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A spin-$\frac{1}{2}$ positive-energy relativistic wave equation*

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A relativistic wave equation which describes a particle having mass m, spin $\frac{1}{2}$, and only positive-energy eigenvalues is presented. The theory apparently evades the conditions which prevented minimal coupling to electromagnetism in the spinless positive-energy theory of Dirac. A more general method for reducing the Klein-Gordon operator is pointed out.

I. INTRODUCTION

In 1971 Dirac[1] proposed a relativistic equation describing a massive, spin-zero positive-energy particle. Remarkably, although the theory admits a conserved current, with positive-definite probability density, the particle cannot be coupled minimally to the electromagnetic field.

Interest in this structure has expanded following the appearance of a paper[2] by Biedenharn, Han, and van Dam in which they presented a generalization and interpretation of Dirac's new theory. They have constructed a series of relativistic wave equations which are of successively higher order in the momentum operators (Dirac's equation being the lowest-order case) and which describe, for a given order, positive-energy particles of a corresponding higher spin, s. Each such relativistic equation describes, as well, particles of all spins lower than s, and shares the no-minimal-coupling facet of Dirac's theory.

Of considerably greater importance is Biedenharn, Han, and van Dam's interpretation of the generalized theory as a description of a relativistic composite system bound by harmonic-oscillator forces when viewed within the framework of a new quantum version[2] of the old front-form description[3] of classical relativistic mechanics. The combined theory then exhibits a mass spectrum $m^2 \propto$ spin.

Subsequently, the need for higher-order wave equations has been obviated by the formulation of the theory[4] as a complete Poincaré algebra. This viewpoint has been employed to construct[5] a relativistic theory which exhibits a multiplet structure as well as a mass-spin spectrum, and recently[6] Biedenharn and van Dam have reported a relationship of such quantum front models to the dual resonance model.

The purpose of the present communication is to present a relativistic wave equation, linear in the momentum operators, which is a description of a massive, spin-$\frac{1}{2}$ particle having only a positive-energy spectrum, i.e., the spin-$\frac{1}{2}$ state of the new quantum[2] front model. The equation is only trivially different from one of the secondary equations written down by Dirac[1] for the spinless case. However, it may be of some interest, not only for its spin-$\frac{1}{2}$ character, but also because it appears to neatly evade those pathologies which serve to prevent a minimal electromagnetic coupling in Dirac's theory and in the higher-spin generalizations of Ref. 2. Additionally, and perhaps of some greater interest, the spin-$\frac{1}{2}$ theory makes use of additional generality in the application of the new technique[1] of "factorizing" the Klein-Gordon equation via commutation relations.

In Sec. II, for completeness, for comparison with what follows, and in order to fix the notation, we include a brief review of Dirac's work. In Sec. III the spin-$\frac{1}{2}$ theory is presented, and its coupling to electromagnetism is formulated and discussed in Sec. IV. In these two sections, we carry along a parallel discussion of a different formulation of the spinless theory in order to delineate as clearly as possible the mechanism which avoids the no-coupling restriction in the spin-$\frac{1}{2}$ case. Section V contains a short summary of our results and conclusions. In an appendix we have added a discussion of Dirac's theory of the relativistic electron after the fashion of Secs. III and IV.

II. DIRAC'S POSITIVE-ENERGY THEORY[1]

The massive, spin-zero, positive-energy wave function of Dirac, $\psi(x^\mu, q_1, q_2)$, is a single-component wave function involving, in addition to the Minkowski space-time, two dimensionless harmonic-oscillator degrees of freedom described by the commuting dynamical variables q_1 and q_2. Let η_1 and η_2 denote the commuting dynamical conjugates,[7] so that $[q_j, \eta_k] = i\delta_{jk}$ $(j, k = 1, 2)$ and define a column matrix of four rows by

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} . \tag{1}$$

Then the new relativistic equation proposed by Dirac reads[8]

$$(\alpha_\mu P^\mu + i\beta m)Q\psi(x^\mu, q_j) = 0 , \tag{2}$$

where m is a positive number, and the real 4×4 matrices α_μ and β, satisfying the relations

$$\alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2\beta g_{\mu\nu} , \tag{3}$$

may be taken to be

$$\alpha_0 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} , \quad \alpha_1 = \begin{bmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}$$
$$\alpha_2 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} , \quad \alpha_3 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} , \tag{4}$$
$$\beta = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} .$$

The similarity of Eq. (2) to the usual Dirac equation for the electron is completely superficial. Equation (2) is actually four equations on the single-component wave function ψ. Let

$$T_a = \sum_b (\alpha_\mu P^\mu + i\beta m)_{ab} Q_b . \tag{5}$$

Then Eq. (2) reads

$$T_a \psi = 0, \quad a = 1, 2, 3, 4 . \tag{6}$$

The condition that there exist a wave function ψ which is simultaneously a solution to the four equations (6) is that[1]

$$[T_a, T_b]\psi = 0 . \tag{7}$$

The commutator may be directly evaluated to yield

$$[T_a, T_b] = \beta_{ab}(P^2 - m^2) , \tag{8}$$

and since β has nonzero matrix elements, any solution ψ of Eq. (2) satisfies as well the Klein-Gordon equation

$$(P^2 - m^2)\psi = 0 . \tag{9}$$

Dirac's new equation is not a Lorentz invariant; rather it is Lorentz covariant. If it is satisfied in any one frame, then it is satisfied in every Lorentz frame. The wave function ψ transforms under the Lorentz group generated by

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} , \tag{10}$$

where $L_{\mu\nu}$ denotes the usual space-time generators and the antisymmetric Hermitian operator $S_{\mu\nu}$ generates Lorentz transformations on the space of functions of q_1 and q_2. Let the operators $\vec{J} \equiv (J_1, J_2, J_3)$ and $\vec{K} \equiv (K_1, K_2, K_3)$ generate rotations and pure Lorentz boosts, respectively, on this same manifold. Then these operators may be defined to be[1]

$$\begin{aligned} J_1 &= -S_{23} = \tfrac{1}{2}(q_1 q_2 + \eta_1 \eta_2) , \\ J_2 &= -S_{31} = \tfrac{1}{4}(q_1^2 + \eta_1^2 - q_2^2 - \eta_2^2) , \\ J_3 &= -S_{12} = \tfrac{1}{2}(q_2 \eta_1 - q_1 \eta_2) , \\ K_1 &= S_{10} = \tfrac{1}{4}(q_1^2 - \eta_1^2 - q_2^2 + \eta_2^2) , \\ K_2 &= S_{20} = \tfrac{1}{2}(\eta_1 \eta_2 - q_1 q_2) , \\ K_3 &= S_{30} = \tfrac{1}{2}(q_1 \eta_1 + \eta_2 q_2) . \end{aligned} \tag{11}$$

With this realization of the Lie algebra, the Casimir operators of the Lorentz group become[8]

$$\begin{aligned} F &\equiv \tfrac{1}{4} S_{\mu\nu} S^{\mu\nu} = \tfrac{1}{2}(J^2 - K^2) \equiv -\tfrac{3}{8} , \\ G &\equiv \tfrac{1}{8} \epsilon^{\mu\nu\alpha\beta} S_{\mu\nu} S_{\alpha\beta} = \vec{J}\cdot\vec{K} = \vec{K}\cdot\vec{J} \equiv 0 , \end{aligned} \tag{12}$$

which defines a Majorana representation.[9] As is well known, there exists in this case a unique vector operator, V_μ, which together with $S_{\mu\nu}$ serves to generate the Lie algebra of SO(3, 2):

$$\begin{aligned} &[V_\mu, V_\nu] = iS_{\mu\nu} \\ &[V_\mu, S_{\alpha\beta}] = i(g_{\mu\beta} V_\alpha - g_{\mu\alpha} V_\beta) \\ &[S_{\mu\nu}, S_{\alpha\beta}] = i(g_{\mu\alpha} S_{\nu\beta} - g_{\mu\beta} S_{\nu\alpha} + g_{\nu\beta} S_{\mu\alpha} - g_{\nu\alpha} S_{\mu\beta}) . \end{aligned} \tag{13}$$

With the realization (11), the operators V_0 and $\vec{V} \equiv (V_1, V_2, V_3)$ are[10]

$$\begin{aligned} V_0 &= \tfrac{1}{4}(q_1^2 + q_2^2 + \eta_1^2 + \eta_2^2) , \\ V_1 &= \tfrac{1}{2}(-q_1 \eta_1 + q_2 \eta_2) , \\ V_2 &= \tfrac{1}{2}(q_1 \eta_2 + q_2 \eta_1) , \\ V_3 &= \tfrac{1}{4}(q_1^2 + q_2^2 - \eta_1^2 - \eta_2^2) , \end{aligned} \tag{14}$$

satisfying the identity

$$D \equiv V_\mu V^\mu \equiv -\tfrac{1}{2} , \tag{15}$$

which of course follows also from Eqs. (12) and (13).

From Eq. (2), Dirac obtains equations which are of second order in the η_j but linear in P^μ by multiplying Eq. (2) with any 4×4 matrix Ω and contracting with the transpose of Q:

$$Q^T \Omega [\alpha_\mu P^\mu + i\beta m] Q\psi = 0 . \tag{16}$$

Fifteen independent equations may be obtained in this way ($\Omega = \alpha_1 \alpha_2 \alpha_3$ gives identically zero), and these may be chosen to be

$$(V_\mu P^\mu - \tfrac{1}{2} m)\psi = 0 , \tag{17a}$$

$$(m V_\mu - \tfrac{1}{2} P_\mu + i S_{\mu\nu} P^\nu)\psi = 0 , \tag{17b}$$

$$(im S_{\mu\nu} + V_\mu P_\nu - V_\nu P_\mu)\psi = 0 , \tag{17c}$$

and

$$W^{\mu}\psi = 0 \ , \tag{17d}$$

where W^{μ} is the Pauli-Lubanski operator following from Eq. (10):

$$W^{\mu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}P_{\beta} \ . \tag{18}$$

It follows from Eq. (17d) that the solutions to Eq. (2) carry zero spin. Equation (17a) is just Majorana's equation,[9] and Dirac's new equation effectively projects out of the Majorana spectrum the timelike [via Eq. (9)] spinless solution.[1,2] The fact that Eq. (2) has only positive-energy solutions has been shown directly,[1] but it is sufficient to recall that the timelike solutions of Majorana's equation have positive energy only.[11]

Dirac has remarked,[1] without explicit proof, that his new equation does not admit the possibility of minimal coupling to electromagnetism, this in spite of the existence of the conserved (Majorana) current

$$J^{\mu}(x) = \int dq_1 dq_2 \psi^* V^{\mu}\psi \ . \tag{19}$$

In their recent paper,[2] Biedenharn, Han, and van Dam have presented a proof of this statement along with a very thorough discussion of Dirac's system. It is not necessary to review here the details of their proof since we shall recover the relevant results in another fashion in Sec. IV. It is sufficient to remark that from Eq. (2) with the replacement $P_{\mu} \to \Pi_{\mu} \equiv P_{\mu} + eA_{\mu}$ they are able to obtain in particular the relations

$$\mathfrak{F}_{\mu\nu}^{\text{dual}} V^{\nu}\psi = 0 \ , \tag{20a}$$

$$\mathfrak{F}_{\mu\nu}^{\text{dual}} S^{\mu\nu}\psi = 0 \ , \tag{20b}$$

where $\mathfrak{F}_{\mu\nu}$ is the usual electromagnetic field tensor. It then follows[2] that a coupling to arbitrary electromagnetic fields implies that ψ is invariant under the complete SO(3, 2) group generated by the operators V_{μ} and $S_{\mu\nu}$. ψ is then not a function of the q_j, and the wave equation then implies that ψ is a constant. The higher-order, higher-spin generalizations which are presented in Ref. 2 all lead to Eqs. (20) and thus to the same no-coupling conclusion. The spin-$\frac{1}{2}$ equation which we shall discuss is different from that of Ref. 2 so that the no-coupling conditions may be avoided.

III. A FREE SPIN-$\frac{1}{2}$ POSITIVE-ENERGY THEORY

The basic equation which defines the positive-energy spin-$\frac{1}{2}$ theory is

$$(m V_{\mu} - P_{\mu} + iS_{\mu\nu}P^{\nu})\psi(x^{\mu}, q_1, q_2) = 0 \ , \tag{21}$$

where ψ is again a single function of the indicated arguments. However, in order to facilitate the discussion of electromagnetism in Sec. IV, and for comparison with Sec. II, we shall analyze the following equation:

$$(m V_{\mu} - \kappa P_{\mu} + iS_{\mu\nu}P^{\nu})\psi = 0 \ , \tag{22}$$

where κ is an arbitrary real number. Equation (22) reduces to Eq. (21) above for $\kappa = 1$; for $\kappa = \frac{1}{2}$ it reduces to Eq. (17b) and is thus a simple generalization of that equation. However, we shall show

(i) that the only permissible values of κ are $\frac{1}{2}$ and 1;

(ii) that these values imply spins 0 and $\frac{1}{2}$, respectively;

(iii) that with $\kappa = \frac{1}{2}$ the *entire* physical content of Sec. II may be recovered; and

(iv) in Sec. IV, that the no-coupling conditions, Eqs. (20), obtain for the spinless case but do not arise for the case of spin $\frac{1}{2}$, i.e., $\kappa = 1$.

In Eq. (22) we are again presented with four equations for the single function ψ. Define

$$T_{\mu} \equiv m V_{\mu} - \kappa P_{\mu} + iS_{\mu\nu}P^{\nu} \ . \tag{23}$$

The condition that a solution to Eq. (22) exists is that

$$[T_{\mu}, T_{\nu}]\psi = 0 \ . \tag{24}$$

By means of the Lie algebra (13), the commutator may be expressed as

$$[T_{\mu}, T_{\nu}] = i(m^2 - P^2)S_{\mu\nu} + P_{\nu}T_{\mu} - P_{\mu}T_{\nu} \ , \tag{25}$$

and the last two terms, the "extra" terms when compared to the form of Eq. (8), vanish when applied to a solution of Eq. (22). Then the condition that ψ be a solution of Eq. (22) is that

$$(P^2 - m^2)S_{\mu\nu}\psi = 0 \ . \tag{26}$$

Since the $S_{\mu\nu}$ generate the Majorana representation of the Lorentz group, we may contract Eq. (26) with $S^{\mu\nu}$ and use Eq. (12) to obtain

$$(P^2 - m^2)\psi = 0 \ . \tag{27}$$

Therefore, the six equations (26) imply either that ψ vanishes or that it satisfies the Klein-Gordon equation.

Before continuing the analysis, we shall develop a few convenient identities. We remark that in any representation, such as that of Majorana, which contains a unique vector, V_{μ}, we must have

$$V^{\mu}S_{\mu\nu} = i\lambda V_{\nu} \ , \tag{28}$$

with λ a constant. Contracting both sides on the left with V_{ν} and using (13), we immediately obtain

$$-\frac{i}{2} S_{\mu\nu}S^{\mu\nu} = i\lambda V^{\mu}V_{\mu} \ . \tag{29}$$

Equations (12) and (15) then yield

$$\lambda = -2F/D = -\tfrac{3}{2} \ . \tag{30}$$

Similar arguments yield

$$S_{\mu\nu}V^{\nu}=i\lambda V_{\mu}\ , \tag{31}$$

as well as

$$\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}V_{\beta}=0\ . \tag{32}$$

Finally, multiplying (32) with V^{σ} and using (13) twice yields

$$\epsilon^{\mu\sigma\alpha\beta}S_{\alpha\beta}=i\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}S_{\beta}{}^{\sigma}. \tag{33}$$

Now continuing the analysis, consider[12] the contraction of Eq. (22) with P^{μ}:

$$(m\,V_{\mu}P^{\mu}-\kappa P^{2})\psi=0 \tag{34}$$

implies, via Eq. (27), Majorana's equation[9]

$$(V_{\mu}P^{\mu}-\kappa m)\psi=0\ , \tag{35}$$

while contracting Eq. (22) with V^{μ} and using (28) yields

$$[mD-(\kappa+\lambda)V^{\mu}P_{\mu}]\psi=0\ . \tag{36}$$

Equations (35) and (36) together imply that the only consistent set of equations (22) are those for which the constant κ satisfies the quadratic equation:

$$\kappa^{2}+\lambda\kappa-D=0\ . \tag{37}$$

For the values of λ and D obtaining, this implies $\kappa=\frac{1}{2}$ or $\kappa=1$ as the only possibilities.

In particular, with κ restricted to these values, and therefore positive, Eqs. (27) and (35) restrict ψ to be a timelike solution of the usual form of Majorana's equation. It follows again[11] that ψ has only positive-energy eigenvalues.

Equation (22), like that of Dirac, is not a Lorentz invariant but is rather Lorentz covariant, the wave function ψ transforming via the generators $M_{\mu\nu}$ given in Eq. (10). The spin carried by the field for the two allowed values of κ may be computed with the Pauli-Lubanski operator from Eq. (18). We have

$$W^{\mu}W_{\mu}\equiv-\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}P^{2}+S_{\mu\alpha}S^{\mu\beta}P^{\alpha}P_{\beta}\ , \tag{38a}$$

or, using Eq. (12),

$$W^{2}=-2FP^{2}+S_{\mu\alpha}S^{\mu\beta}P^{\alpha}P_{\beta}\ . \tag{38b}$$

Saturating Eq. (22) with $S^{\mu\alpha}P_{\alpha}$ and using (31) and (35) yields

$$(S_{\mu\alpha}S^{\mu\beta}P^{\alpha}P_{\beta})\psi=(\lambda\kappa m^{2})\psi\ . \tag{39}$$

Thus, with (27),

$$W^{2}\psi=-m^{2}(2F-\lambda\kappa)\psi\ , \tag{40}$$

or, with the values at hand,

$$W^{2}\psi=\begin{cases}0, & \kappa=\frac{1}{2}\\ -\frac{3}{4}m^{2}\psi, & \kappa=1\end{cases} \tag{41}$$

so that the solutions to Eq. (22) are spinless if $\kappa=\frac{1}{2}$ and carry spin $\frac{1}{2}$ if $\kappa=1$.

We should remark at this point that we are not discussing an empty set of solutions. For $\kappa=\frac{1}{2}$, as we remarked earlier, Eq. (22) becomes Eq. (17b), so that any solution of Dirac's positive energy equation is also a solution of that equation. Dirac[1] has given the general momentum eigenfunction solution of his equation:

$$\psi_{D}=N\psi_{0}(q,p)\exp(-ip^{\mu}x_{\mu})\ , \tag{42}$$

where N is a normalization factor, and where

$$\psi_{0}(q,p)=\exp\left\{-\frac{1}{2(p_{0}+p_{3})}[m(q_{1}{}^{2}+q_{2}{}^{2})+ip_{1}(q_{1}{}^{2}-q_{2}{}^{2})-2ip_{2}(q_{1}q_{2})]\right\}\ . \tag{43}$$

The general momentum eigenfunction solution of the spin-$\frac{1}{2}$ equation (21) is

$$\psi=(Aq_{1}+Bq_{2})\psi_{0}(q,p)\exp(-ip^{\mu}x_{\mu})\ , \tag{44}$$

with A and B arbitrary. The spin-$\frac{1}{2}$ character is reflected in the two degrees of freedom A and B which transform one into the other under the action of the Lorentz group in the manner appropriate to a spin-$\frac{1}{2}$ system.[4]

The wave function (42) has been interpreted[2] as a description of the ground state of a relativistic composite system bound by a harmonic-oscillator interaction when analyzed in the new quantum front form of relativistic mechanics. The wave function (44) is just the first excited state of that system. Unfortunately, while the complete spectrum of the interacting system has been given a unified description as an interacting algebraic formulation of the Poincaré group,[4] no single relativistic wave equation describing the complete system has yet emerged.[13] Equation (22), as we have shown, also cannot be applied piecewise to the higher-spin excitations.[14]

It remains in this section only to show the relation of the theory defined by Eq. (22) to the complete formulation of the (noninteracting) positive-energy theory of Dirac outlined in Sec. II. We have so far obtained from Eq. (22) Eqs. (9) and, for $\kappa=\frac{1}{2}$, Eqs. (17a) and (17b). We shall obtain below Eqs. (17c) and (17d) for the spinless case.

We contract Eq. (22) with $\frac{1}{2}\epsilon^{\sigma\alpha\beta\mu}S_{\alpha\beta}$ and use Eqs. (32) and (18) to obtain

$$(-\kappa W^\sigma + \tfrac{1}{2}i\epsilon^{\sigma\alpha\beta\mu}S_{\alpha\beta}S_\mu{}^\nu P_\nu)\psi = 0 \; . \qquad (45)$$

The identity (33) then yields

$$(1-\kappa)W^\sigma\psi = 0 \; . \qquad (46)$$

For $\kappa=\frac{1}{2}$ (spin zero) this equation is just (17d); note, however, that *in the spin-$\frac{1}{2}$ case* ($\kappa=1$) *no statement obtains*. It is exactly this mechanism which will eventually permit the evasion of the no-coupling statements (20).

It follows further from Eq. (46) that

$$(1-\kappa)\epsilon^{\mu\nu\alpha\beta}P_\alpha W_\beta\psi = 0 \; , \qquad (47)$$

a statement which may be reduced to read

$$(1-\kappa)[iS_{\mu\nu}P^2 - iP_\nu S_{\mu\alpha}P^\alpha + iP_\mu S_{\nu\alpha}P^\alpha]\psi = 0 \; . \qquad (48)$$

Using now (27), and the wave equation (22) in the last two terms, we obtain

$$(1-\kappa)[imS_{\mu\nu} + P_\mu V_\nu - P_\nu V_\mu]\psi = 0 \; , \qquad (49)$$

which recovers Eq. (17c) for the spinless case, and again implies no statement for the case of spin $\frac{1}{2}$.

We have shown that the physical content of Sec. II, exclusive of electromagnetism, may be obtained from Eq. (17b), and that a considerably less restricted theory may be obtained from Eq. (21) for the case of spin $\frac{1}{2}$.

IV. MINIMAL ELECTROMAGNETIC COUPLING

Consider the equation, for $\kappa=\frac{1}{2}, 1$,

$$(mV_\mu - \kappa\Pi_\mu + iS_{\mu\nu}\Pi^\nu)\psi = 0 \; , \qquad (50)$$

which follows from Eq. (22) via the minimal coupling replacement $P_\mu \to \Pi_\mu \equiv P_\mu + eA_\mu$. We define the operator

$$\tilde{T}_\mu \equiv mV_\mu - \kappa\Pi_\mu + iS_{\mu\nu}\Pi^\nu \; , \qquad (51)$$

and consider the conditions on any solution of Eq. (50) following from

$$[\tilde{T}_\mu, \tilde{T}_\nu]\psi = 0 \; . \qquad (52)$$

The commutator may be evaluated by means of the algebra (13) and the definition

$$[\Pi_\mu, \Pi_\nu] = ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv ie\mathfrak{F}_{\mu\nu} \; . \qquad (53)$$

After considerable algebra, we may obtain

$$[\tilde{T}_\mu, \tilde{T}_\nu] = i(m^2-\Pi^2)S_{\mu\nu} + ie\kappa(\kappa-1)\mathfrak{F}_{\mu\nu} + e(\kappa-\tfrac{1}{2})(S_{\mu\alpha}\mathfrak{F}^\alpha{}_\nu - S_{\nu\alpha}\mathfrak{F}^\alpha{}_\mu) - \tfrac{1}{2}ie(S_{\mu\alpha}S_{\nu\beta}+S_{\nu\beta}S_{\mu\alpha})\mathfrak{F}^{\alpha\beta} - \Pi_\mu\tilde{T}_\nu + \Pi_\nu\tilde{T}_\mu \; . \qquad (54)$$

The last two terms vanish when applied to any solution of (50), so that the (six) conditions upon ψ read

$$[(\Pi^2-m^2)S_{\mu\nu} - e\kappa(\kappa-1)\mathfrak{F}_{\mu\nu} + ie(\kappa-\tfrac{1}{2})(S_{\mu\alpha}\mathfrak{F}^\alpha{}_\nu - S_{\nu\alpha}\mathfrak{F}^\alpha{}_\mu) + \tfrac{1}{2}e(S_{\mu\alpha}S_{\nu\beta}+S_{\nu\beta}S_{\mu\alpha})\mathfrak{F}^{\alpha\beta}]\psi = 0 \; . \qquad (55)$$

whose forms are different for the spinless ($\kappa=\frac{1}{2}$) and spin-$\frac{1}{2}$ ($\kappa=1$) cases.

We may immediately obtain one independent condition by contracting Eq. (55) with $S^{\mu\nu}$. Now, the identities (28) and (31) along with Eqs. (13) and (15) may be used to obtain the results

$$S^{\mu\nu}S_{\mu\alpha} = i(\lambda+1)S_\alpha{}^\nu - V_\alpha V^\nu + Dg_\alpha^\nu \qquad (56a)$$

and

$$S^{\mu\nu}S_{\mu\alpha}S_{\nu\beta} = (D-\lambda^2-\lambda)S_{\alpha\beta} + iV_\alpha V_\beta - iD(\lambda+1)g_{\alpha\beta} \; , \qquad (56b)$$

so that, with (12), we obtain from (55) the condition

$$\{4F(\Pi^2-m^2) + e[(D-\lambda^2-\lambda-\tfrac{1}{2}) - \kappa(\kappa-1) - 2(\kappa-\tfrac{1}{2})(\lambda+\tfrac{1}{2})]S_{\mu\nu}\mathfrak{F}^{\mu\nu}\}\psi = 0 \; . \qquad (57)$$

The numerical values peculiar to this realization then imply that the condition on the spinless case is

$$[(\Pi^2-m^2) + eS_{\mu\nu}\mathfrak{F}^{\mu\nu}]\psi = 0 \; , \qquad (58a)$$

while that obtaining for the spin-$\frac{1}{2}$ case is

$$[(\Pi^2-m^2) + \tfrac{1}{2}eS_{\mu\nu}\mathfrak{F}^{\mu\nu}]\psi = 0 \; . \qquad (58b)$$

It is immediately apparent that unless something further intervenes, the physics of the spinless case will be dubious indeed.

We remark at this point that contracting the wave equation (50) with Π^μ and using the conditions (58) results in the interacting Majorana equation

$$(V_\mu\Pi^\mu - \kappa m)\psi = 0 \; . \qquad (59)$$

The final operator identity necessary to our analysis may be obtained from the observation that Eq. (32) implies as well the statement

$$\epsilon^{\mu\alpha\nu\sigma}\epsilon^{\sigma\rho\beta\tau}S_{\rho\beta}V_\tau = 0 \; , \qquad (60a)$$

which may be written

$$S^{\mu\alpha}V^\nu - S^{\mu\nu}V^\alpha + S^{\alpha\nu}V^\mu = 0 \; . \qquad (60b)$$

Taking the commutator of this equation with V^β and using (13), we may obtain

$$S_{\mu\alpha}S_{\nu\beta}-S_{\nu\alpha}S_{\mu\beta}=S_{\mu\nu}S_{\alpha\beta}+iS_{\mu\nu}g_{\alpha\beta}+iS_{\nu\alpha}g_{\mu\beta}-iS_{\mu\alpha}g_{\nu\beta}\ , \tag{61}$$

an identity which may be used to simplify the last term in the constraint equations (55). The results are the equations

$$[S_{\mu\nu}(\Pi^2-m^2+\tfrac{1}{2}eS_{\alpha\beta}\mathfrak{F}^{\alpha\beta})-e\kappa(\kappa-1)\mathfrak{F}_{\mu\nu} + ie(\kappa-1)(S_{\mu\alpha}\mathfrak{F}^{\alpha}{}_{\nu}-S_{\nu\alpha}\mathfrak{F}^{\alpha}{}_{\mu})]\psi=0\ . \tag{62}$$

It is then clear that for $\kappa=1$, i.e., the spin-$\frac{1}{2}$ case, the *only* condition on a solution of Eqs. (50) is the *single* equation (58b). However, for the spinless case, there are additional constraints implied by (62) which may be conveniently extracted using Eq. (58a). We obtain

$$(1-\kappa)[S_{\mu\nu}S_{\alpha\beta}\mathfrak{F}^{\alpha\beta}-\tfrac{1}{2}\mathfrak{F}_{\mu\nu}+i(S_{\mu\alpha}\mathfrak{F}^{\alpha}{}_{\nu}-S_{\nu\alpha}\mathfrak{F}^{\alpha}{}_{\mu})]\psi=0\ , \tag{63}$$

where the factor $(1-\kappa)$ has been inserted to emphasize the applicability to the spinless case only.

Rather than proceed with (63) at this point, we shall obtain the (complete) additional information about the spinless case directly from the wave equation (50). We contract that equation with $\frac{1}{2}\epsilon^{\sigma\alpha\beta\mu}S_{\alpha\beta}$ and proceed exactly as in Sec. III to obtain the analog of Eq. (46):

$$(1-\kappa)W^{\sigma}(\Pi)\psi=0\ , \tag{64}$$

where

$$W^{\sigma}(\Pi)\equiv\tfrac{1}{2}\epsilon^{\sigma\alpha\beta\mu}S_{\alpha\beta}\Pi_{\mu}\ . \tag{65}$$

It follows that

$$(1-\kappa)\epsilon^{\mu\nu\alpha\beta}V_{\alpha}W_{\beta}(\Pi)\psi=0\ , \tag{66a}$$

which simplifies, with (28), to read

$$(1-\kappa)[iS_{\mu\nu}(V_{\alpha}\Pi^{\alpha})+\tfrac{1}{2}V_{\mu}\Pi_{\nu}-\tfrac{1}{2}V_{\nu}\Pi_{\mu}]\psi=0\ . \tag{66b}$$

Using now Eq. (59), for the case of $\kappa=\frac{1}{2}$, inside (66b), we obtain the analog also of Eq. (49):

$$(1-\kappa)[imS_{\mu\nu}+V_{\mu}\Pi_{\nu}-V_{\nu}\Pi_{\mu}]\psi=0\ . \tag{67}$$

This equation embodies all of the additional information about the spinless case, i.e., Eqs. (63), and (64) as well, may be recovered from (67). To interpret these conditions directly, it is only necessary to contract Eq. (67) with $\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\Pi_{\beta}$ and use (64) to obtain

$$(1-\kappa)\mathfrak{F}^{\text{dual}}_{\mu\nu}V^{\nu}\psi=0\ , \tag{68a}$$

from which follows immediately

$$(1-\kappa)\mathfrak{F}^{\text{dual}}_{\mu\nu}S^{\mu\nu}\psi=0\ . \tag{68b}$$

We have recovered the results[2] already quoted in Eq. (20).

We conclude that minimal coupling is inconsistent for the spinless case. Again, however, *no such statement obtains in the spin-$\frac{1}{2}$ case.*

V. CONCLUDING REMARKS

We have shown that the complete physical content of Dirac's spin-zero, positive-energy theory, including the no-minimal-coupling conditions, follows from one of his secondary equations, (17b). We have also shown that a slightly different equation, (21), defines a positive-energy spin-$\frac{1}{2}$ theory which apparently evades the inconsistency with minimal coupling to electromagnetism.

We shall summarize the (interacting) spin-$\frac{1}{2}$ theory. The defining equation is

$$(mV_{\mu}-\Pi_{\mu}+iS_{\mu\nu}\Pi^{\nu})\psi=0\ , \tag{69}$$

while the six conditions that there exist a simultaneous solution to the four equations (69) collapse to the single equation

$$(\Pi^2-m^2+\tfrac{1}{2}eS_{\mu\nu}\mathfrak{F}^{\mu\nu})\psi=0\ . \tag{70}$$

Also following from (69) and (70) is the Majorana equation (with interaction)

$$(V_{\mu}\Pi^{\mu}-m)\psi=0\ . \tag{71}$$

In the free theory ($A_{\mu}\equiv0$) Eq. (69) effectively projects out of the Majorana spectrum the timelike, spin-$\frac{1}{2}$ case. As for the interacting theory, Barut and Kleinert[15] have shown that the timelike, spin-$\frac{1}{2}$ solution of the interacting Majorana equation (71) possesses a gyromagnetic ratio of -1, which value is consistent with Eq. (70). We must emphasize, however, that we have not *proven* that Eq. (69) in fact possesses a nontrivial set of solutions for arbitrary electromagnetic fields; we have only shown the absence of any inconsistency and we claim nothing further. (However, in an investigation of the simplest possible case, that of scattering from a static Coulomb potential, we have obtained a formal, perturbative solution; others are being sought.)

We present this model not only to show the existence of a linear, spin-$\frac{1}{2}$ case, but also to underscore the fact that there exists a completely uninvestigated class of physical models which may be obtained via a reduction of the Klein-Gordon equation, not by "factorization," but by commutation relations. Our generalization of Dirac's procedure[1] has consisted essentially in the investigation of a particular case in which the consistency of equations of the type

$$T_a\psi=0\quad(a=1,2,\ldots,n) \tag{72}$$

requires that the operator

$$[T_a,T_b]=B_{ab}(P^2-m^2)+C_{abc}T_c\ , \tag{73}$$

where B_{ab} and C_{abc} are operators antisymmetric in a and b, should vanish on ψ. Inclusion of the last term of (73) represents the generalization.

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APPENDIX

We present here what we believe is an amusing note. It consists of the observation that Dirac's relativistic equation for the electron may be cast into the form

$$\tilde{T}_\mu \psi = 0 , \tag{A1}$$

with

$$\tilde{T}_\mu \equiv m(\tfrac{1}{2}\gamma_\mu) + \tfrac{1}{2}\Pi_\mu + i\Sigma_{\mu\nu}\Pi^\nu , \tag{A2}$$

where

$$\Sigma_{\mu\nu} \equiv -\tfrac{1}{4} i[\gamma_\mu, \gamma_\nu] , \tag{A3}$$

and the γ_μ are the usual Dirac matrices. Equation (A1) may be written as the four equations ($j = 1, 2, 3$)

$$(m\gamma_0 + \Pi_0 + 2i\Sigma_{0j}\Pi^j)\psi = 0 , \tag{A4a}$$

$$(m\gamma_j + \Pi_j - 2i\Sigma_{0j}\Pi^0 - 2i\Sigma_{kj}\Pi^k)\psi = 0 . \tag{A4b}$$

In our conventions,

$$\gamma^0\gamma_0 = 1 ,$$

$$\gamma^j\gamma_j = 1 \text{ (no sum)} ,$$

$$\gamma^0\Sigma_{0j} = -\tfrac{1}{2} i\gamma_j ,$$

$$\gamma^j\Sigma_{0j} = \tfrac{1}{2} i\gamma_0 \text{ (no sum)} ,$$

and

$$\gamma^j\Sigma_{kj} = \tfrac{1}{2} i\gamma_k \text{ (no sum, and } k \neq j) .$$

Then γ^0 times (A4a) becomes

$$(m + \gamma^0\Pi_0 + \gamma_j\Pi^j)\psi = 0 ,$$

or

$$(\gamma^\mu\Pi_\mu + m)\psi = 0 ,$$

while each of Eqs. (A4b) multiplied by the corresponding γ^j yields (no sum on j)

$$\left(m + \gamma^j\Pi_j + \gamma_0\Pi^0 + \sum_{k\neq j} \gamma_k\Pi^k\right)\psi = 0 ,$$

or again

$$(\gamma^\mu\Pi_\mu + m)\psi = 0 . \tag{A5}$$

Therefore *each* of Eqs. (A1) may be written as (A5), so that the known interacting Dirac (positron) solutions are the solutions of (A1). [The more familiar electron equation may be obtained instead with the opposite choice of sign for the last two terms of $\tilde{T}_\mu$. For convenience below, we shall retain the unusual normalizations and the signs of (A2).]

We may push the analogy further, since $V_\mu = \frac{1}{2}\gamma_\mu$ and $\Sigma_{\mu\nu}$ from Eq. (A3) also obey the algebra of SO(3, 2), Eq. (13), and define a (nonunitary, reducible) representation of the Lorentz group with $F = \frac{3}{4}$, $G = -\frac{3}{4} i\gamma_5$, $D = 1$, and $\lambda = -\frac{3}{2}$ as in Eqs. (12), (15), and (28). Proceeding exactly as in Sec. III, we would obtain (27), while (40), with now $\kappa = -\frac{1}{2}$ for this representation, yields the expected result

$$W^2\psi = -\tfrac{3}{4} m^2\psi . \tag{A6}$$

Continuing as in Sec. IV, we obtain from (57) the expected, solitary condition

$$(\Pi^2 - m^2 - e\Sigma_{\mu\nu}\mathfrak{F}^{\mu\nu})\psi = 0 . \tag{A7}$$

Viewed in this fashion, Majorana's equation is the natural generalization of Dirac's equation (A5) to a unitary representation, while (21) is a similar generalization of the same theory written as (A1).

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Investigations in two-dimensional vector-meson field theories

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We study several field theories in one space and one time dimension. Among them are a neutral vector meson with and without mass coupled to fermions and a massive self-coupled Yang-Mills field. For the former we discuss methods of solving the field theory and for the latter we attempt to establish its nonrenormalizability.

I. INTRODUCTION

This paper contains a collection of results from studies of vector-meson field theories in a space of one time and one space dimension. These investigations were motivated by the hope of obtaining additional insight into results obtained and conjectured for more realistic models. We had in mind the asymptotic freedom of gauge theories investigated in four-dimensional space-time by Gross, Wilczek, and Politzer,[1] and the possible complete screening of the quantum numbers of the gauge group, which has been widely conjectured and investigated in the Schwinger model,[2] or 2-dimensional massless quantum electrodynamics, by Casher, Kogut, and Susskind.[3] Both effects are suspected of being related to the severe infrared divergence of the theories. In addition to their kinematic simplicity, gauge theories in two dimensions have charges with the dimensions of mass and we hoped to investigate how the presence of a mass scale in the theory (even when the bare particles may have zero mass) affects the properties mentioned above.

In the following section we discuss the field theory of a massless charged fermion field coupled to a neutral vector meson. This model has been solved and discussed in a number of papers dating back to the original work of Schwinger over ten years ago.[2,4] We are accordingly brief and emphasize the points where our approach is somewhat different from previous ones. In particular our method of solution solves both the cases of nonvanishing bare mass of the vector meson and two-dimensional quantum electrodynamics (QED) and immediately enables us to study the Green's functions for states produced by a scalar source in both cases. The complete suppression of fermion-antifermion pairs discussed for two-dimensional QED by Casher, Kogut, and Susskind[3] does not occur if the vector meson has a bare mass.

The gauge-field generalization of this model does not yield to solution by the same techniques because of the non-Abelian nature of the couplings. The pure Yang-Mills model in two dimensions is trivial because one can choose a gauge where the space components of the gauge field are zero, and there is no nonlinear coupling of the meson field.[5] If we add spin-$\frac{1}{2}$ fermions to such a theory, there are only non-Abelian Coulomb forces between the fermions in lowest order. In this gauge the lowest second-order self-energy of the mesons comes entirely from the creation and annihilation of a fermion-antifermion pair just as in the Abelian case, and the charged vector mesons get masses in the same way. We have not pursued these