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Asymptotic freedom and the Lee model

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Using dimensional regularization we renormalize the Lee model in arbitrary space-time dimension D. We compute $\beta(g)$ and $\gamma(g)$, the coefficient functions of the Callan-Symanzik equation, in closed form and show that the model is asymptotically free when $D < 4$. In addition, we demonstrate a strict correlation between the sign of $\beta(g)$ and the presence of a ghost state: There is no ghost when $\beta(g) < 0$. Finally, we study an extended Lee model with two coupling constants and study the behavior of the effective coupling constants in the deep-Euclidean region.

I. INTRODUCTION

The Lee model[1] was originally formulated because it is a field theory in which mass, wave-function, and charge renormalizations can be easily carried out in closed form. It is therefore reasonable to use the Lee model again to illustrate the more contemporary aspects of renormalization theory such as the renormalization group, the Callan-Symanzik equation,[2] and asymptotic freedom.[3]

Accordingly, in our treatment of the Lee model we have replaced the conventional use of a momentum cutoff by the more modern technique of dimensional regularization[4] to ensure that the renormalized quantities can be expressed as finite integrals. By renormalizing in D dimensions where D is arbitrary, we obtain a continuum of theories. The physical nature of these theories depends crucially on the choice of D. For example, when $D<4$, the Lee model is asymptotically free.

We have organized this paper as follows. In Sec. II we write down the Feynman rules for the Lee model in D dimensions and then use these rules to renormalize the model. In Sec. III we compute in closed form the two coefficient functions $\beta(g)$ and $\gamma(g)$ of the Callan-Symanzik equation. We find that $\beta(g)$ is a two-term polynomial: $\beta(g)=ag+bg^3$. This form for $\beta(g)$ is reminiscent of that discovered by Crewther, Shei, and Yan[5] for $\beta(e)$ in Schwinger's model of two-dimensional quantum electrodynamics.[6] For comparison, the expressions are

$$\beta(e) = -e + e^3/\pi$$

for electrodynamics in two dimensions, and

$$\beta(g) = -g - (2+2\pi)g^3$$

for the Lee model in two dimensions.

For some values of D, $\beta(g)$ has a zero for $g>0$. The nontrivial zero of β occurs when the unrenormalized coupling constant is infinite, as predicted by Wilson.[7] Furthermore, for a suitable choice of D, $\beta(g)$ can be made either positive or negative between its zeros. When $3<D<4$ the latter case obtains.

It is well known that in four dimensions the Lee model develops a ghost state (a state of negative norm) as the momentum cutoff is removed. In Sec. IV we show that in D dimensions there is a strict correlation between the sign of $\beta(g)$ and the presence of a ghost: There is a ghost when and only when $\beta(g)>0$. This means that there is no ghost when the theory is asymptotically free. This result supports a recent conjecture of Salam and Strathdee.[8]

In Sec. V we extend the Lee model by introducing a new particle N_1 which behaves like the N. The model now has two coupling constants. We graph the trajectories that are traced out in the effective coupling constant plane as the model approaches the deep-Euclidean region.

II. RENORMALIZATION IN D DIMENSIONS

The Lee model is a field theory of three fictitious particles N, V, and θ, which interact according to

$$V \rightleftharpoons N+\theta\,. \tag{1}$$

This model is soluble because the crossed reaction $V+\theta \rightleftharpoons N$ is forbidden by a superselection rule.

A suitable Hamiltonian describing such a model in four dimensions, where the fields associated with these particles are spinless, is given in Schweber[9] as

$$H = H_0 + H_1, \tag{2}$$

$$H_0 = m_{V_0} \int d^3p\, V^\dagger(\vec{p})V(\vec{p}) + m_{N_0} \int d^3p\, N^\dagger(\vec{p})N(\vec{p}) + \int d^3k\, \omega(\vec{k})\theta^\dagger(\vec{k})\theta(\vec{k}), \tag{3}$$

$$H_1 = \lambda_0(2\pi)^{-3/2} \int \frac{d^3k\, f(\vec{k}^2)}{2\omega(\vec{k})} \int d^3p [V^\dagger(\vec{p})N(\vec{p}-\vec{k})\theta(\vec{k}) + N^\dagger(\vec{p}-\vec{k})\theta^\dagger(\vec{k})V(\vec{p})]. \tag{4}$$

In the above equation $\omega(\vec{k}) = (\vec{k}^2 + \mu^2)^{1/2}$; μ is the bare mass of the θ particle; $f(\vec{k}^2)$ is a momentum cutoff which tends to zero as $\vec{k}^2 \to \infty$; m_{V_0} and m_{N_0} are the bare masses of the V and N particles; and $\theta(\vec{k})$, $N(\vec{p})$, and $V(\vec{p})$ together with their adjoints obey commutation and anticommutation relations as given by Schweber.[9]

It is easy to derive the Feynman rules for this Hamiltonian and to generalize them to D dimensions. A complete set of rules is

$[\omega - \omega(\vec{k}) + i\epsilon]^{-1}$ for a θ propagator,

$[\omega - m_{N_0} + i\epsilon]^{-1}$ for an N propagator,

$[\omega - m_{V_0} + i\epsilon]^{-1}$ for a V propagator,

$\lambda_0[2\omega(\vec{k})]^{-1}$ for each vertex,

$-i(2\pi)^{-D} \int d\omega\, d^{D-1}k$ for each loop integration.

It is not necessary to use a cutoff in D dimensions, but to include it one must multiply by an additional factor of $f(\vec{k}^2)$ at each vertex.

A. The V propagator

The diagrams contributing to the V propagator are a string of $N\theta$ loops connected by V lines. We evaluate the $N\theta$ loop A using the Feynman rules and obtain

$$A = -i\lambda_0^2(2\pi)^{-D} \int d\omega\, d^{D-1}k [2\omega(\vec{k})]^{-1} \times (E - \omega - m_{N_0} + i\epsilon)^{-1}[\omega - \omega(\vec{k}) + i\epsilon]^{-1}.$$

Performing the ω integration by closing the contour gives

$$A = \lambda_0^2(2\pi)^{1-D} \int d^{D-1}k [2\omega(\vec{k})]^{-1} \times [E - \omega(\vec{k}) - m_{N_0} + i\epsilon]^{-1}. \tag{5}$$

In the above expressions E is the energy that passes through the loop.

The V propagator $G_V(E)$ is obtained by summing the N loops as a geometric series. The result is

$$G_V^{-1}(E) = E - m_{V_0} + i\epsilon - \lambda_0^2(2\pi)^{1-D} \int d^{D-1}k [2\omega(\vec{k})]^{-1} \times [E - \omega(\vec{k}) - m_{N_0} + i\epsilon]^{-1}. \tag{6}$$

B. Mass renormalization

The renormalized mass of the V particle is the pole in $G_V(E)$. If we choose for simplicity to make m, the renormalized mass of the V particle, equal to that of the N particle, then we have

$$m = m_{V_0} - \lambda_0^2(2\pi)^{1-D} \int d^{D-1}k [2\omega^2(\vec{k})]^{-1}.$$

This integral may be evaluated using the formula

$$\int_0^\infty x^\alpha dx (1+x^2)^{-\beta} = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\alpha)\Gamma(\beta - \frac{1}{2}\alpha - \frac{1}{2})}{2\Gamma(\beta)} \tag{7}$$

and the expression

$$S(D-1) = \frac{2\pi^{D/2}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}D - \frac{1}{2})}, \tag{8}$$

where $S(D)$ is the surface area of a sphere in D dimensions. This gives

$$m = m_{V_0} - \lambda_0^2 \mu^{D-3}\sqrt{\pi}\, \Gamma\left(\frac{3-D}{2}\right)(2\sqrt{\pi})^{-D}. \tag{9}$$

The masses of the θ and the N particle undergo no renormalization. Hence in the above expression μ is the mass of the physical as well as the bare θ particle.

C. Wave-function renormalization

The wave-function renormalization constant Z for the V field is the residue of the V-particle pole in $G_V(E)$. Z is thus given by

$$Z^{-1} = 1 + \lambda_0^2(2\pi)^{1-D} \int d^{D-1}k [2\omega^3(\vec{k})]^{-1}.$$

This integral may be evaluated using Eqs. (7) and (8) with the result that

$$Z^{-1} = 1 + 2\lambda_0^2 \mu^{D-4}\Gamma\left(\frac{4-D}{2}\right)(2\sqrt{\pi})^{-D}. \tag{10}$$

D. Coupling-constant renormalization

The renormalized coupling constant is conventionally defined by requiring that the exact $N\theta$ scattering amplitude at threshhold ($E = m_N + \mu$) be equal to the Born approximation to the scattering amplitude in which the unrenormalized coupling constant λ_0 and mass m_{V_0} are replaced by their renormalized counterparts λ and m_V. In the Lee model the NV scattering amplitude is merely the V propagator $G_V(E)$ multiplied by λ_0^2. We thus obtain the equation

$$\lambda^2=\frac{\lambda_0^2}{1+\lambda_0^2(2\pi)^{1-D}\int d^{D-1}k[2\omega^2(\vec{k})]^{-1}[\omega(\vec{k})-\mu]^{-1}},$$

which reduces to

$$\lambda^2=\frac{\lambda_0^2}{1+2\lambda_0^2\mu^{D-4}(D-3)^{-1}[\Gamma(\frac{1}{2}(4-D))+\Gamma(\frac{1}{2}(5-D))\sqrt{\pi}\,](2\sqrt{\pi}\,)^{-D}} \tag{11}$$

when the integral is evaluated using Eqs. (7) and (8).

The three results in Eqs. (9)–(11) agree with the conventional expressions for the renormalized quantities when $D=4$.[9]

III. THE CALLAN-SYMANZIK EQUATION

It is most convenient to use a dimensionless coupling constant in the context of the renormalization group; to wit, we define g by

$$g=\lambda\mu^{(D-4)/2}(2\sqrt{\pi}\,)^{-D/2}. \tag{12}$$

In terms of g, the two coefficient functions $\beta(g)$ and $\gamma(g)$ of the Callan-Symanzik equation are defined by[2]

$$\beta(g)=\mu\frac{\partial g}{\partial\mu},$$

$$\gamma(g)=\tfrac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z.$$

Using the results of Sec. II, specifically Eqs. (9)–(11), we obtain

$$\beta(g)=\frac{D-4}{2}g-\frac{D-4}{D-3}\left[\Gamma\left(\frac{4-D}{2}\right)+\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\right]g^3, \tag{13}$$

$$\gamma(g)=\frac{2\Gamma(\frac{1}{2}(6-D))g^2}{1-[2/(D-3)][\Gamma(\frac{1}{2}(5-D))\sqrt{\pi}+2\Gamma(\frac{1}{2}(6-D))]g^2}. \tag{14}$$

In terms of these coefficients the Callan-Symanzik equation for the Lee model may be written as

$$\left[\mu\frac{\partial}{\partial\mu}+2m\frac{\partial}{\partial m}+\beta(g)\frac{\partial}{\partial g}-n_V\gamma(g)\right]\times\Gamma(k_1\cdots k_{n_\theta},p_1\cdots p_{n_N},q_1\cdots q_{n_V})=\Delta\Gamma, \tag{15}$$

where Γ is the connected, one-particle irreducible Green's function with n_θ, n_N, and n_V external lines for the θ, N, and V fields, respectively. $\Delta\Gamma$ is the Green's function Γ with a mass insertion for each internal line. It is presumed in the above formula that the Green's function Γ does not violate the superselection rules that n_N+n_V and n_V+n_θ remain constant in any scattering process. Otherwise, $\Gamma=0$.

A. Zeros of $\beta(g)$

Apart from the trivial zero of β at $g=0$, β may have one other zero for $g>0$ as long as

$$B\equiv\frac{2}{D-3}\left[\Gamma\left(\frac{5-D}{2}\right)\sqrt{\pi}+\Gamma\left(\frac{4-D}{2}\right)\right]>0.$$

This can never happen for $D<3$; it is true for $3<D<4$, and it is true for an infinite number of finite intervals when $D>4$. The boundaries of these regions are obtained by solving the equation

$$\Gamma\left(\frac{5-D}{2}\right)\sqrt{\pi}+\Gamma\left(\frac{4-D}{2}\right)=0.$$

This equation has an infinite number of solutions, the smallest of which is $D\simeq 4.425$. Thus, $B<0$ when $4<D\simeq 4.425$ and $B>0$ when $5>D\gtrsim 4.425$. The remaining regions are determined in a similar manner.

At the nontrivial zero the expression for the anomalous dimension $\gamma(g)$ in Eq. (14) simplifies considerably:

$$\gamma(g)=\frac{4-D}{2}. \tag{16}$$

This result supports the contention of Callan and Gross[10] that γ cannot vanish at a zero of β unless the zero occurs at $g=0$. Of course, when $D=4$, $\gamma(g)$ in Eq. (16) does vanish, but for this special value of D $\beta(g)$ has only one zero at $g=0$. The three distinct zeros of β merge as D approaches 4.

As g approaches the nontrivial fixed point, the unrenormalized coupling constant λ_0 diverges, as can be seen by examining Eq. (11).[11] This result was conjectured by Wilson[7] and observed to take place in two-dimensional quantum electrodynamics.[5] But as λ_0 diverges the product $\lambda_0^2G_V(E)$ simplifies:

$$\lim_{\lambda_0\to\infty}[\lambda_0^2G_V(E)]^{-1}=-(2\pi)^{1-D}\int d^{D-1}k[2\omega(\vec{k})]^{-1}[E-\omega(\vec{k})-m+i\epsilon]^{-1}. \tag{17}$$

Observe that the V-particle pole disappears leaving the contribution from the $N\theta$ cut. This means that difficult calculations in the Lee model that have been performed in the past, such as the determination of the $V\theta$ scattering amplitude[12] and the NV bound state,[13] become somewhat simpler to perform.

B. Asymptotic freedom

When β passes through the origin with a negative slope, the theory is said to be asymptotically free. We give detailed plots of β in the next section, but a glance at Eq. (13) indicates that the theory is asymptotically free when $D<4$.

IV. GHOSTS AND ASYMPTOTIC FREEDOM

It is well known that the conventional Lee model in four dimensions exhibits a ghost state as the cutoff is removed. The ghost state appears as a pole in the V-particle propagator. The state is called a ghost because the residue of this pole is negative. The presence of a pole implies that the S matrix is nonunitary and that the renormalized Hamiltonian is not Hermitian.

A simple way to detect the presence of a ghost is to look for a violation of conservation of probability in the renormalization constant Z. Since Z is the probability of finding a bare V state in a physical V state, it must lie between 0 and 1. A ghost is present if and only if $Z<0$ or $Z>1$.

We have observed a direct correlation between the sign of β and the presence of a ghost, namely, that when β is negative Z lies between 0 and 1 and when β is positive either $Z<0$ or $Z>1$. We illustrate this property of the theory by plotting in Fig. 1 graphs of Z and β as functions of g for various values of D.

We observe that there is no ghost having the quantum numbers of a V particle when the theory is asymptotically free. This result supports the conjecture in a recent paper by Salam and Strath-

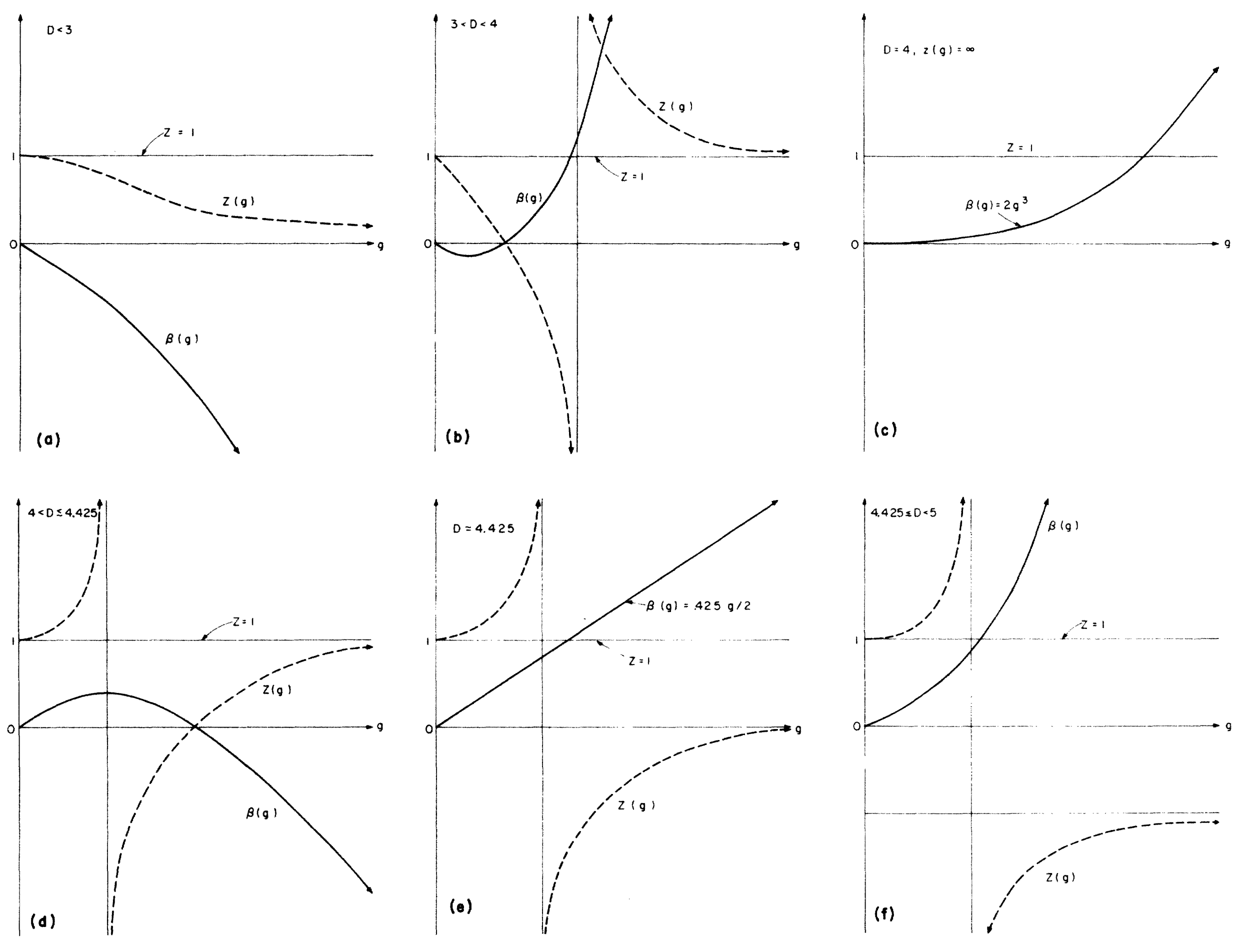

FIG. 1. A schematic graph showing the dependence of the functions $\beta(g)$ in Eq. (13) and $Z(g)$ in Eq. (10) upon the dimensionless renormalized coupling constant g. The graphs are drawn for six distinct regions of D: (a) $D<3$, (b) $3<D<4$, (c) $D=4$, (d) $4<D\lesssim 4.425$, (e) $D\simeq 4.425$, and (f) $4.425\lesssim D<5$. Observe that when $\beta\leq 0$, Z is interpretable as a physical probability because $0\leq Z\leq 1$. We have omitted $D=3$ because for that value of D, $\beta(g)=\infty$.

dee[8] that field theories which are asymptotically free have no Bogoliubov-Redmond[14] ghosts. (The V-particle ghost is a Bogoliubov-Redmond ghost.)

One can carry this point still further by considering the differential equation for the effective coupling constant $g(t)$:

$$\frac{dg(t)}{dt}=\beta(g(t)), \quad g(0)=g. \tag{18}$$

The solution to this initial-value problem is

$$g^2(t)=\frac{g^2(0)e^{(D-4)t}}{1+g^2(0)[2/(D-3)][\Gamma(\frac{1}{2}(4-D))+\sqrt{\pi}\,\Gamma(\frac{1}{2}(5-D))](e^{(D-4)t}-1)}. \tag{19}$$

The physical behavior of the Lee model in the deep-Euclidean region is obtained by allowing $t\to\infty$ in this equation. But if g^2 is negative as t approaches ∞, then $g(t)$ is imaginary. We interpret this to mean that the effective Hamiltonian is non-Hermitian in this limit. In such a situation one would not be surprised to find a ghost. Indeed, the condition that $\lim_{t\to\infty}g^2(t)>0$ is precisely the condition that there be no ghosts in the model, i.e., that $\beta<0$. We illustrate this point by plotting in Fig. 2 $g^2(t)$ versus t for various values of D.

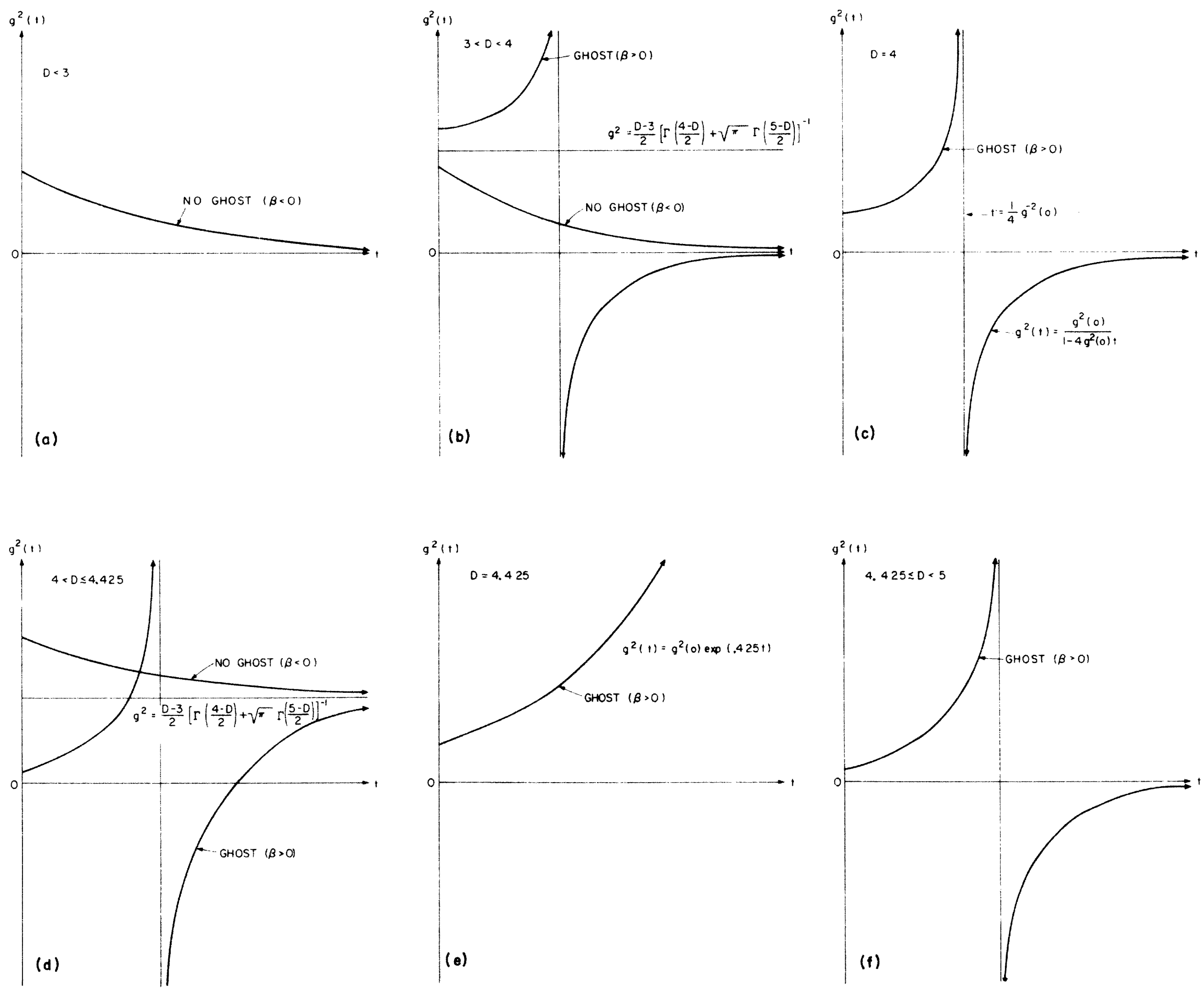

FIG. 2. A schematic graph showing the dependence of $g^2(t)$, the square of the effective coupling constant in Eq. (18), upon t. The graphs are drawn for the same six regions of D as in Fig. 1. The deep-Euclidean region is attained as $t\to\infty$. Observe that when $\beta<0$, $g^2(t)$ approaches $g^2(\infty)$ along positive values, but that when $\beta>0$, $g^2(t)$ approaches $g^2(\infty)$ along negative values. When $g^2(t)$ is negative, $g(t)$ is imaginary and the model has a ghost.

V. LEE MODEL WITH TWO COUPLING CONSTANTS

To illustrate the kind of behavior one might expect of a theory with more than one coupling constant we construct an extended Lee model. We introduce a new particle N_1 with nearly the same properties as the N. The extended model now has two fundamental interactions:

$$\begin{aligned} &V \leftrightarrow N+\theta \text{ (bare coupling } \lambda_0), \\ &V \leftrightarrow N_1+\theta \text{ (bare coupling } \kappa_0). \end{aligned} \tag{20}$$

The Hamiltonian for the extended model in four dimensions is the same as that for the ordinary model with several additional terms. Specifically, H_0 in Eq. (3) is augmented by the term

$$m_{N_0}\int d^3p\, N_1^\dagger(\vec{p})N_1(\vec{p}) \tag{21}$$

and H_I in Eq. (4) by the term

$$\kappa_0(2\pi)^{-3/2}\int \frac{d^3k\, f(\vec{k}^2)}{2\omega(\vec{k})}\int d^3p[V^\dagger(\vec{p})N_1(\vec{p}-\vec{k})\theta(\vec{k}) + N_1^\dagger(\vec{p}-\vec{k})\theta^\dagger(\vec{k})V(\vec{p})]. \tag{22}$$

For simplicity we have chosen the same masses for the N and N_1 particles.

The Feynman rules for the extended model in D dimensions are obvious generalizations of the rules for the ordinary Lee model. From these we deduce the amplitude B for an $N_1\theta$ loop. B has the same form as A in Eq. (5) with λ_0^2 replaced with κ_0^2. The full V-particle propagator in terms of A and B is simply

$$G_V^{-1}(E)=E-m_{V_0}+i\epsilon-A-B.$$

We perform mass, wave-function, and coupling-constant renormalization as in Sec. II. The equation for the renormalized coupling constants is

$$(\lambda^2,\kappa^2)=\frac{(\lambda_0^2,\kappa_0^2)}{1+2(\lambda_0^2+\kappa_0^2)\mu^{D-4}(D-3)^{-1}[\Gamma(\frac{1}{2}(4-D))+\Gamma(\frac{1}{2}(5-D))\sqrt{\pi}\,](2\sqrt{\pi}\,)^{-D}}, \tag{23}$$

which is the generalization of Eq. (11).

Finally, we define a pair of dimensionless coupling constants by

$$(g,h)=(\lambda,\kappa)\mu^{(D-4)/2}(2\sqrt{\pi}\,)^{-D/2}, \tag{24}$$

and compute the functions β and β_1:

$$\begin{aligned} \beta &\equiv \mu\frac{\partial g}{\partial\mu} \\ &=\frac{D-4}{2}g-\frac{D-4}{D-3}\left[\Gamma\left(\frac{4-D}{2}\right)+\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\right] \\ &\qquad\times g(g^2+h^2), \\ \beta_1 &\equiv \mu\frac{\partial h}{\partial\mu} \\ &=\frac{D-4}{2}h-\frac{D-4}{D-3}\left[\Gamma\left(\frac{4-D}{2}\right)+\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\right] \\ &\qquad\times h(g^2+h^2). \end{aligned} \tag{25}$$

The simultaneous zeros of these functions are the point located at $g=h=0$ and the circle satisfying the equation

$$g^2+h^2=\frac{D-3}{2}\left[\Gamma\left(\frac{4-D}{2}\right)+\sqrt{\pi}\,\Gamma\left(\frac{5-D}{2}\right)\right]^{-1}, \tag{26}$$

for those values of dimension D for which the right-hand side of Eq. (26) is positive, namely $3<D<4$ and $4<D\lesssim 4.425$. The locations of these

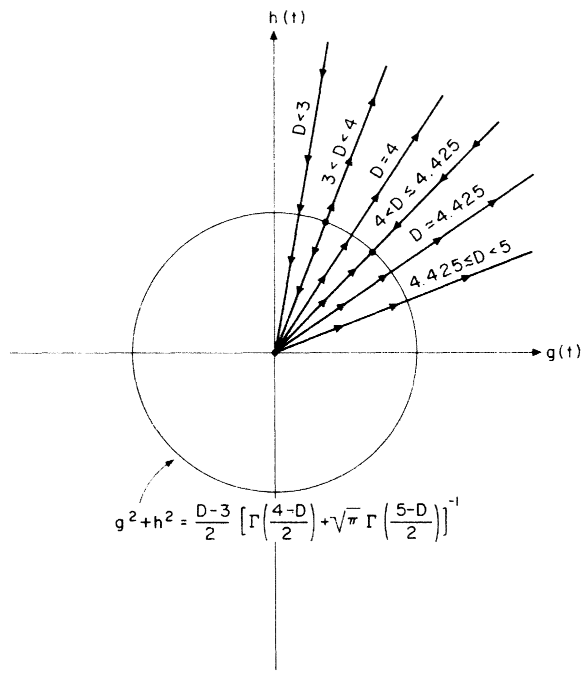

FIG. 3. A schematic graph showing the dependence of $g(t)$ and $h(t)$ in Eq. (27) upon the variable t. As t increases, the point $(g(t),\, h(t))$ moves along the radial lines in the direction indicated by the arrows. The heavy black dots indicate fixed points [points where $g(t)$ and $h(t)$ are independent of t] in the coupling-constant plane. The origin is a fixed point for all D; the indicated circle is a continuous line of fixed points when $3<D<4$ and when $4<D\lesssim 4.425$.

"critical" points are indicated in Fig. 3.

Corresponding to Eq. (18), the equations for the effective coupling constants $g(t)$ and $h(t)$ are

$$\dot{g}(t) = \beta(g,h)\,, \qquad \dot{h}(t) = \beta_1(g,h)\,, \tag{27}$$

whose solutions are straight lines in the coupling-constant plane in Fig. 3. Observe that depending on the dimension D the point $(g(t),h(t))$ traces out a trajectory as $t\to\infty$ which may approach the origin when the model is asymptotically free, the point at ∞, or the circle.[15]

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[11]It should be noted that at the nontrivial zero of $\beta(g)$ the symmetry-breaking solution of the Lee model discovered by G. S. Guralnik and C. R. Hagen [Phys. Rev. 149, 1017 (1966)] probably does not exist. This is because $\lambda_0\to\infty$ at the fixed point of β and this violates the condition for symmetry breaking. The disappearance of symmetry breaking when β has a zero has also been noted for scalar electrodynamics by S. Coleman and E. Weinberg [Phys. Rev. D 7, 1888 (1973)].

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[15]β and β_1 in Eq. (25) are the components of the gradient of a single function $V(g,h)$. Thus, the trajectories in the coupling-constant plane in Fig. 3 are examples of "gradient flow" [see D. J. Wallace and R. Zsia, University of Southampton, U.K., report (unpublished)]. Models having gradient flow may not exhibit limit cycles, ergodic behavior, or spiral critical points.