frac{(\vec{p}_\mu+\vec{p}_\nu+\vec{p}_\sigma)}{m_\mu+m_\nu+m_\sigma}\right\},\end{aligned} \tag{4.4c}$$

where $u_{\mu\nu}^{(0)}$ is a function of $\vec{r}_{\mu\nu}=\vec{r}_\mu-\vec{r}_\nu$, $\vec{p}_{\mu\nu}=(m_\nu\vec{p}_\mu-m_\mu\vec{p}_\nu)/M_{\mu\nu}$ (with $M_{\mu\nu}=m_\mu+m_\nu$), $\vec{s}_\mu$, and $\vec{s}_\nu$ which is rotationally invariant, symmetric in its two subscripts, zero if the subscripts are equal, and vanishes sufficiently rapidly as $\vec{r}_{\mu\nu}\to\infty$, so that all terms are separable. Consequently, $u_{\mu\nu}^{(0)}$ also commutes with $\vec{R}_{\mu\nu}=(m_\mu\vec{r}_\mu+m_\nu\vec{r}_\nu)/M_{\mu\nu}$, $\vec{P}_{\mu\nu}=\vec{p}_\mu+\vec{p}_\nu$, and $\vec{J}_{\mu\nu}=\vec{r}_\mu\times\vec{p}_\mu+\vec{r}_\nu\times\vec{p}_\nu+\vec{s}_\mu+\vec{s}_\nu$. This particular choice is of interest in that it includes the N-particle results of Zhivopistsev, Perolomov, and Shirokov[26] as a special case, and shows that the presence of three-body terms in $U^{(1)}$ is generally required if $U^{(1)}$ is to be both separable and covariant.[25] Since $U^{(0)}$ is given, and $U^{(1)}$ and $\vec{V}^{(1)}$ satisfy the Lie algebra, (a) implies that $U^{(1)}$ and $\vec{V}^{(1)}$ satisfy the first-order relationship between U, $\vec{V}$, and $h(\beta)$, and hence determine $h^{(1)}(\beta)$ to within an "integration constant" [Eq. (2.29c)]. The result is

$$h^{(1)}(\beta)=h^{(1)}(0)+\beta U^{(1)}(\text{int})+i\beta[\omega_\pi^{(1)},U^{(0)}]+i\beta[\zeta_\pi^{(1)},h^{(0)}(\beta)]\ , \tag{4.5a}$$

where $\Pi^{(1)}$ $(=\omega_\pi^{(1)}+\beta\zeta_\pi^{(1)})$ is an arbitrary scalar function of internal variables only,

$$\begin{aligned}U^{(1)}(\text{int})=\tfrac{1}{2}\sum_{\mu\nu}\Bigg\{&-\frac{(\pi_{\mu\nu}^{+})^2}{2M_{\mu\nu}}u_{\mu\nu}^{(0)}+\frac{i(m_\nu-m_\mu)}{4m_\mu m_\nu M_{\mu\nu}}[(\vec{\rho}_{\mu\nu}^{\,-}\cdot\vec{\pi}_{\mu\nu}^{\,+})(\pi_{\mu\nu}^{-})^2+(\pi_{\mu\nu}^{-})^2(\vec{\pi}_{\mu\nu}^{\,+}\cdot\vec{\rho}_{\mu\nu}^{\,-}),u_{\mu\nu}^{(0)}]\\&+\frac{i}{4M_{\mu\nu}{}^2}[(\vec{\rho}_{\mu\nu}^{\,-}\cdot\vec{\pi}_{\mu\nu}^{\,+})(\vec{\pi}_{\mu\nu}^{\,-}\cdot\vec{\pi}_{\mu\nu}^{\,+})+(\vec{\pi}_{\mu\nu}^{\,+}\cdot\vec{\pi}_{\mu\nu}^{\,-})(\vec{\pi}_{\mu\nu}^{\,+}\cdot\vec{\rho}_{\mu\nu}^{\,-}),u_{\mu\nu}^{(0)}]-\frac{i}{2M_{\mu\nu}}\left[\left(\frac{\vec{\sigma}_\mu}{m_\mu}-\frac{\vec{\sigma}_\nu}{m_\nu}\right)\times\vec{\pi}_{\mu\nu}^{\,-}\cdot\vec{\pi}_{\mu\nu}^{\,+},u_{\mu\nu}^{(0)}\right]\Bigg\}\\&+\frac{i}{2}\sum_{\mu\nu\sigma}\left\{\frac{(\vec{\pi}_\mu+\vec{\pi}_\nu+\vec{\pi}_\sigma)}{m_\mu+m_\nu+m_\sigma}\cdot[\vec{\rho}_{\mu\nu}^{\,+}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]+[\vec{\rho}_{\mu\nu}^{\,+}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]\cdot\frac{(\vec{\pi}_\mu+\vec{\pi}_\nu+\vec{\pi}_\sigma)}{m_\mu+m_\nu+m_\sigma}\right\}\ ,\end{aligned} \tag{4.5b}$$

and

$$\vec{\rho}_{\mu\nu}^{\,-}=\vec{\rho}_\mu-\vec{\rho}_\nu\ ,$$

$$\vec{\pi}_{\mu\nu}^{\,-}=(m_\nu\vec{\pi}_\mu-m_\mu\vec{\pi}_\nu)/M_{\mu\nu}\ ,$$

$$\vec{\rho}_{\mu\nu}^{\,+}=(m_\mu\vec{\rho}_\mu+m_\nu\vec{\rho}_\nu)/M_{\mu\nu}\ ,$$

$$\vec{\pi}_{\mu\nu}^{\,+}=\vec{\pi}_\mu+\vec{\pi}_\nu\ .$$

Clearly $h^{(1)}(\beta)$ is a scalar function of internal variables only. Moreover, as we have learned in Sec. III, $U^{(1)}(\text{int})$ may be found directly from $U^{(1)}$ by inspection. (In the last sum, replace $\vec{R}_{\mu\nu}$ with $\vec{R}_{\mu\nu}-\vec{r}_\xi$, $\xi\neq\mu$, ν, or σ, which does not change the triple sum, as $[\vec{r}_\xi u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]$ is antisymmetric in ν and σ, and proceed as before.) Hence, it is easy to check.

With regard to Eq. (4.1a), one simply considers as physically acceptable only those relativistic corrections, i.e., $U^{(1)}$, which are manifestly separable when written in terms of individual particle variables by means of the nonrelativistic definition of c.m. variables, with the restriction that the assumed translationally invariant vector function $\vec{W}^{(1)}$ be such that $\vec{V}^{(1)}$ is also separable. (A separable $U^{(0)}$ is assumed to be given.) By (b), $U^{(1)}$ satisfies the Lie algebra, and hence is then a separable, relativistic correction.[25]

The connection between separability and relativistic c.m. variables is through the function ϕ, which depends upon the form of $\vec{V}$, and hence $\vec{W}$. Separability restricts the form of $\vec{W}$, and generally forbids $\vec{W}$ from being set equal to zero. (The two-body case would be an exception, for example.) In the example above,

$$\vec{W}^{(1)} = \tfrac{1}{2}\sum_{\mu\nu} (\vec{R}_{\mu\nu} - \vec{R})u_{\mu\nu}^{(0)} = \tfrac{1}{2}\sum_{\mu\nu} \vec{\rho}_{\mu\nu}^{\,+} u_{\mu\nu}^{(0)} , \tag{4.6a}$$

$$C \equiv -\frac{\beta}{M}\int_0^{\vec{P}} d\vec{P}\cdot\vec{W}^{(1)} = -\frac{\beta}{2M}\sum_{\mu\nu} \vec{\rho}_{\mu\nu}^{\,+}\cdot\vec{P}u_{\mu\nu}^{(0)} , \tag{4.6b}$$

$$[C, \vec{\rho}_\mu] = -\frac{\beta}{M}\sum_{\sigma} \vec{\rho}_{\mu\sigma}^{\,+}\cdot\vec{P}[u_{\mu\sigma}^{(0)}, \vec{\rho}_\mu] , \tag{4.6c}$$

$$[C, \vec{\pi}_\mu] = -\frac{\beta}{M}\sum_{\sigma} \vec{\rho}_{\mu\sigma}^{\,+}\cdot\vec{P}[u_{\mu\sigma}^{(0)}, \vec{\pi}_\mu] - i\beta\frac{m_\mu P}{M}\left(\sum_{\sigma}\frac{u_{\mu\sigma}^{(0)}}{M_{\mu\sigma}} - \frac{U^{(0)}}{M}\right) , \tag{4.6d}$$

$$[C, \vec{\sigma}_\mu] = -\frac{\beta}{M}\sum_{\sigma} \vec{\rho}_{\mu\nu}^{\,+}\cdot\vec{P}[u_{\mu\sigma}^{(0)}, \vec{\sigma}_\mu] . \tag{4.6e}$$

Using Eqs. (4.6b)–(4.6e) in Eq. (2.26b) and Eqs. (2.27a)–(2.27c) yields ϕ, $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$, respectively, which then serve to define the relativistic c.m. variables $\vec{\rho}_\mu$, $\vec{\pi}_\mu$, and $\vec{\sigma}_\mu$.

It should be clear from the above example that there are many other acceptable forms for U and $\vec{V}$ which are both separable and covariant,[25] and that these all could be treated in the same way as the above example.

ACKNOWLEDGMENTS

We wish to thank the physics faculty at the University of Washington, where part of this work was done, for the hospitality extended to us during our stay. One of us (R.A.K.) would like to thank Leon Heller of the Los Alamos Scientific Laboratory for his interest in this work, and for many clarifying conversations.

Special thanks from R.A.K. are due to J. L. Friar of Brown University for many stimulating and helpful discussions, particularly those concerning the "correction" to the FW EM interaction Hamiltonian, and for bringing Refs. 15 and 16 to our attention.

We are also indebted to Dr. F. Coester for recalling to us the important role of separability in these problems.

APPENDIX A

Three of the nine commutation relations between $\vec{\mathcal{P}}$, $\vec{\mathcal{J}}$, $\vec{\mathcal{K}}$, and $\mathcal{H}$ given in Eq. (2.2) may be derived from five others. These together with the commutator of $\vec{\mathcal{J}}$ with itself form the six independent commutation relations of the Poincaré group

$$[\mathcal{P}_i, \mathcal{P}_j] = 0 , \tag{A1a}$$

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k, \quad [\mathcal{J}_i, \mathcal{P}_j] = i\epsilon_{ijk}\mathcal{P}_k, \quad [\mathcal{J}_i, \mathcal{K}_j] = i\epsilon_{ijk}\mathcal{K}_k , \tag{A1b}$$

$$[\mathcal{K}_i, \mathcal{P}_j] = i\delta_{ij}\mathcal{H}/c^2 , \quad [\mathcal{K}_i, \mathcal{K}_j] = -i\epsilon_{ijk}\mathcal{J}_k/c^2 . \tag{A1c}$$

The three dependent commutation relations follow as a result of commuting $\vec{\mathcal{P}}$ with the three commutation relations above which contain $\vec{\mathcal{K}}$. Beginning with Eq. (A1b), we have

$$[\mathcal{P}_l, [\mathcal{J}_i, \mathcal{K}_j]] = i\epsilon_{ijk}[\mathcal{P}_l, \mathcal{K}_k] , \tag{A2a}$$

or using the Jacobi identity and Eqs. (A1),

$$\delta_{jl}[\mathcal{J}_i, \mathcal{H}] = 0 . \tag{A2b}$$

Since this must be true for all i, j, and l, it follows that

$$[\mathcal{J}_i, \mathcal{H}] = 0 . \tag{A2c}$$

Similarly, from the first commutation relation of Eq. (A1c), we have

$$[\mathcal{P}_l, [\mathcal{K}_i, \mathcal{P}_j]] = i\delta_{ij}[\mathcal{P}_l, \mathcal{H}]/c^2 , \tag{A3a}$$

and from the Jacobi identity and Eqs. (A1), this is equivalent to

$$\delta_{il}[\mathcal{P}_j, \mathcal{H}] = \delta_{ij}[\mathcal{P}_l, \mathcal{H}] , \tag{A3b}$$

since i, j, and l are arbitrary,

$$[\mathcal{P}_i, \mathcal{H}] = 0 . \tag{A3c}$$

Finally, from the last commutation relation of Eq. (A1c),

$$[\mathcal{P}_l, [\mathcal{K}_i, \mathcal{K}_j]] = -i\epsilon_{ijk}[\mathcal{P}_l, \mathcal{J}_k]/c^2 , \tag{A4a}$$

or again using the Jacobi identity and Eqs. (A1), Eq. (A4a) becomes

$$\delta_{jl}\{[\mathcal{K}_i, \mathcal{H}] - i\mathcal{P}_i\} = \delta_{il}\{[\mathcal{K}_j, \mathcal{H}] - i\mathcal{P}_j\} . \tag{A4b}$$

Since i, j, and l are arbitrary,

$$[\mathcal{K}_i, \mathcal{H}] = i\mathcal{P}_i \ . \qquad \text{(A4c)}$$

APPENDIX B

We show that a solution exists for $\xi^{(n)}$, for all $n \geq 1$. The condition to be satisfied is Eq. (2.20), which results from the requirement that the $\vec{P}$ curl of the $\vec{P}$ gradient of $\xi^{(n)}$ vanish, i.e.,

$$\epsilon_{klm}[R_l, [R_m, \xi^{(n)}]] = 0, \quad n \geq 1 \ . \qquad \text{(B1a)}$$

Since $\xi^{(n)}$ commutes with $\vec{\mathcal{P}}$ [Eq. (2.16a)], this is equivalent to

$$\epsilon_{klm}[\mathcal{K}_l^{(0)}, [\mathcal{K}_m^{(0)}, \xi^{(n)}]] = 0, \quad n \geq 1 \ . \qquad \text{(B1b)}$$

Contracting with ϵ_{ijk} on k and using Eq. (2.19a) yields Eq. (2.20), given here as

$$\left[\kappa_i^{(0)}, \kappa_j^{(n)} - k_j^{(n)} - \sum_{\alpha=1}^{n-1} (k_j^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\kappa_j^{(n-\alpha)})\right] - (i \leftrightarrow j) = 0 \ , \qquad \text{(B1c)}$$

where the sum is identically zero for $n=1$. The demonstration has two parts, $n=1$, and $n>1$, and depends only upon $[\mathcal{K}_i, \mathcal{K}_j] = -i\epsilon_{ijk}\mathcal{J}_k/c^2$. We will need to expand the commutator of $\vec{\mathcal{K}}$ with itself in powers of $1/c^2$

$$\left[\mathcal{K}_i^{(0)} + \sum_{n=1} \frac{\mathcal{K}^{(n)}}{c^{2n}}, \ \mathcal{K}_j^{(0)} + \sum_{m=1} \frac{\mathcal{K}^{(m)}}{c^{2m}}\right] = -\frac{i\epsilon_{ijk}\mathcal{J}_k}{c^2} \ , \qquad \text{(B2a)}$$

or, equivalently,

$$[\mathcal{K}_i^{(0)}, \mathcal{K}_j^{(0)}] = 0 \ , \qquad \text{(B2b)}$$

$$[\mathcal{K}_i^{(0)}, \mathcal{K}_j^{(1)}] - [\mathcal{K}_j^{(0)}, \mathcal{K}_i^{(1)}] = -i\epsilon_{ijk}\mathcal{J}_k \ , \qquad \text{(B2c)}$$

$$[\mathcal{K}_i^{(0)}, \mathcal{K}_j^{(n)}] - [\mathcal{K}_j^{(0)}, \mathcal{K}_i^{(n)}] + \sum_{\alpha=1}^{n-1} [\mathcal{K}_i^{(\alpha)}, \mathcal{K}_j^{(n-\alpha)}] = 0 \ , \quad n \geq 2 \ . \qquad \text{(B2d)}$$

Recall that $\vec{\kappa}^{(0)} = \vec{\mathrm{k}}^{(0)}$ [Eq. (2.18a)]. Both $\vec{\kappa}$ and $\vec{\mathrm{k}}$ satisfy Eq. (B2a). The former follows because the commutation relations of Eq. (2.7) are satisfied, the latter because the commutation relations of Eq. (2.4) are satisfied.

The first part of the demonstration, $n=1$, is simple. Using Eq. (B2c), Eq. (B1c) is trivially satisfied:

$$\{[\kappa_i^{(0)}, \kappa_j^{(1)}] - [\kappa_j^{(0)}, \kappa_i^{(1)}]\} - \{[k_i^{(0)}, k_j^{(1)}] - [k_j^{(0)}, k_i^{(1)}]\} = -i\epsilon_{ijk}\mathcal{J}_k + i\epsilon_{ijk}\mathcal{J}_k = 0 \ . \qquad \text{(B3a)}$$

The second part, $n>1$, is more complicated.

Using Eq. (B2d) repeatedly, Eq. (B1c) can eventually be shown to be satisfied. Let

$$C_{ij} \equiv \left[\kappa_i^{(0)}, \kappa_j^{(n)} - k_j^{(n)} - \sum_{\alpha=1}^{n-1} (k_j^{(\alpha)}\xi^{(n-\alpha)} - \xi^{(\alpha)}\kappa_j^{(n-\alpha)})\right] - (i \leftrightarrow j) \ .$$

Then,

$$\begin{aligned} C_{ij} = &- \sum_{\alpha=1}^{n-1} [\kappa_i^{(\alpha)}, \kappa_j^{(n-\alpha)}] + \sum_{\alpha=1}^{n-1} [k_i^{(\alpha)}, k_j^{(n-\alpha)}] - \sum_{\alpha=1}^{n-1} \left\{ - \sum_{\beta=1}^{\alpha-1} [k_i^{(\beta)}, k_j^{(\alpha-\beta)}]\xi^{(n-\alpha)} + \xi^{(\alpha)} \sum_{\beta=1}^{n-\alpha-1} [\kappa_i^{(\beta)}, \kappa_j^{(n-\alpha-\beta)}] \right\} \\ &- \sum_{\alpha=1}^{n-1} \left\{ k_j^{(\alpha)}\left[\kappa_i^{(n-\alpha)} - k_i^{(n-\alpha)} - \sum_{\gamma=1}^{n-\alpha-1} (k_i^{(\gamma)}\xi^{(n-\alpha-\gamma)} - \xi^{(\gamma)}\kappa_i^{(n-\alpha-\gamma)})\right] \right. \\ &\left. - \left[\kappa_i^{(\alpha)} - k_i^{(\alpha)} - \sum_{\beta=1}^{\alpha-1} (k_i^{(\beta)}\xi^{(\alpha-\beta)} - \xi^{(\beta)}\kappa_i^{(\alpha-\beta)})\right] K_j^{(n-\alpha)} - (i \leftrightarrow j) \right\} . \end{aligned} \qquad \text{(B3b)}$$

Only the six double sums survive the initial direct cancellation, and these will be shown to cancel in pairs.

$$C_{ij} = C_{ij}^{(1)} + C_{ij}^{(2)} + C_{ij}^{(3)} \ ,$$

$$C_{ij}^{(1)} = \sum_{\alpha=2}^{n-1} \sum_{\beta=1}^{\alpha-1} [k_i^{(\beta)}, k_j^{(\alpha-\beta)}]\xi^{(n-\alpha)} - \sum_{\alpha=1}^{n-2} \sum_{\gamma=1}^{n-\alpha-1} (k_i^{(\alpha)}k_j^{(\gamma)} - k_j^{(\alpha)}k_i^{(\gamma)})\xi^{(n-\alpha-\gamma)} \ ,$$

$$C_{ij}^{(2)} = \sum_{\alpha=1}^{n-2} \sum_{\gamma=1}^{n-\alpha-1} (k_i^{(\alpha)}\xi^{(\gamma)}\kappa_j^{(n-\alpha-\gamma)} - k_j^{(\alpha)}\xi^{(\gamma)}\kappa_i^{(n-\alpha-\gamma)}) - \sum_{\alpha=2}^{n-1} \sum_{\beta=1}^{\alpha-1} (k_i^{(\beta)}\xi^{(\alpha-\beta)}\kappa_j^{(n-\alpha)} - k_j^{(\beta)}\xi^{(\alpha-\beta)}\kappa_i^{(n-\alpha)}),$$

$$C_{ij}^{(3)} = -\sum_{\alpha=1}^{n-2} \sum_{\beta=1}^{n-\alpha-1} \xi^{(\alpha)}[\kappa_i^{(\beta)}, \kappa_j^{(n-\alpha-\beta)}] + \sum_{\alpha=2}^{n-1} \sum_{\beta=1}^{\alpha-1} \xi^{(\beta)}(\kappa_i^{(\alpha-\beta)}\kappa_j^{(n-\alpha)} - \kappa_j^{(\alpha-\beta)}\kappa_i^{(n-\alpha)}) \ .$$

In the second double sum of $C_{ij}^{(1)}$, let $\alpha'=\alpha+\gamma$. It then becomes

$$-\sum_{\alpha'=2}^{n-1}\sum_{\alpha=1}^{\alpha'-1}(k_i^{(\alpha)}k_j^{(\alpha'-\alpha)}-k_j^{(\alpha)}k_i^{(\alpha'-\alpha)})\xi^{(n-\alpha')}$$
$$=-\sum_{\alpha'=2}^{n-1}\sum_{\alpha=1}^{\alpha'-1}[k_i^{(\alpha)},k_j^{(\alpha'-\alpha)}]\xi^{(n-\alpha')}\ ,$$

so that $C_{ij}^{(1)}=0$. Similarly, let $\alpha-\beta=\gamma$ in the second double sum of $C_{ij}^{(2)}$. The result is

$$-\sum_{\beta=1}^{n-2}\sum_{\gamma=1}^{n-\beta-1}(k_i^{(\beta)}\xi^{(\gamma)}\kappa_j^{(n-\beta-\gamma)}-k_j^{(\beta)}\xi^{(\gamma)}\kappa_i^{(n-\beta-\gamma)}),$$

which exactly cancels the first sum, so that $C_{ij}^{(2)}=0$. Finally, with $\alpha-\beta=\beta'$, and $n-\alpha=n-\alpha'-\beta'$ in the second double sum of $C_{ij}^{(3)}$, it too exactly cancels the first double sum, leaving $C_{ij}^{(3)}=0$:

$$\sum_{\alpha'=1}^{n-2}\sum_{\beta'=1}^{n-\alpha'-1}\xi^{(\alpha')}(\kappa_i^{(\beta')}\kappa_j^{(n-\alpha'-\beta')}-\kappa_j^{(\beta')}\kappa_i^{(n-\alpha'-\beta')})$$
$$=\sum_{\alpha'=1}^{n-2}\sum_{\beta'=1}^{n-\alpha'-1}\xi^{(\alpha')}[\kappa_i^{(\beta')},\kappa_j^{(n-\alpha'-\beta')}]\ .$$

This completes the demonstration that Eq. (2.20) or (B1c) is satisfied for all $n\geq 1$. Consequently, Eq. (2.19a) has a solution for all $n\geq 1$, and that solution is given by Eq. (2.19b).

APPENDIX C

We show that, with the assumption of an energy gap between the "ground state" and the first excited state and time-reversal invariance, the matrix elements of the 4-current for a "loosely bound" system of particles to order $1/c^2$ are in the form given by Lapidus and Chou.[3] This current is conserved to order $1/c^2$, and the LET for Compton scattering follows from an argument by Lapidus and Chou.[3]

Consider the charge density as expressed in terms of constituent particle variables,

$$j_0=\sum_\mu\left\{\epsilon_\mu\delta^3(\vec{r}_\mu)+\frac{1}{4m_\mu c^2}(g_\mu^s-g_\mu^l)\nabla_\mu{}^2\delta^3(\vec{r}_\mu)-\frac{1}{2m_\mu c^2}(g_\mu^s-g_\mu^l)\vec{s}_\mu\cdot[\vec{p}_\mu\times\vec{\nabla}_\mu\delta^3(\vec{r}_\mu)+\text{H.c.}]\right\},\tag{C1a}$$

where ϵ_μ (μ_μ^0) is the charge (intrinsic magnetic moment) of particle μ,

$$g_\mu^s=\begin{cases}0\ , & s_\mu=0\\ \dfrac{\mu_\mu^0}{s_\mu}+\dfrac{\epsilon_\mu}{2m_\mu s_\mu}\ , & s_\mu\neq 0\end{cases}\tag{C1b}$$

$$g_\mu^l=\frac{\epsilon_\mu}{2m_\mu}\ ,\tag{C1c}$$

and $\vec{\nabla}_\mu$ is the gradient with respect to $\vec{r}_\mu$. The matrix elements of j_0 will be determined to order $1/c^2$ in terms of relativistic c.m. variables characterized by $\Pi^{(1)}=0$ and $\beta\vec{V}^{(1)}=0$, so that the corresponding state vectors take the form[7] $|\vec{P}\rangle\otimes|S,M\rangle\equiv|\vec{P};S,M\rangle$, where $\vec{P}$, S, and M represent the total momentum, spin, and spin projection of the composite system, and all other quantum numbers have been suppressed.

$$\langle\vec{P}_2;S,M_2|j_0|\vec{P}_1;S,M_1\rangle=\left\langle\vec{P}_2;S,M_2\left|\sum_\mu\left(\delta^3(\vec{R}+\vec{\rho}_\mu)+\frac{i^2}{4m_\mu c^2}(g_\mu^s-g_\mu^l)\left[\left(\pi_\mu+\frac{m_\mu}{M}P\right)_k,\left[\left(\pi_\mu+\frac{m_\mu}{M}P\right)_k,\delta^3(\vec{R}+\vec{\rho}_\mu)\right]\right]\right.\right.$$
$$-\frac{i}{2m_\mu c^2}(g_\mu^s-g_\mu^l)\vec{\sigma}_\mu\cdot\left\{\left(\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}\right)\times\left[\left(\vec{\pi}_\mu+\frac{m_\mu}{M}\vec{P}\right),\delta^3(\vec{R}+\vec{\rho}_\mu)\right]+\text{H.c.}\right\}$$
$$\left.\left.+\frac{i}{c^2}[\phi^{(1)},\epsilon_\mu\delta^3(\vec{R}+\vec{\rho}_\mu)]\right)\right|\vec{P}_1;S,M_1\right\rangle\ ,\tag{C2a}$$

where repeated latin indices are summed, and $\phi^{(1)}$ is given by Eq. (2.26b) with $\Pi^{(1)}=0$ and $\beta\to 0$. Expanding $\delta^3(\vec{R}+\vec{\rho}_\mu)$ about $\vec{R}$,

$$\delta^3(\vec{R}+\vec{\rho}_\mu)=\delta^3(\vec{R})+\rho_\mu^k\nabla_R^k\delta^3(\vec{R})+\frac{\rho_\mu^k\rho_\mu^l\nabla_R^k\nabla_R^l\delta^3(\vec{R})}{2!}+\cdots,\tag{C2b}$$

where $\vec{\nabla}_R$ is the gradient with respect to $\vec{R}$, is equivalent to a power series in $\vec{P}_2-\vec{P}_1$ when considered between states of total momentum $\vec{P}_1$ and $\vec{P}_2$. Since we are interested in Compton scattering from a composite system, for which an energy gap is assumed, in the limit as the momentum transfer approaches zero, the power series converges, and only terms to order $(\vec{P}_2-\vec{P}_1)^2$ are important.

Time-reversal invariance is invoked in the following way. Let $\tau_Q^{(1)}$ denote a vector operator in standard form, constructed only from the internal variables such that it is either even or odd under time reversal. Let $S_Q^{(1)}$ denote the spin operator $\vec{S}$ of the composite system, expressed in standard form. The matrix elements of $\tau_Q^{(1)}$ between the states $|SM\rangle$ and $|SM'\rangle$ are related to those of $S_Q^{(1)}$ by the Wigner-Eckart theorem[27] as follows:

$$\langle S, M'|\tau_Q^{(1)}|S, M\rangle = \frac{\langle S\|\tau^{(1)}\|S\rangle}{\langle S\|S^{(1)}\|S\rangle}\langle S, M'|S_Q^{(1)}|S, M\rangle \ , \tag{C3a}$$

where M and M' are the projection of S in the direction of quantization, $\langle S\|\tau^{(1)}\|S\rangle$ and $\langle S\|S^{(1)}\|S\rangle$ are the reduced matrices of $\vec{\tau}$ and $\vec{S}$, and $S\neq 0$, so that the quotient of the two reduced matrices is well defined. Using the phase convention of Wigner,[28] time-reversal invariance implies that

$$\langle S, M'|\tau_Q^{(1)}|S, M\rangle = (-)^{M'-M}\theta(\vec{\tau})\langle S, -M|\tau_Q^{(1)}|S, -M'\rangle \ , \tag{C3b}$$

$$\langle S, M'|S_Q^{(1)}|S, M\rangle = (-)^{M'-M+1}\langle S, -M|S_Q^{(1)}|S, -M'\rangle \ , \tag{C3c}$$

where θ is either $+1$ or -1 according to whether its argument is even or odd under time reversal. So, from Eq. (C3a), we have

$$[\theta(\vec{\tau})+1]\langle S, M'|\tau_Q^{(1)}|S, M\rangle = 0 \ , \tag{C3d}$$

where, since the reduced matrices are independent of M and M' and M and M' are arbitrary, the replacement $M'\to -M$ and $M\to -M'$ has been made. Consequently, by comparison, those matrix elements of internal variables $\tau_Q^{(1)}$ which are zero by time reversal invariance are easily identified.

With the above remarks and assumptions in mind, Eq. (C2) can be evaluated in a straightforward, though careful, manner to yield

$$(\vec{P}_2; S, M_2|j_0|\vec{P}_1; S, M_1\rangle$$
$$= \left\langle \vec{P}_2; S, M_2\right|\left[\epsilon\delta^3(\vec{R}) + \tau_{ij}(P_2-P_1)_i(P_2-P_1)_j\delta^3(\vec{R}) + \frac{i}{Mc^2}\left(g - \frac{\epsilon}{2M}\right)\vec{S}\cdot\vec{P}_2\times\vec{P}_1\delta^3(\vec{R})\right]\left|\vec{P}_1; S, M_1\right\rangle + O(P^3) \ , \tag{C4a}$$

where

$$\epsilon = \sum_\mu \epsilon_\mu \ , \tag{C4b}$$

$$M = \sum_\mu m_\mu \ , \tag{C4c}$$

$$\vec{S} = \sum_\mu (\vec{\rho}_\mu\times\vec{\pi}_\mu + \vec{\sigma}_\mu) \ , \tag{C4d}$$

$$\tau_{ij} = -\tfrac{1}{2}\sum_\mu \epsilon_\mu\rho_\mu^i\rho_\mu^j - \sum_\mu \frac{1}{4m_\mu c^2}(g_\mu^s - g_\mu^l)\delta_{ij} + \sum_\mu \frac{1}{2m_\mu c^2}(g_\mu^s - g_\mu^l)[(\sigma_\mu\times\pi_\mu)^i\rho_\mu^j + \text{H.c.}]$$
$$+ \sum_{\mu\nu}\frac{\epsilon_\mu}{8m_\nu Mc^2}[\rho_\mu^i(\rho_\nu^j\vec{\pi}_\nu^{\,2} + \text{H.c.}) + \text{H.c.}] - \sum_{\mu\nu}\frac{\epsilon_\mu}{2m_\nu Mc^2}(\sigma_\nu\times\pi_\nu)^i\rho_\mu^j \ . \tag{C4e}$$

The gyromagnetic ratio of the composite system g, defined as μ/S where μ and S are the magnetic moment and spin of the composite system, enters the calculation by way of the Wigner-Eckart theorem already discussed above. Assuming no exchange currents, the magnetic moment of the composite system is given by

$$\mu = \left\langle S, M=S\right|\sum_\mu (g_\mu^l\vec{l}_\mu + g_\mu^s\vec{\sigma}_\mu)\cdot\hat{n}\left|S, M=S\right\rangle \ , \tag{C5a}$$

where $\hat{n}$ is the direction of quantization, and

$$\vec{l}_\mu = \vec{\rho}_\mu\times\vec{\pi}_\mu \ , \tag{C5b}$$

so that, from Eq. (C3a),

$$\mu = \sum_\mu\left(g_\mu^l\frac{\langle S\|\vec{l}_\mu\|S\rangle}{\langle S\|\vec{S}\|S\rangle} + g_\mu^s\frac{\langle S\|\vec{\sigma}_\mu\|S\rangle}{\langle S\|\vec{S}\|S\rangle}\right)S \ . \tag{C5c}$$

This serves to define the gyromagnetic ratio of the composite system in terms of those of the constituent particles. Furthermore, in deriving Eq. (C4), explicit use of the assumed energy gap has been made to relate the following matrix elements to order $1/c^2$:

$$0 = \langle\vec{P}_2; S, M_2|i[h, m_\mu\rho_\mu^i\rho_\mu^j]|\vec{P}_1; S, M_1\rangle/c^2$$
$$= \langle\vec{P}_2; S, M_2|\rho_\mu^i\pi_\mu^j + \pi_\mu^i\rho_\mu^j|\vec{P}_1; S, M_1\rangle/c^2 \ , \tag{C6}$$

where h is the internal Hamiltonian given by Eq. (2.29) with $\Pi^{(1)}=0$ and $\beta\to 0$. Equation (C4) is seen to be more general than the corresponding equation of Lapidus and Chou,[3] owing to an explicit account of the internal structure of the composite system.

The matrix elements of the current operator $\vec{j}$ need only be determined to order $1/c$. This is easily accomplished from the techniques discussed above. In terms of the constituent particle variables,

$$\vec{j}=\sum_{\mu}\{[g_{\mu}^{l}\vec{p}_{\mu}\delta^{3}(\vec{r}_{\mu})+\text{H.c.}]+g_{\mu}^{s}\vec{s}_{\mu}\times\vec{\nabla}_{\mu}\delta^{3}(\vec{r}_{\mu})\}\ , \tag{C7a}$$

from which it is easily shown that, in terms of the c.m. variables to order $1/c$,

$$\langle\vec{P}_2;S,M_2|\vec{j}|\vec{P}_1;S,M_1\rangle=\left\langle\vec{P}_2;S,M_2\left|\left[\frac{\epsilon}{2M}(\vec{P}_1+\vec{P}_2)\delta^3(\vec{R})+ig\vec{S}\times(\vec{P}_2-\vec{P}_1)\delta^3(\vec{R})\right]\right|\vec{P}_1;S,M_1\right\rangle+O(P^2)\ , \tag{C7b}$$

where all terms have been previously defined. Equation (C7b) is to be compared with Eq. (45) of Lapidus and Chou.[3] Because the matrix elements of the 4-current for the composite system are in the form given by Eq. (45) of Lapidus and Chou[3] the LET for Compton scattering follows in the same way as their general proof.

---

*Work performed under the auspices of the U. S. Atomic Energy Commission.

†Present address.

[1]L. L. Foldy, Phys. Rev. 122, 275 (1961).

[2]B. Bakamjian and L. H. Thomas, Phys. Rev. 92, 1300 (1953). This paper contains an early discussion of relativistic c.m. variables.

[3]F. E. Low, Phys. Rev. 96, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *ibid.* 96, 1433 (1954); L. I. Lapidus and Chou K.-C., Zh. Eksp. Teor. Fiz. 39, 1286 (1960) [Sov. Phys.—JETP 12, 898 (1961)].

[4]S. D. Drell and A. C. Hearn, Phys. Rev. Lett. 16, 908 (1966); S. B. Gerasimov, Yad. Fiz. 2, 598 (1965) [Sov. J. Nucl. Phys. 2, 430 (1966)].

[5]G. Barton (unpublished); G. Barton and N. Dombey, Phys. Rev. 162, 1520 (1967).

[6]S. J. Brodsky and J. R. Primack, Phys. Rev. 174, 2071 (1968); Ann. Phys. (N.Y.) 52, 315 (1969).

[7]H. Osborn, Phys. Rev. 176, 1514 (1968); 176, 1523 (1968).

[8]R. A. Krajcik and L. L. Foldy, Phys. Rev. Lett. 24, 545 (1970).

[9]F. E. Close and L. A. Copley, Nucl. Phys. B19, 477 (1970).

[10]F. E. Close and H. Osborn, Phys. Rev. D 2, 2127 (1970).

[11]H. Grotch and R. Kashuba, Phys. Rev. A 5, 527 (1972).

[12]M. K. Liou and M. I. Sobel, Phys. Rev. C 7, 2044 (1973).

[13]J. L. Friar, private communication; J. L. Friar, Phys. Lett. 43B, 108 (1973). The expanded version of this letter is correct, however: J. L. Friar, Ann. Phys. (N. Y.) 81, 332 (1973).

[14]S. Gartenhaus and C. Schwartz, Phys. Rev. 108, 482 (1957). See in particular H. Osborn, Phys. Rev. 176, 1514 (1968).

[15]Yu. M. Shirokov, Zh. Eksp. Teor. Fiz. 36, 474 (1959) [Sov. Phys.—JETP 9, 330 (1959)]. Corrections to Eq. (22): The term

$$-\frac{(m_2-m_1)}{2m_1m_2M}\left(\vec{p}^2\,\vec{P}\cdot\frac{\partial}{\partial\vec{p}}+\vec{p}'^2\,\vec{P}\cdot\frac{\partial}{\partial\vec{p}'}\right)W_0(\vec{p},\vec{p}')$$

is missing, and the spin-dependent terms should have the opposite sign.

[16]B. S. Bhaker, Phys. Rev. C 2, 905 (1970). Corrections to Eq. (2.14): The spin-dependent terms should have the opposite sign, and are too small by a factor of 2, i.e., $+\frac{1}{8}m^2$ should be $-\frac{1}{4}m^2$.

[17]E. Inonu and E. P. Wigner, Proc. Natl. Acad. Sci. U.S.A. 39, 510 (1953). See in particular H. Bacry and J. Levy-LeBlond, J. Math. Phys. 9, 1605 (1968), for an illuminating discussion of contractions of Lie algebras.

[18]V. Bargmann, Ann. Math. 59, 1 (1954). See in particular J. Levy-LeBlond, J. Math. Phys. 4, 776 (1963), for an excellent discussion of the Galilean group.

[19]E. Inonu and E. P. Wigner, Nuovo Cimento 9, 705 (1952); M. Hamermesh, Ann. Phys. (N.Y.) 9, 518 (1960).

[20]J. D. Louck, J. Mol. Spectrosc. 15, 68 (1965).

[21]By condition (3), $[\xi^{(n)},\vec{P}]=0$, so that one can see from Eq. (2.19a) that the integrand of the integral in Eq. (2.19b) commutes with $\vec{P}$. (Using the commutator $[\mathcal{K}_i,\mathcal{H}]=i\mathcal{P}_i$, one can show that the condition $[\xi^{(l)},\vec{P}]=0$, $l=1,2,\ldots,n$, serves to limit the form of $h^{(m)}(\beta)$, $m=1,2,\ldots,n-1$.) Consequently, one can define an integral of the form $I(\vec{P})=\int_0^{\vec{P}}d\vec{P}\cdot\vec{f}$, where $[f_i,P_j]=0$ and $[R_i,f_j]-(i\leftrightarrow j)=0$, as that operator which satisfies the commutation equation $[\vec{R},I(\vec{P})]=i\vec{f}$, and whose matrix matrix elements are zero between a state of total momentum zero and any other state.

[22]Note that $\vec{\mathcal{K}}$ as defined by Close and Osborn (Ref. 10) is the negative of ours.

[23]R. A. Krajcik, thesis, Case Western Reserve University, 1970 (unpublished).

[24]For a discussion of separability and scattering theory in a three-particle system, refer to F. Coester, Helv. Phys. Acta 38, 7 (1965).

[25]L. L. Foldy and R. A. Krajcik, Phys. Rev. Lett. 32, 1025 (1974).

[26]F. A. Zhivopistsev, A. M. Perolomov, and Yu. M. Shirokov, Zh. Eksp. Teor. Fiz. 36, 478 (1959) [Sov. Phys.—JETP 9, 333 (1959)]. See also N. S. Zelenskaya and Yu. M. Shirokov, *ibid.* 41, 1934 (1961) [14, 1374 (1962)]. In the latter reference, change the sign of the first spin-dependent term in Eq. (2).

[27]See for example A. Messiah, *Quantum Mechanics II* (North-Holland, Amsterdam, 1962).

[28]See for example E. P. Wigner, *Group Theory* (Academic, New York, 1959).