Geodesic synchrotron radiation in the Kerr geometry by the method of asymptotically factorized Green's functions*

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The scalar, electromagnetic, and gravitational geodesic synchrotron radiation (GSR) spectra are determined for the case of a test particle moving on a highly relativistic circular orbit about a rotating (Kerr) black hole. One finds that the spectral shape depends only weakly on the angular momentum parameter a/M of the black hole, but the total radiated power drops unexpectedly for $a/M \gtrsim 0.95$ and vanishes for $a/M \to 1$. A spin-dependent factor (involving the inner product of the polarization of a radiated quantum with the source) is isolated to explain the dependence of the spectral shape upon the spin of the radiated field. Although the scalar wave equation is solved by separation of variables, this procedure is avoided for the vector and tensor cases by postulating there a sum-over-states expansion for the Green's function similar to that found to hold in the scalar case. The terms in this sum, significant for GSR, can then be evaluated in the geometric optics approximation without requiring the use of vector or tensor spherical harmonics.

I. INTRODUCTION

Gravitational synchrotron radiation (GSR) emanating from the center of the galaxy has been proposed[1] as an explanation of the high intensity of gravity waves detected by Weber.[2] The existence of GSR under astrophysically unrealistic conditions has been demonstrated in the Schwarzschild geometry[3,4] by showing that radiation from particles in highly relativistic orbits is beamed strongly into the equatorial plane in high harmonics of the fundamental (orbital) frequency (ω_0) of the particle motion. A portion of the gravitational[5] and virtually all of the scalar[3] and electromagnetic radiation occurs in these narrowly beamed high harmonic modes.

GSR from particles in a circular orbit about a Schwarzschild black hole at the center of the galaxy is an unsatisfactory astrophysical model for two major reasons. First, the only circular orbits emitting GSR are the physically unrealistic orbits near $r=3M$; such an orbit represents a highly energetic and carefully aimed particle scattered through an angle $\Delta\varphi \gg 2\pi$. Second, as pointed out by Bardeen,[6] a rotating black hole is a much better model of the galactic center. In fact, the accretion of matter after the initial collapse will tend to increase the angular momentum of the black hole almost up to the causal limit $a^2=M^2$. In a more detailed model, Thorne[7] finds a limit $a \leqslant 0.9982M$.

For this second reason, it is important to study GSR in the Kerr[8] geometry since it is the expected final state of collapse of any rotating body. However, the only particle orbits which might occur naturally that this paper considers are bound stable circular orbits. As had been anticipated by Bardeen[9] and Goebel,[10] these orbits, which are not highly relativistic, radiate primarily quadrupole radiation and are not sources of GSR. Further details of the radiation from these orbits are given by Bardeen, Press, and Teukolsky.[11] The sources of GSR considered in this paper are relativistic, unstable, unbound, circular geodesics. Although these orbits are unphysical, they allow the study of the influence of the black hole's angular momentum on the radiation process. These calculations have served as useful preliminaries to further computations involving noncircular motion which will be reported later, but which failed to find more interesting sources of GSR.

Scalar radiation from a point particle of mass $\mu \ll M$ in a circular orbit is examined by solving the scalar wave equation in a Kerr background. Scalar radiation is easy to study since the scalar wave equation,

$$-\Phi^{;\mu}{}_{;\mu}=4\pi f T\,, \tag{1.1}$$

separates[12] with well-known angular functions.[13] T is the trace of the stress-energy tensor of the source and f is a coupling constant. In Sec. II, the Green's function for this equation is written in the form

$$G^{+}(x,y)=i\int d\omega\sum_{lm}\Phi^{\rm up}_{lm\omega}(x)\frac{\omega}{|\omega|}[\Phi^{\rm out}_{lm\omega}(y)]^{*}\,, \tag{1.2}$$

where the $\Phi_{lm\omega}$ are solutions of the homogeneous scalar wave equations satisfying certain specified boundary conditions, and $r(x)>r(y)$, where r is the Boyer-Lindquist coordinate. [A similar form with different boundary conditions on the $\Phi_{lm\omega}$ holds for $r(x)<r(y)$.] The states $\Phi_{lm\omega}$ have completeness

and orthonormal properties in terms of the indefinite Klein-Gordon inner product on future null infinity $\mathscr{I}^+$. This expansion of the Green's function as a sum of factors then leads to the following expression for the energy radiated by a source $fT(y)$:

$$\frac{dW^{(0)}}{d\omega}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\omega\,|\langle\Phi^{\text{out}}_{lm\omega},fT\rangle|^2\,. \tag{1.3a}$$

The inner product of $\Phi^{\text{out}}_{lm\omega}$ and the source term which appears here is an integration over all spacetime outside the horizon. Analogously, we expect the electromagnetic and gravitational energy spectra to take the form

$$\frac{dW^{(1)}}{d\omega}=\sum_{\lambda}\omega\,|\langle A^{\text{out}}_{\alpha}(\lambda\omega),J^{\alpha}\rangle|^2\,, \tag{1.3b}$$

$$\frac{dW^{(2)}}{d\omega}=\sum_{\lambda}\omega\,|\langle h^{\text{out}}_{\alpha\beta}(\lambda\omega),T^{\alpha\beta}\rangle|^2\,. \tag{1.3c}$$

In the above equations, the sum is to be taken over a complete set of orthonormal eigenstates labeled by λ and the superscript on the left-hand side of the equation denotes the spin of the field being considered. A^{out}_{α} and $h^{\text{out}}_{\alpha\beta}$ respectively are the solutions of the homogeneous electromagnetic and gravitational wave equations obeying boundary conditions similar to Φ^{out}.

We do not prove that Eqs. (1.3) are correct, but merely derive them from a conjecture that the Green's functions for the vector and tensor equations can be written, at least asymptotically for $r(x)\to\infty$, in a factorized form similar to Eq. (1.2) [see Eqs. (3.19) and (3.38)]. That such factorized Green's functions exist is apparent when the wave equations for the potentials can be solved by separation of variables as is the case for test fields in the Schwarzschild geometry.[14] In more general spacetimes where the wave equations are not known to separate, the above energy spectra formulas are somewhat speculative in that they depend on the existence of Green's functions which can be written in the factorized form at least in asymptotic regions. We conjecture only asymptotic validity as x lends the spatial infinity for general stationary metrics, as we see no other obvious analog for the conditions $r(x)>r(y)$ defining the range of validity of Eq. (1.2) in the Kerr metric.

For the purpose of studying the radiation emission by a highly relativistic test particle, Eqs. (1.3) for the energy spectra are ideally suited. An energetic particle typically radiates at high frequencies $M\omega\gg 1$, thereby allowing one to study scalar, vector, and tensor radiation in a unified fashion. Misner *et al.*[3] previously pointed out that the homogeneous scalar, electromagnetic, and gravitational radial equations are identical in the Schwarzschild geometry when $M\omega=Mm\omega_0\gg 1$. Similarly, the various homogeneous Kerr perturbation equations agree in the high-frequency limit, as can be seen by inspecting Teukolsky's separated wave equations[15] in the Kerr background. Hence, using the methods described below, we are able to reduce high-frequency electromagnetic and gravitational radiation problems in the Kerr geometry to scalar calculations (with a modified source term).

Following the Isaacson[16] analysis of high-frequency radiation, one may use the WKB approximation to find the desired homogeneous solutions to the vector and tensor wave equations. It can be shown that in a suitable gauge

$$A^{\text{out}}_{\alpha}=e_{\alpha}\Phi^{\text{out}}\,, \tag{1.4}$$

$$h^{\text{out}}_{\alpha\beta}=e_{\alpha\beta}\Phi^{\text{out}}\,, \tag{1.5}$$

at frequencies $M\omega\gg 1$. Φ^{out} is the aforementioned solution of the homogeneous scalar wave equation, and e_α ($e_{\alpha\beta}$) is a polarization vector (tensor) orthogonal to the direction of propagation of radiation. With the aid of (1.4) and (1.5), (1.3) reduce to

$$\frac{dW^{(0)}}{d\omega}=\sum_{l,m}\omega\,|\langle\Phi^{\text{out}}_{lm\omega},fT\rangle|^2\,, \tag{1.6a}$$

$$\frac{dW^{(1)}}{d\omega}=\sum_{l,m}\omega\,|\langle\Phi^{\text{out}}_{lm\omega},e_{\alpha}J^{\alpha}\rangle|^2\,, \tag{1.6b}$$

$$\frac{dW^{(2)}}{d\omega}=\sum_{l,m}\omega\,|\langle\Phi^{\text{out}}_{lm\omega},e_{\alpha\beta}T^{\alpha\beta}\rangle|^2\,. \tag{1.6c}$$

By comparing energy formulas (1.6), one finds major differences amongst the scalar, electromagnetic, and gravitational high-frequency energy spectra. Since the radiation of a particle in a highly relativistic circular orbit is beamed into a narrow cone centered about the particle's direction of motion, the polarization vector (tensor) of the radiation is nearly orthogonal to the particle current. At high frequencies $e_\alpha J^\alpha$ (and $e_{\alpha\beta}T^{\alpha\beta}$) is much smaller in magnitude than T by a factor ψ^2 (ψ^4) involving the angular width ψ of the radiated beam. Since $\psi\propto\omega^{-1/2}$, Eqs. (1.6) imply that scalar energy is radiated more efficiently at high frequencies than either electromagnetic or gravitational energy.

Certain features of the GSR spectra, however, are independent of spin. For an unbound circular geodesic orbit with large energy-at-infinity per unit mass, γ, the radiation spectra have an exponential cutoff $\exp(-2\omega/\omega_{\text{crit}})$. The critical frequency is a high harmonic $\omega_{\text{crit}}=m_{\text{crit}}\omega_0$ of the fundamental ω_0, with

$$m_{\text{crit}} = \frac{2\sqrt{3}}{\pi}\left(\frac{r_\gamma + 3M}{\sqrt{r_\gamma M}}\right)\gamma^2 . \quad (1.7)$$

Throughout the paper r_γ will denote the radius of the prograde null circular orbit, which, for arbitrary a, is given by

$$r_\gamma = 2M\left\{1 + \cos\left[\tfrac{2}{3}\arccos\left(\frac{-a}{M}\right)\right]\right\} . \quad (1.8)$$

(For retrograde orbits let $a \to -a$.) As in the case of Schwarzschild GSR, equatorial beaming occurs although the half-width $\Delta\vartheta$ of the beam widens slowly as a increases. Specifically

$$\Delta\vartheta = |m|^{-1/2}(1 - a^2\omega_0^{\,2})^{-1/4} , \quad (1.9)$$

where

$$\omega_0 = M^{1/2}(r_0^{\,3/2} + M^{1/2}a)^{-1} \quad (1.10)$$

is the frequency of the prograde geodesic circular orbit at radius r_0 (see Appendix A). For $\gamma \gg 1$ one will have $r_0 \simeq r_\gamma$.

A detailed analysis of the scalar radiation spectrum is presented in the next section. In Sec. III, conjectured analogs of the Green's function "factorization" (1.2) are formulated for vector and tensor wave equations, and their application in the high-frequency limit is outlined. The specific application of this formalism to radiation from energetic circular Kerr orbits is given in Sec. IV, leading to approximate analytic power formulas for high-frequency electromagnetic and gravitational radiation. These approximate power spectra, in the limit $a \to 0$, are in excellent agreement with previously derived exact Schwarzschild formulas.[17] Conclusions are drawn in Sec. V.

II. SCALAR RADIATION

To learn the details of the scalar power spectrum, one must solve the relativistic wave equation

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\, g^{\alpha\beta}\frac{\partial\Phi}{\partial x^\beta}\right) = -4\pi f\sqrt{-g}\, T . \quad (2.1)$$

Here f is the coupling constant and $g_{\alpha\beta}$ are the components of the Kerr metric tensor in the Boyer-Lindquist[18] coordinate system with line element

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\varphi)^2 \\ &+ \frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)d\varphi - a dt]^2 \\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 , \end{aligned} \quad (2.2)$$

where $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\theta$ so $(-g)^{1/2} = \rho^2\sin\theta$. When the source of the scalar waves is a point particle of mass $\mu \ll M$ on world line $z(\tau)$,

$$\sqrt{-g}\, T = \mu \int d\tau\, u^\mu u_\mu \delta^4(x - z(\tau)) . \quad (2.3)$$

Brill *et al.*[19] have shown that the solution to the wave equation (2.1) is

$$\Phi(\vec{r}, t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty}\chi_{lm}(r)Z_l^m(\theta,\varphi)e^{-i\omega t}d\omega , \quad (2.4)$$

with

$$Z_l^m(\theta,\varphi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} S_{ml}(-a^2\omega^2, \cos\theta)e^{im\varphi} . \quad (2.5)$$

$S_{ml}(-a^2\omega^2, \cos\theta)$ is an oblate spheroidal angular function chosen so that $Z_l^m(\Omega)$ satisfies the normalization convention $\int |Z_l^m|^2 d\Omega = 1$. The angular equation,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_{ml}}{d\theta}\right) + \cos^2\theta\left(\frac{m^2}{\sin^2\theta} - a^2\omega^2\right)S_{ml} = QS_{ml} , \quad (2.6)$$

is studied in some detail in Appendix B.

The radial factor $\chi_{lm}(r)$ is the solution of the differential equation

$$\begin{aligned} -\Delta\frac{d}{dr}\left(\Delta\frac{d\chi_{lm}}{dr}\right) + [\Delta(Q + m^2 + a^2\omega^2) - a^2m^2 + 4Ma\omega r \\ - (r^2 + a^2)^2\omega^2]\chi_{lm} = 4\pi f\Delta(\rho^2 T)_{lm\omega} , \end{aligned} \quad (2.7)$$

where

$$(T\rho^2)_{lm\omega} = \frac{1}{2\pi}\int e^{i\omega t}Z_l^{m*}(\Omega)(\rho^2 T)d\Omega\, dt , \quad (2.8)$$

and Q is given by (B11). Introduce a new radial coordinate and a new radial function:

$$r^* = \int_{r_\gamma}^{r}\frac{(r^2 + a^2)dr}{\Delta} , \quad \chi_{lm}(r) = \frac{u_{lm}(r)}{(r^2 + a^2)^{1/2}} . \quad (2.9)$$

Then the radial equation (2.7) becomes a one-dimensional Schrödinger-type equation:

$$-\frac{d^2u_{lm\omega}}{dr^{*2}} + \omega^2 V(r)u_{lm\omega} = \frac{4\pi f\Delta}{(r^2 + a^2)^{3/2}}(\rho^2 T)_{lm\omega} . \quad (2.10)$$

The effective potential $V(r)$ is given by

$$V(r) = -1 + (r^2 + a^2)^{-2}[4Mabr - a^2b^2 + \Delta(\overline{Q} + b^2 + a^2)] + (r^2 + a^2)^{-3}\omega^{-2}\left[\Delta(3r^2 - 4Mr + a^2) - \frac{3\Delta^2 r^2}{r^2 + a^2}\right] , \quad (2.11)$$

with $b \equiv m/\omega$ and $\overline{Q} \equiv Q/\omega^2$.

To calculate the scalar power radiated to infinity one uses

$$P_{\text{out}} = -\lim_{r\to\infty} \int T^r{}_t\, \rho^2\, d\Omega\ , \tag{2.12a}$$

where

$$T^{\mu\nu} = \frac{1}{4\pi}\left(\Phi^{;\mu}\Phi^{;\nu} - \tfrac{1}{2}g^{\mu\nu}\Phi_{;\lambda}\Phi^{;\lambda}\right). \tag{2.12b}$$

From Eqs. (2.4), (2.9), and the above relations, it follows that the total scalar energy radiated is

$$W^{(0)}_{\text{out}} = \int P^{(0)}_{\text{out}}\, dt$$

$$= \lim_{r\to\infty} \int_0^\infty d\omega \sum_{m=-l}^{l} \sum_{l=0}^{\infty} \omega^2 |u_{lm\omega}|^2 . \tag{2.13}$$

This allows one to define the energy spectrum

$$\frac{dW^{(0)}_{\text{out}}}{d\omega} = \lim_{r\to\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \omega^2 |u_{lm\omega}|^2 . \tag{2.14}$$

To derive these energy formulas, the asymptotic form $du_{lm\omega}/dr \sim i\omega u_{lm\omega}$ for outgoing waves when $r\to\infty$ and the reality condition $(u_{lm\omega}Z_l^m)^* = u_{l\,-m\,-\omega}Z_l^{-m}$ have been used.

The solution of the Schrödinger-type radial equation (2.10) can be given in the Green's function form:

$$u_{lm\omega}(r) = \int G(r^*, r_s^*)\left[\frac{4\pi f\Delta}{(r^2+a^2)^{3/2}}(\rho^2 T)_{lm\omega}\right]_{r=r_s} dr_s^* . \tag{2.15}$$

The Green's function, as in Ref. 4, is formed by matching solutions of the homogeneous equation $-u'' + \omega^2 V u = 0$ with a discontinuity prescribed by the source δ function in $-G'' + \omega^2 VG = \delta(r^* - r_s^*)$. The result is

$$G(r^*, r_s^*) = \frac{i}{2}\frac{\omega}{|\omega|} \times \begin{cases} L(r_s^*)R(r^*)\,, & r^* > r_s^* \\ L(r^*)R(r_s^*)\,, & r^* < r_s^* \end{cases} . \tag{2.16}$$

Here R and L are scattering solutions of the homogeneous equation. They incorporate the boundary conditions[20] implicit in the asymptotic forms

$$L(r^*) \sim \begin{cases} |k_+|^{-1/2}[\exp(-ik_+r^*) + \mathcal{S}\exp(ik_+r^*)], & r^* \to +\infty \\ |k_-|^{-1/2}\,\mathcal{T}\exp(-ik_-r^*)\,, & r^* \to -\infty \end{cases} \tag{2.17a}$$

and

$$R(r^*) \sim \begin{cases} |k_+|^{-1/2}\exp(ik_+r^*)\,, & r^* \to +\infty \\ |k_-|^{-1/2}[\mathcal{T}^{-1}\exp(ik_-r^*) - (\mathcal{S}/\mathcal{T})^*\exp(-ik_-r^*)]\dfrac{k_-}{|k_-|}\dfrac{k_+}{|k_+|}\,, & r^* \to -\infty \end{cases} \tag{2.17b}$$

where $k_+ = \omega$, $k_- = \omega - ma(r_+{}^2 + a^2)^{-1}$, and $r_+ = M + (M^2 - a^2)^{1/2}$. The scattering and transmission amplitudes are denoted by $\mathcal{S}$ and $\mathcal{T}$, respectively.

Define free wave states (solutions of the homogeneous wave equation) as follows:

$$\Phi^{\text{up}}_{lm\omega} = R(r^*)(r^2+a^2)^{-1/2}Z_l^m(\theta,\varphi)e^{-i\omega t}\,, \tag{2.18a}$$

$$\Phi^{\text{out}}_{lm\omega} = L^*(r^*)(r^2+a^2)^{-1/2}Z_l^m(\theta,\varphi)e^{-i\omega t}. \tag{2.18b}$$

Thus $\Phi^{\text{up}}_{lm\omega}$ is the scattering solution whose initial state ($t\to-\infty$) is a wave coming "up" from the horizon, and $\Phi^{\text{out}}_{lm\omega}$, formed from the complex conjugate of $L(r^*)$, denotes the solution which has an *out*going state ($t\to+\infty$) consisting of a single wave approaching $r^* = +\infty$ (no "down the black hole" component). See Fig. 1. These two solutions are related to each other by the condition that they asymptotically coincide (in both amplitude and phase) at future null infinity $\mathcal{I}^+$. In terms of these two free wave states, the solution to wave equation (2.15) is

$$\Phi(x) = +i\int_{-\infty}^{\infty} d\omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Phi^{\text{up}}_{lm\omega}(x)\frac{\omega}{|\omega|}\langle\Phi^{\text{out}}_{lm\omega}, fT\rangle \tag{2.19}$$

when $r > r_s$. Here the "inner product" is defined by

$$\langle\Phi, fT\rangle = \int d^4x_s(-g)^{1/2}\Phi^*(x_s)fT(x_s)\,. \tag{2.20}$$

This form shows that each state of the emitted wave is generated by the inner product of the source current with an associated wave state, and suggests an immediate generalization, which will be described in Sec. III, to the cases of the corresponding vector and tensor wave equations.

Using Eq. (2.14) and asymptotic forms (2.17), one finds that the spectrum of energy radiated out to $r = +\infty$ is given by

$$\frac{dW^{(0)}_{\text{out}}}{d\omega} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \omega\,|\langle\Phi^{\text{out}}_{lm\omega}, fT\rangle|^2 . \tag{2.21}$$

In the case of circular particle motion, (2.3) becomes

$$\sqrt{-g}\,T = -\mu\left(\frac{dz^0}{d\tau}\right)^{-1}\delta(r-r_0)\,\delta\left(\theta-\frac{\pi}{2}\right)\delta(\varphi-\omega_0 t)\,, \tag{2.22}$$

where orbital frequency $\omega_0 = d\varphi/dt$ is related to the radius of the orbit r_0 by (A9). By evaluating the inner product, (2.21) simplifies to

$$\frac{dW^{(0)}_{\text{out}}}{d\omega} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{f^2\mu^2 4\pi^2\omega}{(dz^0/d\tau)^2}\frac{|L(r_0^*)|^2}{r_0^{\,2}+a^2}\,|Z_l^m(\pi/2,0)|^2 \times[\delta(m\omega_0-\omega)]^2\,. \tag{2.23a}$$

Thus, the total scalar energy radiated is

$$W^{(0)}_{\text{out}} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{4\pi^2 f^2\mu^2\omega}{(dz^0/d\tau)^2}\frac{|L(r_0^*)|^2|Z_l^m(\pi/2,0)|^2}{r_0^{\,2}+a^2}\,\delta(0)\,, \tag{2.23b}$$

with $\omega = m\omega_0$. The scalar power formula follows from Eq. (2.23) by assuming that the particle radiates in the interval $-T \leq t \leq T$ where $T \gg M$. Then,

$$W^{(0)}_{\text{out}} = \int P^{(0)}_{\text{out}}\,dt = 2TP^{(0)}_{\text{out}} \tag{2.24a}$$

and

$$\delta(0) = T/\pi\,, \tag{2.24b}$$

so that

$$P^{(0)}_{\text{out}} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{2\pi f^2\mu^2 m\omega_0}{(dz^0/d\tau)^2}\frac{|L(r_0^*)|^2|Z_l^m(\pi/2,0)|^2}{r_0^{\,2}+a^2} \tag{2.25}$$

is the scalar power radiated by a point particle in a circular orbit.

Introduced as being the solution of $-u'' + \omega^2 V u = 0$ which represents a wave incident from $r^* = \infty$ scattered by the potential $V(r)$, $L(r_0^*)$ can be found by using the WKB approximation. In particular,

$$L(r_0^*) \simeq e^{-i\pi/4}[\omega\kappa(r_0^*)]^{-1/2}e^{-m\omega\theta(r_0^*)}\,. \tag{2.26a}$$

The barrier penetration factor $\theta(r_0^*)$ is

$$\theta(r_0^*) = \int_{r_0^*}^{r_{\text{tp}}^*}\kappa\,dr^*\,, \tag{2.26b}$$

where r_{tp}^*, defined by $V(r_{\text{tp}}) = 0$, is the classical turning point at the outer edge of the potential barrier and

$$\kappa(r_0^*) = [V(r_0^*)]^{1/2}\,. \tag{2.26c}$$

From (2.25) it is easy to see that a necessary condition for GSR, where a large amount of power is radiated at high frequencies $M\omega \gg 1$, is that

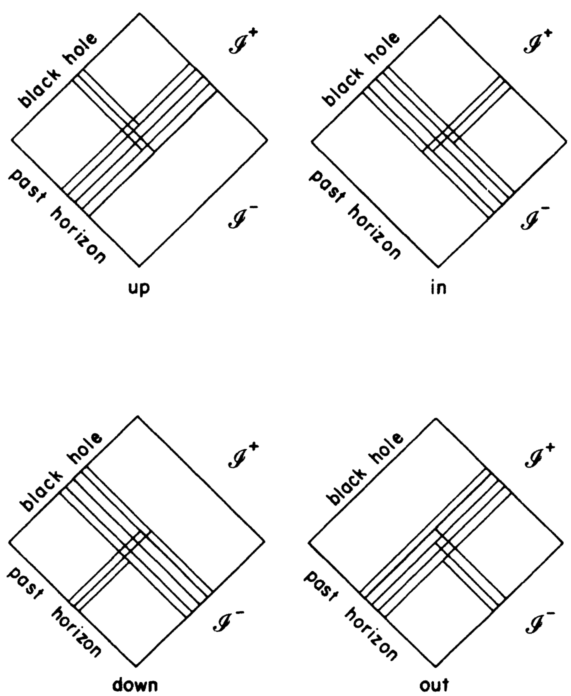

FIG. 1. The boundary conditions defining $\Phi^{\text{up}}_{lm\omega}$, $\Phi^{\text{in}}_{lm\omega}$, $\Phi^{\text{down}}_{lm\omega}$, and $\Phi^{\text{out}}_{lm\omega}$ are illustrated by sketching wave packets built from them, e.g., $\Phi^{\text{in}} = \int d\omega\, f^*(\omega)\Phi^{\text{in}}_{lm\omega}$, on Penrose conformal diagrams of the Kerr geometry. The scattering states Φ^{up} and Φ^{in} are characterized by the behavior of the incident wave packet: Φ^{up} is a wave initially coming "up" from the horizon and Φ^{in} consists of incident "ingoing" radiation. The labels "down" and "out" refer to the characteristic feature of the outgoing state; the entire wave packet is going "down" the black hole in the former case and "out" to infinity in the latter.

$\theta(r_0) \ll M$ in Eq. (2.26). Clearly this condition can only be satisfied if $V(r_0) \ll 1$, and, using Eqs. (2.11) and (A9), one sees that high-frequency radiation is exponentially damped unless $r_0 \simeq r_\gamma$.

In Appendix A it is noted that the radius of the last stable orbit r_{ls} approaches r_γ only when $a \to M$, so that $a^2 \simeq M^2$ is the only possible candidate for GSR among stable circular orbits. Specifically, when $a = (1-\alpha^2)M$ with $\alpha^2 \ll 1$

$$r_{ls} \simeq [1 + (2\alpha^2)^{1/3}]M\,, \tag{2.27}$$

while the edge of the barrier is given by

$$r_{tp} \simeq [1 + \tfrac{3}{2}(2\alpha^2)^{1/3}]M\,. \tag{2.28}$$

By approximating the potential by a parabola in the region under the barrier, one can show that[11]

$$\omega\theta(r_0^*) = \frac{m\omega_0 M\sqrt{3}}{2}\int_1^{3/2} \frac{[(x-\frac{1}{2})(\frac{3}{2}-x)]^{1/2}}{x^2}dx$$

$$\simeq 0.12\, m\omega_0 M\,. \tag{2.29}$$

Thus, there is a negligible amount of radiation in high-frequency modes from particles in stable orbits even when $a^2 \simeq M^2$.

In accordance with the necessary condition $r_0 \simeq r_\gamma$, GSR does exist for the case

$$r_0 = r_\gamma(1+\delta)\,,\quad \delta \ll (r_\gamma/M) - 1\,, \tag{2.30}$$

where, from (A12), the energy per unit rest mass satisfies

$$\gamma^2 = \frac{r_\gamma - M}{6r_\gamma\delta} \gg 1\,. \tag{2.31}$$

When $\alpha^2 \ll 1$ these relations take the form $r_\gamma \simeq [1 + (2/\sqrt{3})\alpha]M$, $\delta \ll (2/\sqrt{3})\alpha$, and $\gamma^2 \simeq 3^{-3/2}(\alpha/\delta) \gg 1$.

Since the barrier penetration factor must obey $\omega\theta(r_0) \leq 1$ at high frequencies to allow GSR, it must be demonstrated that the validity criterion

$$\left|\frac{d\omega\kappa/dr^*}{\omega^2\kappa^2}\right|_{r=r_0} = \frac{d\omega^2V/dr^*}{|\omega^2V|^{3/2}} \ll 1 \tag{2.32}$$

is satisfied before the WKB approximation can be used. At high frequencies the last two terms in Eq. (2.11) are negligible so that, by using (A9), the potential can be shown to be

$$V(r) \simeq \frac{\Delta(\bar{Q} + r_0^3M^{-1}) - (r^2 - ar_0^{3/2}M^{-1/2})^2}{(r^2+a^2)}\,, \tag{2.33}$$

where the radius of the circular orbit is given by Eq. (2.31). With this expression for the potential, the circular orbit relations in Appendix A, and some lengthy algebra, one finds

$$V(r_0) \simeq \frac{r_0^4 W}{(r_0^2+a^2)^2}\,, \tag{2.34a}$$

$$\frac{dV(r_0)}{dr^*} \simeq \frac{-\Delta_0(r_0^2-a^2)(r_0-M)^2r_0^2}{2(r_0^2+a^2)^4\gamma^2M}\,, \tag{2.34b}$$

when $r_0 \simeq r_\gamma$. In (2.34), W is defined by

$$W = \frac{\sqrt{3}\,\Delta_\gamma}{\omega r_\gamma^3}\left(1 + 2k + \frac{4}{\pi}\frac{m}{m_{crit}}\right), \tag{2.35a}$$

with $\Delta_\gamma = r_\gamma^2 - 2Mr_\gamma + a^2$, $k \equiv l - m$, and

$$m_{crit} = \frac{2\sqrt{3}}{\pi}\left(\frac{r_\gamma + 3M}{(r_\gamma M)^{1/2}}\right)\gamma^2\,. \tag{2.35b}$$

The above can be substituted into inequality (2.32) to find that the WKB approximation is valid whenever

$$\frac{1}{m^{1/2}}\left\{\frac{(r^2-a^2)(r-M)^2(r+3M)}{8(3M^5)^{1/4}(r^2+a^2)\Delta^{1/2}(r^{3/2}+aM^{1/2})^{1/2}}\right\}_{r=r_0}\frac{(4/\pi)m/m_{crit}}{[1+2k+(4/\pi)m/m_{crit}]^{3/2}} \ll 1\,. \tag{2.36}$$

Since the factor in curly brackets has values in the range $0 \leq \{\ \}_{r_0} \leq 3^{-1/2}$ when evaluated at $r = r_0 \simeq r_\gamma$ for any $|a/M| \leq 1$, it can immediately be seen that WKB methods are valid at all frequencies (for all m and k, that is) when $m_{crit} \gg 1$. Thus, to find $L(r_0^*)$, one simply must calculate the penetration factor (2.26b). In the region near the peak of the potential, $V(r)$ is approximately a parabola:

$$V(r) \simeq V(r_\gamma) + \frac{1}{2}\frac{d^2V(r_\gamma)}{dr^{*2}}(r^* - r_\gamma)^2$$

$$\simeq \frac{r_\gamma^4W}{(r_\gamma^2+a^2)^2} + \frac{3\Delta_\gamma^2r_\gamma^2}{(r_\gamma^2+a^2)^4}(r^*-r_\gamma^*)^2\,. \tag{2.37}$$

When this parabolic form for $V(r)$ is substituted into (2.26b), integration from the classical turning point

$$r_{tp} \simeq r_\gamma[1 + (\tfrac{1}{3}W)^{1/2}] \tag{2.38}$$

to r_0 yields

$$\omega\theta(r_0) \simeq \frac{\pi}{4}\left(1 + 2k + \frac{4}{\pi}\frac{m}{m_{crit}}\right) \equiv \frac{\pi\epsilon}{4}\,, \tag{2.39}$$

so that

$$|L(r_0^*)|^2 \simeq \frac{r_0^2+a^2}{\omega^{1/2}r_0^{1/2}3^{1/4}\Delta^{1/2}}\frac{e^{-\pi\epsilon/2}}{\epsilon^{1/2}}\,. \tag{2.40}$$

To obtain Eq. (2.40), (2.34a) has been employed to find $\kappa(r_0)$.

With the aid of Eqs. (2.40) and (B15), power formula (2.25) becomes

$$P_{\text{out}}^{(0)}(k,m)=\frac{8f^2\mu^2(3Mr_\gamma)^{1/2}(r_\gamma-M)}{\pi^{3/2}r_\gamma{}^2(r_\gamma+3M)^2}\epsilon^{-1/2}\left|\frac{m}{m_{\text{crit}}}\right|\frac{k!}{[(k/2)!]^2 2^k}e^{-\pi\epsilon/2}\,,\quad k \text{ even}$$
$$=0\,,\quad k \text{ odd} \tag{2.41}$$

where ϵ, r_γ, and m_{crit} are given by (2.39), (A6), and (2.35b), respectively. The total power radiated is

$$P_{\text{out}}^{(0)}=\sum_{m=0}^{\infty}\sum_{\substack{k=0\\(\text{even})}}^{\infty}P_{\text{out}}^{(0)}(k,m)\,, \tag{2.42}$$

and a power spectrum can be defined via

$$\frac{dP_{\text{out}}^{(0)}(\omega)}{d\omega}=\frac{1}{\omega_0}\sum_{\substack{k=0\\(\text{even})}}^{\infty}P_{\text{out}}^{(0)}(k,m=\omega/\omega_0)\,, \tag{2.43}$$

with $\omega_0=\Delta\omega$ the frequency interval between modes m and $m+1$, so that by definition

$$P_{\text{out}}^{(0)}=\sum_{m=0}^{\infty}\frac{dP_{\text{out}}^{(0)}(\omega)}{d\omega}\Delta\omega\simeq\int\frac{dP_{\text{out}}^{(0)}(\omega)}{d\omega}\omega_0\,dm\,. \tag{2.44}$$

The power spectrum series converges very rapidly with the $k=0$ mode accounting for over 99.9% of the power radiated. Hence,

$$\omega_0\frac{dP_{\text{out}}^{(0)}}{d\omega}=\frac{8f^2\mu^2(3M)^{1/2}(r_\gamma-M)e^{-\pi\epsilon/2}}{\pi^{3/2}r_\gamma{}^{3/2}(r_\gamma+3M)^2\epsilon^{1/2}}\left|\frac{m}{m_{\text{crit}}}\right|, \tag{2.45}$$

with

$$\epsilon=1+\frac{4}{\pi}\frac{m}{m_{\text{crit}}}.$$

The important features of the scalar power spectrum will be discussed after the vector and tensor spectra have been found.

III. ASYMPTOTICALLY FACTORIZED GREEN'S FUNCTIONS AT HIGH FREQUENCIES

In order to generalize Eq. (2.11) to the vector (electromagnetic radiation) and tensor (gravitational radiation) cases, some new techniques are required. We seek methods that exploit the simple relations between scalar, vector, and tensor wave equations that are known to exist in the geometrical optics limit. This avoids the difficulties that still remain in the separation of variables[15] for vector and tensor wave equations in the Kerr geometry. It also, by being more direct, adds physical insight to the high-frequency solutions even in the Schwarzschild geometry where complete solutions by separation of variables have been given.[5] The first step will be to formalize some properties of the Green's function solution of the scalar wave equation given in Eq. (2.19), so the corresponding vector and tensor equations can be written with all the normalization factors correct. Then the high-frequency terms in this expansion, which alone are of interest for GSR, can be approximated in a geometrical optics limit.

Consider two solutions of the scalar wave equation,

$$\Box\psi=-4\pi S,\quad \Box\phi=-4\pi T, \tag{3.1}$$

and form from them the vector

$$j^\mu=\frac{1}{4\pi i}(\psi^*\nabla^\mu\phi-\phi\nabla^\mu\psi^*) \tag{3.2}$$

whose divergence is

$$j^\mu{}_{;\mu}=i(\psi^*T-S^*\phi)\,. \tag{3.3}$$

Upon integrating this divergence over a 4-volume V with boundary $\partial V=\Sigma$, one finds

$$i(\langle\psi,T\rangle_V-\langle S,\phi\rangle_V)=\oint_{\partial V}j^\mu d^3\Sigma_\mu\equiv\langle\psi,\phi\rangle_{\partial V}\,. \tag{3.4}$$

On the left of this equation occur 4-dimensional inner products $\langle\ ,\ \rangle_V$ as defined in Eq. (2.20). On the right occurs a 3-dimensional Klein-Gordon inner product. This equation will lead to a relation between the volume integrations in the Green's function equation (2.19) and the 3-dimensional integrals characterizing the normalization and completeness of the functions in which that Green's function was factored. A Green's function $G(x,y)$ of the scalar wave equation will satisfy

$$\Box_x G(x,y)=-4\pi\delta(x,y)\,, \tag{3.5}$$

where the δ function[21] satisfies $\langle\delta(x,y),\phi(y)\rangle_M=\phi(x)$, and M is the entire spacetime volume. Let $G^+(x,y)$ [respectively $G^-(x,y)$] be the Green's function[22] which vanishes unless x is in the causal future [past] of y, and in Eq. (3.4) choose

$$\psi(y)=G^-(y,x)=[G^+(x,y)]^*. \tag{3.6}$$

The result is

$$\Phi(x)=\int_M G^+(x,y)T(y)(-g)^{1/2}d^4x$$
$$=\langle\delta(y,x),\Phi(y)\rangle_V-i\langle G^-(y,x),\Phi(y)\rangle_{\partial V}\,. \tag{3.7}$$

The first equality here is just the usual definition of the retarded solution of the wave equation with source T. For the second equality one assumed that T vanished except within the volume V, so this integral over M could be identified with $\langle G^-(y,x), T(y)\rangle_V$ to which Eq. (3.4) was applied with the result shown.

Refer now to Fig. 2 to see that the surface integral $\langle\ ,\ \rangle_{\partial V}$ in Eq. (3.7) will vanish when x is in V, so the second equality verifies the first. But when x lies outside V, the δ function vanishes throughout V so $\langle\delta(y,x_{out}),\Phi(y)\rangle_V=0$, and only the surface integral contributes. For consistency one must then have

$$\Phi(x_{out}) = -i\langle G^-(y,x_{out}),\Phi(y)\rangle_{\partial V}\ . \tag{3.8}$$

Next introduce the "factorization" of the Green's function shown in Eq. (2.19), which holds only when $x=x_>$ is farther out than y spatially:

$$G^+(x_>,y)=[G^-(y,x_>)]^* = i\int_{-\infty}^{\infty}d\omega\sum_{l,m}\Phi^{up}_{lm\omega}(x_>)[\Phi^{out}_{lm\omega}(y)]^*\frac{\omega}{|\omega|}\ . \tag{3.9}$$

This form of G^- may be used in Eq. (3.8) with the result

$$\Phi(x_>)=\int_{-\infty}^{\infty}d\omega\frac{\omega}{|\omega|}\sum_{l,m}\Phi^{up}_{lm\omega}(x_>)\langle\Phi^{out}_{lm\omega},\Phi\rangle_{\partial V}\ . \tag{3.10}$$

In the inner product $\langle\Phi^{out},\Phi\rangle_{\partial V}$, Φ vanishes on the parts of ∂V near the past horizon and near past null infinity $\mathscr{I}^-$ (see Fig. 2), while Φ^{out} vanishes on the part near the future horizon, so the integral can be restricted to the boundary near future null infinity $\mathscr{I}^+$. But near $\mathscr{I}^+$, both Φ^{up} and Φ^{out} have been defined in Eqs. (2.17) and (2.18) to have a common asymptotic form,

$$\Phi^{up}_{lm\omega}\sim\Phi^{out}_{lm\omega}\sim\Phi_{lm\omega}\sim|\omega|^{-1/2}e^{i\omega r}r^{-1}Z^m_l(\theta,\varphi)e^{-i\omega t}\ , \tag{3.11}$$

and Eq. (3.10) is a completeness requirement on the wave functions near $\mathscr{I}^+$.

The normalization conditions on $\Phi_{lm\omega}$ can be seen by choosing $\Phi=\Phi_{l'm'\omega'}$ in Eq. (3.10). One evidently must have

$$\langle\Phi_{lm\omega},\Phi_{l'm'\omega'}\rangle_{\mathscr{I}^+}=\frac{\omega}{|\omega|}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\ . \tag{3.12}$$

This condition is readily verified from Eq. (3.11), provided one orients $\mathscr{I}^+$ correctly as the limit of a part of ∂V:

$$\langle\psi,\phi\rangle_{\mathscr{I}^+}=\lim_{r\to\infty}\frac{1}{4\pi i}\int_{-\infty}^{\infty}dt\oint d\Omega\, r^2(\psi^*\partial_r\phi-\phi\partial_r\psi^*)\ . \tag{3.13}$$

The principal result of the foregoing discussion for our present purposes is the link between the Kirchhoff formula (3.7) and the normalization (3.12) required for a factorization (3.9). By writing the corresponding vector and tensor Kirchhoff formulas, then, we can determine the normalizations required for similar factorizations of the vector and tensor Green's functions.

In the vector case, the wave equation in the Lorentz gauge is

$$\Delta A^\mu \equiv -A^{\mu;\nu}{}_{;\nu}+R^\mu{}_\nu A^\nu=4\pi J^\mu\ , \tag{3.14}$$

and Green's functions will satisfy

$$\Delta_x G^\mu{}_{\alpha'}(x,x') = -G^\mu{}_{\alpha'}{}^{;\nu}{}_{;\nu}+R^\mu{}_\nu G^\nu{}_{\alpha'} = 4\pi g^\mu_{\alpha'}\delta(x,x')\ , \tag{3.15}$$

where the bitensor notation follows DeWitt and Brehme,[21] except that our $\delta(x,x')$ includes the factors required to make it a biscalar. The 3-dimensional inner product will be defined by

$$\langle\psi^\mu,A_\mu\rangle_\Sigma\equiv\int j^\mu d^3\Sigma_\mu\ , \tag{3.16a}$$

where

$$j^\mu=\frac{1}{4\pi i}g^{\alpha\beta}(\psi^*_\alpha\nabla^\mu A_\beta - A_\beta\nabla^\mu\psi^*_\alpha)\ , \tag{3.16b}$$

and the choice

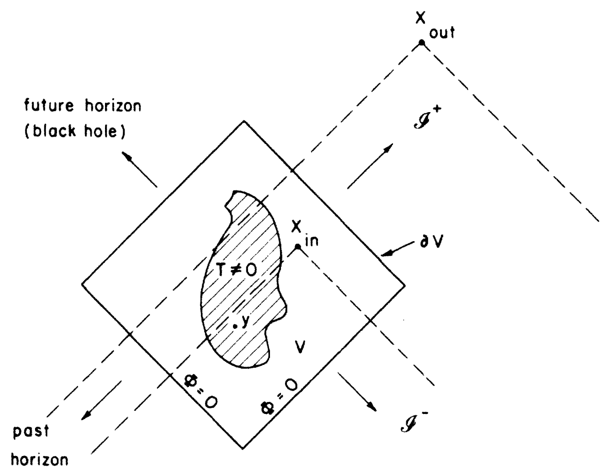

FIG. 2. A retarded solution $\Phi(x)$ generated by a bounded source $T(y)$ by application of the Kirchhoff formula (3.7) over a bound region V. If $x=x_{in}$ is inside V, the δ-function integral $\langle\delta(y,x),\ \Phi(y)\rangle_V$ gives $\Phi(x_{in})$, but it vanishes for x_{out}. The surface integral $\langle G^-(y,x),\ \Phi(y)\rangle_V$ has contributions only from the past cone of x, and vanishes for $x=x_{in}$ since $\Phi(y)$ vanishes on the part of the boundary in this cone. But when $x=x_{out}$ lies outside V but at large r, the boundary approaching $\mathscr{I}^+$ (null future infinity) gives a nonzero contribution (although the other three components of the boundary do not), and Eq. (3.8) holds.

$$\psi_{\alpha'}(x') = G^{-}_{\alpha'\beta}(x', x) = [G^{+}_{\beta\alpha'}(x, x')]^* \quad (3.17)$$

leads to

$$A_\mu(x) = \int G^{+}_{\mu\alpha'} J^{\alpha'}(-g')^{1/2} d^4x'$$
$$= \langle g_\mu^{\alpha'} \delta(x, x'), A_{\alpha'}(x')\rangle_V - i\langle G^{-}_{\alpha'\mu}, A^{\alpha'}\rangle_{\partial V} , \quad (3.18)$$

in complete analogy to Eq. (3.7). We therefore presume that there exists (at least asymptotically) a factorization analogous to Eq. (3.9),

$$G^{+}_{\mu\alpha'}(x_>, x') = i\int_{-\infty}^{\infty} d\omega \sum_{lmPP'} A^{\rm up}_\mu(x_>, lm\omega P) g^{PP'} \frac{\omega}{|\omega|} \times [A^{\rm out}_{\alpha'}(x', lm\omega P')]^* , \quad (3.19)$$

in which P is a polarization state label, and $g^{PP'}$ is a metric (more below) for asymptotic polarization states. The normalization requirement, from Eq. (3.18) with x near $\mathscr{G}^+$, is

$$A^{\rm up}_\mu \sim A^{\rm out}_\mu \sim A_\mu(x, lm\omega P) \sim e_\mu(P)|\omega|^{-1/2} e^{i\omega r_*} r^{-1} Z_l^m(\theta, \varphi) e^{-i\omega t} , \quad (3.20)$$

with

$$e_\mu(P)^* e^\mu(P') = g_{PP'} , \quad (3.21)$$

so that

$$\langle A_\mu(lm\omega P), A^\mu(l'm'\omega'P')\rangle_{\mathscr{G}^+} = g_{PP'} \frac{\omega}{|\omega|} \delta_{ll'}\delta_{mm'}\delta(\omega - \omega') . \quad (3.22)$$

The metric $g_{PP'}$ (reciprocal $g^{PP'}$) appears in Eq. (3.21) since four independent polarization states can only be found if these indefinite (Minkowski) normalizations are allowed.

One computes the power radiated in electromagnetic waves from Eq. (2.12a), now using the Maxwell stress-energy tensor

$$T^{\mu\nu} = \frac{1}{4\pi}(F^{\mu\alpha}F^\nu{}_\alpha - \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}) . \quad (3.23)$$

The leading terms for $r \to \infty$ are given by

$$T^r{}_t \sim (4\pi)^{-1}(A^{\theta;r}A_{\theta;t} + A^{\varphi;r}A_{\varphi;t}) . \quad (3.24)$$

Because this is so precisely analogous to the scalar formula $T^r{}_t = (4\pi)^{-1}\Phi^{;r}\Phi_{;t}$, one sees that the formula for the spectrum, analogous to Eq. (2.21) will be [here $\omega > 0$ and $W_{\rm out} = \int_0^\infty (dW_{\rm out}/d\omega) d\omega$]

$$\frac{dW^{(1)}_{\rm out}}{d\omega} = \sum_{l,P=T} \sum_{m=-l}^{l} \omega |\langle A^{\rm out}_\mu(lm\omega P), J^\mu\rangle|^2 . \quad (3.25)$$

Here the polarization sum is only ("$P = T$") over transverse polarizations (unit vectors in the θ and φ directions) $e_\mu(\perp) = \delta_\mu^{\hat\theta}$ and $e_\mu(\|) = \delta_\mu^{\hat\varphi}$ since the other polarizations do not contribute in Eq. (3.24). For these transverse waves $g_{PP'} = \delta_{PP'}$, so the polarization metric does not appear explicitly in Eq. (3.25).

After defining a tensor wave operator

$$\Delta\phi^{\mu\nu} \equiv -\phi^{\mu\nu;\rho}{}_{;\rho} - 2R^\mu{}_\sigma{}^\nu{}_\tau \phi^{\sigma\tau} \quad (3.26)$$

and an algebraic operation

$$\bar\phi^\mu{}_\nu = \phi^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu \phi^\alpha{}_\alpha , \quad (3.27)$$

Sciama, Waylen, and Gilman[23] write the Einstein equations in the form

$$\Delta g^{\mu\nu} = -16\pi g^{\lambda(\mu}\bar T^{\nu)}{}_\lambda . \quad (3.28)$$

This form has the interesting property that it is preserved under small variations of $T^\mu{}_\nu$, so that when

$$T^\mu{}_\nu \to T^\mu{}_\nu + \delta T^\mu{}_\nu$$

and

$$g^{\mu\nu} \to g^{\mu\nu} - h^{\mu\nu} \quad (3.29)$$

one finds to first order that

$$\Delta h^{\mu\nu} = 16\pi g^{\lambda(\mu}\delta\bar T^{\nu)}{}_\lambda ,$$

provided a gauge with $\bar h_\alpha{}^\beta{}_{;\beta} = 0$ is used. We restrict attention to the case where $g_{\mu\nu}$ is a vacuum spacetime metric. Then the perturbation equation for the $h^{\mu\nu}$ generated by a small perturbation $T^{\mu\nu}$ is

$$\Delta h^{\mu\nu} = 16\pi \bar T^{\mu\nu} , \quad (3.30)$$

and, since $R_{\mu\nu} = 0$, Δ coincides also with the Lichnerowicz-deRham wave operator.[24] In this vacuum background metric case with $R_{\mu\nu} = 0$, Δ has the following properties:

$$g_{\mu\nu}(\Delta\phi^{\mu\nu}) = \Delta(g_{\mu\nu}\phi^{\mu\nu}) , \quad (3.31a)$$

$$\Delta(\phi g^{\mu\nu}) = g^{\mu\nu}(\Delta\phi) , \quad (3.31b)$$

$$(\Delta\phi^{\mu\nu})_{;\nu} = \Delta(\phi^{\mu\nu}{}_{;\nu}) . \quad (3.31c)$$

The wave operators on the right-hand sides are as appropriate, the vector or scalar wave operator, e.g., $\Delta\phi \equiv -\phi^{;\mu}{}_{;\mu}$. Thus any solution of Eq. (3.30) will equivalently satisfy

$$\Delta\bar h^{\mu\nu} = 16\pi T^{\mu\nu} \quad (3.32)$$

and, as a consequence,

$$\Delta(\bar h^{\mu\nu}{}_{;\nu}) = 0 \quad (3.33)$$

since $T^{\mu\nu}{}_{;\nu}=0$. In particular, any purely retarded solution of Eqs. (3.30) or (3.32) will satisfy the required gauge condition

$$\bar{h}^{\mu\nu}{}_{;\nu}=0 \tag{3.34}$$

since Eq. (3.33) has no nonvanishing purely retarded solutions. Note, however, that $h^{\mu\nu}$ cannot be expected to then satisfy $h=0$ even asymptotically for $x\to\mathscr{I}^+$ since $\Delta h=-16\pi T\neq 0$. To achieve $h=0$ near $\mathscr{I}^+$ would require a gauge transformation, $h_{\mu\nu}\to h_{\mu\nu}+2\xi_{(\mu;\nu)}$ with $\Delta\xi^\mu=0$, which would not leave $h^{\mu\nu}=0$ in the asymptotic past.

Note that Δ is self-adjoint, i.e.,

$$\langle\psi^{\mu\nu},\Delta\phi_{\mu\nu}\rangle_M=\langle\Delta\psi^{\mu\nu},\phi_{\mu\nu}\rangle_M$$

holds for tensor fields $\psi^{\mu\nu}$, $\phi^{\mu\nu}$ with compact support. Therefore the Green's functions[24] will have symmetries as in the scalar case (3.17). They satisfy

$$\Delta_x G^{\mu\nu}{}_{\alpha'\beta'}(x,x')=4\pi g^{(\mu}_{(\alpha'}g^{\nu)}_{\beta')}\delta(x,x')\,. \tag{3.35}$$

Define $\langle\psi^{\mu\nu},\phi_{\mu\nu}\rangle_\Sigma\equiv\int_\Sigma j^\alpha d^3\Sigma_\alpha$ with

$$j^\alpha=\frac{1}{4\pi i}g^{\mu\sigma}g^{\nu\tau}(\psi^*_{\mu\nu}\nabla^\alpha\phi_{\sigma\tau}-\phi_{\sigma\tau}\nabla^\alpha\psi^*_{\mu\nu})\,, \tag{3.36}$$

and find

$$\begin{aligned}\bar{h}_{\mu\nu}(x)&=\int_M G^+_{\mu\nu\alpha'\beta'}4T^{\alpha'\beta'}(-g')^{1/2}d^4x'\\ &=\langle g^{\alpha'}_{\mu}g^{\beta'}_{\nu}\delta(x',x),\bar{h}_{\alpha'\beta'}(x')\rangle_V\\ &\quad -i\langle G^-_{\alpha'\beta'\mu\nu},\bar{h}^{\alpha'\beta'}\rangle_{\partial V}\,.\end{aligned} \tag{3.37}$$

Assume a factorization

$$G^+_{\mu\nu\alpha'\beta'}(x_>,x')=i\int d\omega\sum_{lmP'P}\phi^{\rm up}_{\mu\nu}(x_>,lm\omega P)g^{PP'}\frac{\omega}{|\omega|}\times[\phi^{\rm out}_{\alpha'\beta'}(x',lm\omega P')]^* \tag{3.38}$$

and deduce the normalization requirements

$$\langle\phi_{\mu\nu}(lm\omega P),\phi^{\mu\nu}(l'm'\omega'P')\rangle_{\mathscr{I}^+}=\frac{\omega}{|\omega|}g_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\,, \tag{3.39}$$

which are satisfied by

$$\phi^{\rm up}_{\mu\nu}\sim\phi^{\rm out}_{\mu\nu}\sim\phi_{\mu\nu}(lm\omega P)\sim e_{\mu\nu}(P)|\omega|^{-1/2}e^{i\omega r}r^{-1}Z^m_l(\theta,\varphi)e^{-i\omega t}\,, \tag{3.40}$$

provided that

$$e_{\mu\nu}(P)^*e^{\mu\nu}(P')=g_{PP'}\,. \tag{3.41}$$

(This tensor polarization metric will have a 7+, 3− signature.)

An Isaacson effective stress-energy tensor for gravitational wave perturbations in the $\bar{h}^{\mu\nu}{}_{;\nu}=0$ gauge is given by[25]

$$T_{\mu\nu}=(32\pi)^{-1}(h_{\alpha\beta;\mu}\bar{h}^{\alpha\beta}{}_{;\nu})\,. \tag{3.42}$$

In obtaining the formula

$$\frac{dW^{(2)}_{\rm out}}{d\omega}=\frac{1}{8}\sum_{l,P=TT}\sum_{m=-l}^{l}\tfrac{1}{2}\omega|\langle\phi^{\rm out}_{\mu\nu}(lm\omega P),4T^{\mu\nu}\rangle|^2 \tag{3.43}$$

a few minor factors differ from those in the analogous equation (3.25). The $(32\pi)^{-1}$ in Eq. (3.42) replaces $(4\pi)^{-1}$ in (3.24), and the "4"'s, inherited from Eq. (3.37), came from 16π in Eq. (3.32) versus 4π in (3.14); but by choosing $g_{PP'}=2\delta_{PP'}$ for the physical (TT) modes, one finds a factor $g^{PP'}=\frac{1}{2}\delta_{PP'}$ in Eq. (3.43) which makes all numerics cancel.

Let us consider gauge questions and the "bar" operation in (3.42) more explicitly, however. In Eq. (3.38) the sum over polarization states would have to include ten independent states P to give the full Green's function satisfying (3.35). We know, however, that the conserved source, $T^{\mu\nu}{}_{;\nu}=0$ in (3.32) leads via G^+ to a field $\bar{h}^{\mu\nu}$ satisfying $\bar{h}^{\mu\nu}{}_{;\nu}=0$. Thus four states P inconsistent with this may be omitted from the sum

$$\bar{h}_{\mu\nu}(x_>)=i\int d\omega\sum\phi^{\rm up}_{\mu\nu}(x_>,lm\omega P)\frac{\omega}{|\omega|}g^{PP'}\times\langle\phi^{\rm out}_{\alpha'\beta'}(lm\omega P'),4T^{\alpha'\beta'}\rangle \tag{3.44}$$

without changing the solution $\bar{h}_{\mu\nu}(x)$. Then when the radiated energy flux is computed from (3.42), each term in the resulting sum contains a factor $e^{\mu\nu}(P)^*\bar{e}_{\mu\nu}(P')$. One finds from $k_\mu e^{\mu\nu}(P)=0$ that

$$e^{\mu\nu}(P)^*\bar{e}_{\mu\nu}(\text{gauge})=0$$

for each of the four "gauge modes" for which, in Eq. (3.40), one sets

$$e_{\mu\nu}(\text{gauge})=k_{(\mu}C_{\nu)}-\tfrac{1}{2}g_{\mu\nu}k_\alpha C^\alpha\,.$$

Here $k=e_{\hat 0}+e_{\hat r}$ is the radial null vector (at $\mathscr{I}^+$) and C is an arbitrary vector. Omitting these gauge modes from the sum in (3.44) then changes $\bar{h}_{\mu\nu}(x)$, but not the power radiated. Thus only the remaining two "transverse traceless" polarization states need be retained in obtaining Eq. (3.43). These two states can be chosen on $\mathscr{I}^+$ as

$$\begin{aligned}e_{\mu\nu}(+)&=[e_\mu(\perp)e_\nu(\perp)-e_\mu(\|)e_\nu(\|)]\,,\\ e_{\mu\nu}(\times)&=[e_\mu(\perp)e_\nu(\|)+e_\mu(\|)e_\nu(\perp)]\,,\end{aligned} \tag{3.45}$$

which satisfy $e^\mu{}_\mu=0$ and $g_{PP'}=2\delta_{PP'}$, so $\frac{1}{2}e_{\mu\nu}(P)\bar{e}^{\mu\nu}(P')=\delta_{PP'}$, and then no "bars" need finally appear in Eq. (3.43) where the polarization sum is restricted to the two physical components wave modes. Of course the polarizations (3.45) are not well defined (and not continuous) near the poles of the $\theta\varphi$ coordinate system. More refined analysis (tensor spherical harmonics[26,27]) would be required in place of the boundary conditions (3.40) were it desired to separate out two physical modes that were convenient near the poles. In the GSR problem of this paper, where radiation well off the equator is negligible, the states defined by Eq. (3.45) are entirely adequate.

The factorized Green's functions such as (3.38) purport to be exact as indeed was shown for the scalar case (3.9). For the Schwarzschild geometry, exact vector and tensor Green's function of the prescribed form do exist and can be found[14]; it is apparent (but not proved here) that such is the case whenever the wave equations for the electromagnetic and gravitational potentials separate. Less obvious is the supposition that (3.19) and (3.38) are exact in the Kerr geometry, where separated wave equations have been found[15] for only certain scalars formed from linear combinations of electromagnetic field components and components of the Riemann tensor. Separable equations for the potentials may well exist at least at high frequencies, for in that limit inspection of Maxwell's equations reveals[28] that the decoupled equation for ϕ_1 (as well as the equations for ϕ_0 and ϕ_2) may be solved by separation of variables. Hence, for our present goal of studying high-frequency radiation emission near a Kerr black hole, the postulated vector and tensor Green's functions are highly plausible.

In more general spacetimes with wave equations that cannot be expected to be separable, we still conjecture that the factorized Green's functions exist and are exact at least asymptotically, where the condition $x_> \to \infty$ can replace the $r(x_>) > r(x')$ domain condition on the factorization in the Kerr scalar example. (Note that asymptotic validity of the Green's function is sufficient to derive the energy formulas.) A rigorous examination of this conjecture is not attempted here; we know of only limited work in this direction by Clarke and Sciama.[29]

For high frequencies $M\omega \gg 1$, consideration of the geometrical optics limit will now allow an approximate construction of the vector and tensor Green's functions from the scalar factors $\Phi_{lm\omega}$. It is essential in a geometrical optics solution that wave fronts be curved only on scales much larger than a wavelength, a condition not normally satisfied for solutions in the neighborhood of a source, and certainly not for point sources as in the problem at hand. The Green's function factorizations, however, allow us to treat free-field states $\Phi_{lm\omega}$ whose structure is not determined by the source, and for which the approximation is valid. In particular, to compute the power at high frequencies only the states $\Phi^{\text{out}}_{lm\omega}$ for those frequencies are required.

Geometrical optics for gravitational waves is discussed by Isaacson[16] (see also MTW,[30] Chap. 35). To summarize briefly, take the electromagnetic example where $\Delta A^\mu = 0 = A^\mu{}_{;\mu}$ are solved by setting

$$A^\mu = e^\mu \mathcal{A} e^{i\psi}, \tag{3.46}$$

and assume that $k_\mu \equiv \psi_{,\mu}$ is large while all other derivatives are small. The resulting conditions are

$$\psi_{;\mu}\psi^{;\mu} = 0, \tag{3.47a}$$

$$e^\mu{}_{;\beta} k^\beta = 0, \tag{3.47b}$$

$$(\mathcal{A}^2 k^\beta)_{;\beta} = 0, \tag{3.47c}$$

and

$$k_\mu e^\mu = 0. \tag{3.47d}$$

From Eq. (3.47a) one shows that the rays $dx^\mu/d\lambda = k^\mu(x)$ are null geodesics, along which by Eq. (3.47b) the polarization is parallel propagated, maintaining the normalization condition $e_\mu e^\mu = 1$ and gauge condition (3.47d). Intensities are controlled jointly by the red shifts in k^μ along geodesics, and the "conservation of photons" law (3.47c). The scalar and tensor cases are identical except for the replacement of the polarization vector e_μ by 1 or $e_{\mu\nu}$, respectively. Thus if $\Phi = \mathcal{A}e^{i\psi}$ is a scalar solution, corresponding vector and tensor waves may be formed as $A_\mu = e_\mu \mathcal{A} e^{i\psi}$ and $\phi_{\mu\nu} = e_{\mu\nu} e^{i\psi}$ by supplying unit polarization tensors e_μ or $e_{\mu\nu}$ which are parallel propagated along $k_\mu \equiv \psi_{,\mu}$.

In general, carrying out the above outline to construct $\phi^{\text{out}}_{\mu\nu}(x, lm\omega P)$ and $A^{\text{out}}_\mu(x, lm\omega P)$ from $\Phi^{\text{out}}_{lm\omega}$ when $M\omega \gg 1$ presents two difficulties. One is that not all high-frequency waves have the WKB form (3.46); the second is to integrate the equation of parallel propagation.

The first of these problems is not realized in the GSR case, for the free fields $\Phi^{\text{out}}_{lm\omega}$ were found in Sec. II to be WKB solutions; the angular and time dependence is precisely of the form $\mathcal{A}e^{i\psi}$ and the WKB approximation is valid in the radial direction. However, the methods employed here work for the more general case where the free fields are a linear combination of such WKB solutions:

$$\Phi^{\text{out}}_{lm\omega} = \sum_\psi \mathcal{A} e^{i\psi}. \tag{3.48}$$

Then the corresponding tensor would be

$$\phi^{\text{out}}_{\mu\nu}(lm\omega P) = \sum_{\psi} e_{\mu\nu}(P, \psi)\, \alpha e^{i\psi}\,, \tag{3.49}$$

with the polarizations chosen to satisfy $e^{\mu\nu}\psi_{,\nu} = 0$ appropriately in each term. We will find it possible to write this in the form

$$\phi^{\text{out}}_{\mu\nu}(lm\omega P) = e_{\mu\nu}(P)\sum_{\psi} \alpha e^{i\psi}$$

$$= e_{\mu\nu}(P)\,\phi^{\text{out}}_{lm\omega} \tag{3.50}$$

(and similarly in the vector case) by treating $e_{\mu\nu}(P)$ here as a (differential) operator which reconstructs (3.49) from (3.50) to leading order in ω^{-1}, as will be seen in the detailed treatment in the next section.

The second problem, integrating Eq. (3.47b), is simplified by the fact that we treat only point-particle sources, so the spacetime integral

$$\langle \phi^{\text{out}}_{\mu\nu}, T^{\mu\nu}\rangle_{M} = \mu \int d\tau\, u^{\mu} u^{\nu}\, \phi^{\text{out}}_{\mu\nu} \tag{3.51}$$

is seen reduced to an integral over the world line of the source particle. Different points on this world line, a circular orbit, differ only by a time translation and a φ rotation. Both the Kerr metric and the boundary condition on the polarization (3.45) are invariant under these motions, so the polarization states (for congruent rays k^{μ}) at different points of the world line are also related by these simple translations. The absolute orientation of say $e_{\mu\nu}(+)$, parallel transported from $\mathscr{I}^{+}$ back to one point on the world line, is also not required since the sum

$$\sum_{PP'} \bar{e}_{\mu\nu}(P) g^{PP'} e^{\mu\nu}(P')^{*} \tag{3.52}$$

over polarization states which appears in Eq. (3.43) (multiplying $2\omega T^{\mu\nu} T^{\alpha\beta *}$) is invariant under changes in the polarization basis. (This absolute orientation would be required to compute the polarization of the radiation instead of merely the total power. See Hughes and Misner,[31] who find no significant rotation of the polarization axes, no "gravitational Faraday rotations" for radiation near the equatorial plane of the Kerr metric.)

IV. ESTIMATED ELECTROMAGNETIC AND GRAVITATIONAL SPECTRA

In Sec. III, it was shown that the electromagnetic energy spectrum emitted by a source J in the neighborhood of a black hole can be written as

$$\frac{dW^{(1)}_{\text{out}}}{d\omega} = \sum_{P} \sum_{l,m} \omega\, |\langle A^{\text{out}}_{\alpha}(lm\omega P), J^{\alpha}\rangle|^{2}\,, \tag{4.1}$$

where A^{out} is normalized to give

$$A^{\text{out}} \sim e(P)\,|\omega|^{-1/2} e^{i\omega r} r^{-1} Z^{m}_{l}(\theta, \phi) e^{-i\omega t} \tag{4.2}$$

near $\mathscr{I}^{+}$, and the sum over P just includes the two linearly independent transverse polarization states.

When the source of the electromagnetic radiation is a particle with charge q moving on a relativistic geodesic circular orbit at radius $r_0 \simeq r_\gamma$ given by Eq. (A9), the current is

$$(-g)^{1/2} J = q(u^{0})^{-1} u\,\delta(r - r_0)\delta(\theta - \pi/2)\delta(\varphi - \omega_0 t)\,, \tag{4.3}$$

with u the 4-velocity of the particle. With the aid of the above, evaluation of the inner product in Eq. (4.1) yields that

$$P^{(1)}_{\text{out}} = \sum_{P} \sum_{l,m} 2\pi q^{2} m\omega_0 \left(\frac{dt}{d\tau}\right)^{-2} |u \cdot A^{\text{out}}(lm\omega P)|^{2} \tag{4.4}$$

is the total power radiated to infinity in the form of electromagnetic waves.

For the high-frequency ($\omega = m\omega_0 \gg M^{-1}$) modes excited by a relativistic circling particle, the vector potential takes the Isaacson form

$$A^{\text{out}}(lm\omega P) \simeq e(P)\Phi^{\text{out}}(lm\omega)\,, \tag{4.5}$$

where Φ is the scalar wave equation solution discussed in Sec. II and $e(P)$ is a polarization vector normalized to unity. It follows that

$$P^{(1)}_{\text{out}} = \sum_{P} \sum_{l,m} 2\pi q^{2} m\omega_0 \left(\frac{dt}{d\tau}\right)^{-2} \times |u \cdot e(P)\Phi^{\text{out}}(r_0, lm\omega)|^{2}\,, \tag{4.6}$$

so the problem of estimating the electromagnetic spectrum reduces to finding $u \cdot e(P)$ at the source of the radiation.

We choose

$$e(\parallel) = e_{\hat{r}} - k\left(\frac{k_{\hat{r}}}{k_{\hat{t}}^{2}}\right) - e_{\hat{t}}\left(\frac{k_{\hat{r}}}{k_{\hat{t}}}\right)\,, \tag{4.7a}$$

$$e(\perp) = e_{\hat{\theta}} - k\left(\frac{k_{\hat{\theta}}}{k_{\hat{t}}^{2}}\right) - e_{\hat{t}}\left(\frac{k_{\hat{\theta}}}{k_{\hat{t}}}\right) \tag{4.7b}$$

as the two polarization vectors that describe the physical components of the electromagnetic field at the source. In Eq. (4.7), k is the radiation momentum 4-vector, the carets denote components as measured in an orthonormal frame erected by a locally nonrotating (Bardeen) observer,[32] and "parallel" and "perpendicular" label (at the source) the orientation of the polarization states with respect to the orbital plane. At the point of emission, the high-frequency components of the radiation field are strongly beamed $k_{\hat{r}} \sim k_{\hat{\theta}} \ll k_{\hat{\varphi}}$. Hence, second-order terms $(k_{\hat{r}}/k_{\hat{\varphi}})^{2}$, $(k_{\hat{\theta}}/k_{\hat{\varphi}})^{2}$ are negligible and one can verify that to first order the polarization vectors in Eq. (4.7) are both transverse ($e \cdot k = 0$) and orthonormal [$e(P) \cdot e(P') = \delta_{PP'}$] and satisfy a further simplifying gauge condition $e(P) \cdot e_{\hat{t}} = 0$.

The choice (4.7) for the basis vectors will be correct to first order in the beaming angle. The requirements are that these vectors be the result of parallel transporting, from $\mathscr{I}^+$ back to the source, the vectors specified following Eqs. (3.25). For perfect beaming ($k_{\hat{r}} = 0 = k_{\hat{\theta}}$) this parallel transport occurs within the equatorial plane and Eqs. (4.7) are correct from elementary symmetry considerations. That no $e_{\hat{\theta}}$ term occurs in (4.7a), nor any $e_{\hat{r}}$ in (4.7b), is a consequence of the lack of any (gravitational Faraday) chiral rotations occurring in the parallel transport process—a result Hughes and Misner[31] find valid to first order in the beaming angle on the basis of more refined symmetry considerations. The gauge and orthonormality conditions then fix the remaining terms uniquely.

The inner products in Eq. (4.6) are

$$u \cdot e(\|)\Phi |_{x=x_0} = \frac{-u^{\hat{\varphi}}}{k_{\hat{t}}} k_{\hat{r}} \Phi \Big|_{x=x_0} = \frac{iu^{\hat{\varphi}}}{k_{\hat{t}}} \frac{\Delta^{1/2}}{\rho} \frac{\partial \Phi}{\partial r}\Big|_{x=x_0}, \tag{4.8a}$$

$$u \cdot e(\perp)\Phi |_{x=x_0} = \frac{u^{\hat{\varphi}}}{k_{\hat{t}}} k_{\hat{\theta}} \Phi \Big|_{x=x_0} = \frac{-iu^{\hat{\varphi}}}{k_{\hat{t}}} \frac{1}{\rho} \frac{\partial \Phi}{\partial \theta}\Big|_{x=x_0}. \tag{4.8b}$$

In the above, we recall that Φ takes a WKB form for the high-frequency modes considered here, and note that asymptotically $k_{\hat{t}}$ is the eigenvalue of $-ie_{\hat{t}}$, where $e_{\hat{t}}$ are the basis vectors (considered as differential operators) describing the Bardeen frame:

$$e_{\hat{t}} = \frac{1}{\rho}\left(\frac{B}{\Delta}\right)^{1/2}\left(\frac{\partial}{\partial t} + \omega_B \frac{\partial}{\partial \varphi}\right), \tag{4.9a}$$

$$e_{\hat{r}} = \frac{\Delta^{1/2}}{\rho} \frac{\partial}{\partial r}, \tag{4.9b}$$

$$e_{\hat{\theta}} = \frac{1}{\rho} \frac{\partial}{\partial \theta}, \tag{4.9c}$$

$$e_{\hat{\varphi}} = \frac{\rho}{B^{1/2}} \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi}, \tag{4.9d}$$

with $B = (r^2 + a^2)^2 - a^2\Delta \sin^2\theta$ and $\omega_B = 2Mar/B$. The "energy" eigenvalue $k_{\hat{t}}$ in Eq. (4.8), for example, is given by

$$k_{\hat{t}} \Phi |_{x=x_0} = \frac{-i}{\rho}\left(\frac{B}{\Delta}\right)^{1/2}\left(\frac{\partial \Phi}{\partial t} + \omega_B \frac{\partial \Phi}{\partial \varphi}\right)_{x=x_0} \tag{4.10a}$$

or

$$k_{\hat{t}} = \frac{m}{r_0}\left(\frac{B}{\Delta}\right)^{1/2}(\omega_B - \omega_0), \tag{4.10b}$$

so the useful expression

$$\left| u^{\hat{\varphi}}\left(\frac{dt}{d\tau} k_{\hat{t}}\right)^{-1}\right|^2 = \frac{\Delta}{m^2} \tag{4.11}$$

follows with the aid of the geodesic equations.

Combining Eqs. (4.6), (4.8), and (4.11) with some of the relations from Appendix A, we obtain

$$P_{\text{out}}^{(1)} = \sum_{l, m>0} \left[\frac{4\pi M(r^2 + a^2)q^2}{mr^2(Mr)^{1/2}(r+3M)} \left|\frac{dL}{dr^*}\right|^2 |Z_l^m|^2 + \frac{\pi(r-M)q^2}{mr^2(Mr)^{1/2}(r+3M)(r^2+a^2)} |L|^2 \left|\frac{dZ_l^m}{d\theta}\right|^2\right]_{x=x_0}, \tag{4.12}$$

with terms smaller by a factor $(M\omega)^{-1} \ll 1$ neglected. (Notice that the parallel [perpendicular] polarization is the source of electric $(-)^l$ [magnetic $(-)^{l+1}$] parity terms in the sum.) To simplify this expression, note that in the WKB approximation the radial function satisfies

$$\left|\frac{dL(r_0)}{dr^*}\right|^2 \simeq \omega^2 V(r_0) |L(r_0)|^2, \tag{4.13a}$$

or, by Eq. (2.34),

$$\left|\frac{dL(r_0)}{dr^*}\right|^2 \simeq \frac{3^{1/2} m\omega_0 r\Delta}{(r^2+a^2)^2} \epsilon |L(r_0)|^2, \tag{4.13b}$$

with $\epsilon = 1 + 2k + (4/\pi)(m/m_{\text{crit}})$ and $|L(r_0)|^2$ given by Eq. (2.40). Furthermore, from Appendix B we have

$$\left|\frac{dZ_l^m(\pi/2, 0)}{d\theta}\right|^2 \simeq 2mk(1 - a^2\omega_0^2)^{1/2} |Z_l^{m+1}(\pi/2, 0)|^2 \tag{4.14}$$

and

$$|Z_l^m(\pi/2, 0)|^2 \simeq \begin{cases} (1 - a^2\omega_0^2)^{1/4} \dfrac{k!m^{1/2}}{[(k/2)!]^2 2^k}, & l - m \equiv k \text{ even} \\ 0, \quad l - m \equiv k, \text{ odd}, \end{cases} \tag{4.15}$$

so power formula (4.12) becomes

$$P^{(1)}_{\text{out}} = \sum_{m=0}^{\infty} \frac{2q^2(3Mr_\gamma)^{1/2}(r_\gamma - M)}{\pi^{1/2}r_\gamma{}^2(r_\gamma + 3M)^2} \sum_{k\text{ even}} \epsilon^{1/2} \frac{k!}{[(k/2)!]^2 2^k} e^{-\pi\epsilon/2} + \sum_{k\text{ odd}} 2k\epsilon^{-1/2} \frac{(k-1)!}{\{[(k-1)/2]!\}^2 2^{k-1}} e^{-\pi\epsilon/2}. \quad (4.16)$$

As in the scalar case, only the leading term in each sum over k contributes significantly to the total power since higher order terms are smaller by the factor $e^{-2\pi}$. In addition, the leading term in the electric parity expansion (k = even) dominates the magnetic mode leading term by $\sim e^{\pi}$ so the radiation is ~96% plane polarized (electric parity) at the source. To determine the polarization measured at a distant point, generally one must integrate the equations describing the parallel propagation of the polarization vectors along null geodesics. In the synchrotron case, however, the narrow latitudinal beaming of radiation allowed us, following Eqs. (4.7), to argue that this had been done adequately, so we conclude that electromagnetic GSR is about ~96% plane polarized at infinity. An assessment as to the polarization of electromagnetic GSR as a function of angle in the Schwarzschild limit has been made by Breuer and Vishveshwara,[33] who find that the degree of linear polarization is greater than 90% at all latitudes within the half-width of the radiation beam.

As an independent check of the accuracy of this approach to doing vector radiation calculations, consider the Schwarzschild limit of Eq. (4.12) where the first-order perturbation calculations have been done in detail.[5,17,34] In this limit, $r_0 \simeq 3M$ and Eq. (4.12) reduces to

$$P^{(1)}_{\text{out}} = \sum_{l,m>0} \frac{2\pi q^2}{3^{3/2}Mm}\left(\left|\frac{dL}{dr^*}\right|^2 |Y_l^m|^2 + \frac{1}{27M^2}|L|^2\left|\frac{dY_l^m}{d\theta}\right|^2\right)_{x=x_0}, \quad (4.17)$$

which is identical to the leading terms in the $r_0 \simeq 3M$ limit of the exact formula (see Ref. 17, Table I).

Analysis of gravitational radiation emission differs little from the electromagnetic case just considered. Use the gravitational formula from Sec. III,

$$\frac{dW^{(2)}_{\text{out}}}{d\omega} = \sum_P \sum_{l,m} \omega |\langle \phi^{\text{out}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle|^2, \quad (4.18)$$

together with the stress-energy tensor of an orbiting particle

$$(-g)^{1/2}\underline{T} = \mu(u^0)^{-1}u\otimes u\,\delta(r-r_0)\delta(\theta-\pi/2)\delta(\varphi-\omega_0 t) \quad (4.19)$$

to obtain

$$P^{(2)}_{\text{out}} = \sum_P \sum_{l,m>0} 2\pi\mu^2 m\omega_0\left(\frac{dt}{d\tau}\right)^{-2} |u^\alpha u^\beta \phi^{\text{out}}_{\alpha\beta}(r_0, m\omega P)|^2 \quad (4.20\text{a})$$

$$= \sum_P \sum_{l,m>0} 2\pi\mu^2 m\omega_0\left(\frac{dt}{d\tau}\right)^{-2} |u\otimes u : \underline{e}(P)\Phi^{\text{out}}(r_0, lm\omega P)|^2. \quad (4.20\text{b})$$

Now one must evaluate the inner product $u \otimes u : \underline{e}(P)$, where $\underline{e}(P)$ are the two orthogonal transverse traceless polarization tensors normalized to

$$\underline{e}(P):\underline{e}(P') = e_{\alpha\beta}(P)e^{\alpha\beta}(P') = 2\delta_{PP'}. \quad (4.21)$$

Out of the previously introduced polarization vectors (4.7), one can construct these two tensors:

$$\underline{e}(+) = [e(\|)\otimes e(\|) - e(\perp)\otimes e(\perp)], \quad (4.22\text{a})$$

$$\underline{e}(\times) = [e(\|)\otimes e(\perp) + e(\perp)\otimes e(\|)], \quad (4.22\text{b})$$

with "plus" and "cross" describing the relative orientation of the orbital plane and the polarization states at the source. This choice follows from Eq. (3.45) since parallel transport preserves such tensor relations. The above satisfy

$$u\otimes u:\underline{e}(+) = \frac{(u^{\hat\varphi})^2}{k_{\hat t}{}^2}(k_{\hat r}{}^2 - k_{\hat\theta}{}^2), \quad (4.23\text{a})$$

$$u\otimes u:\underline{e}(\times) = \frac{(u^{\hat\varphi})^2}{k_{\hat t}{}^2}\, 2k_{\hat r}k_{\hat\theta}, \quad (4.23\text{b})$$

or, in operator form,

$$u\otimes u:\underline{e}(+)\Phi|_{x=x_0} = \frac{(u^{\hat\varphi})^2}{k_{\hat t}{}^2}\left(-\frac{\Delta}{\rho^2}\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2}\right)_{x=x_0}, \quad (4.24\text{a})$$

$$u\otimes u:\underline{e}(\times)\Phi|_{x=x_0} = \frac{(u^{\hat\varphi})^2}{k_{\hat t}{}^2}\left(2\frac{\Delta^{1/2}}{\rho^2}\frac{\partial^2\Phi}{\partial r\partial\theta}\right)_{x=x_0}. \quad (4.24\text{b})$$

Accordingly, Eq. (4.20) becomes

$$P^{(2)}_{\text{out}} = \sum_{l,m>0} \frac{2\pi\mu^2 m\omega_0}{r^2+a^2}\left(\frac{dt}{d\tau}\right)^2 \frac{\Delta^2}{m^4}\left\{\left[\frac{(r^2+a^2)^2}{\rho^2\Delta}\left|\frac{d^2L}{dr^{*2}}\right||Z_l^m| + \frac{1}{\rho^2}|L|\left|\frac{d^2Z_l^m}{d\theta^2}\right|\right]^2 + 4\frac{(r^2+a^2)^2}{\rho^4\Delta}\left|\frac{dL}{dr^*}\right|^2\left|\frac{dZ_l^m}{d\theta}\right|^2\right\}_{x=x_0}, \quad (4.25)$$

where Eq. (4.11) has been used for simplification. Notice that, analogous to the vector case, the "plus" ("cross") polarization is the source of electric (magnetic) parity terms.

Further reduction of the power formula occurs by using

$$\left|\frac{d^2L(r_0)}{dr^{*2}}\right|^2 \simeq [\,\omega^2 V(r_0)]^2 |L(r_0)|^2 \tag{4.26}$$

and

$$\left|\frac{d^2 Z_l^m(\pi/2,0)}{d\theta^2}\right|^2 \simeq Q|Z_l^m|^2 \simeq m^2(1-a^2\omega_0^2)\epsilon^2|Z_l^m|^2, \tag{4.27}$$

which follows from the computations done in Appendix B. With the above relations and the relativistic circular orbit equations, a lengthy calculation yields

$$P_{\rm out}^{(2)} = \frac{2\pi^{1/2}\mu^2(r_\gamma - M)(3Mr_\gamma)^{1/2}}{r_\gamma^2(r_\gamma+3M)^2}\sum_{m=0}^{\infty}\left(\sum_{k\ \rm even}\epsilon^{3/2}\frac{m_{\rm crit}}{m}\frac{k!}{[(k/2)!]^2 2^k}e^{-\pi\epsilon/2} + \sum_{k\ \rm odd}2k\epsilon^{1/2}\left|\frac{m_{\rm crit}}{m}\right|\frac{(k-1)!}{\{[(k-1)/2]!\}^2 2^{k-1}}e^{-\pi\epsilon/2}\right). \tag{4.28}$$

Again, we have $k=l-m$, $\epsilon=1+2k+(4/\pi)m/m_{\rm crit}$, and $r_\gamma \cong r_0$ the radius of the null orbit. As in the vector case, the radiation is predominantly (~96%) electric parity, so, using the vector polarization arguments, we conclude that ~96% of the observed high-frequency gravitational radiation has "plus" polarization.

V. DISCUSSION

Equations (2.41), (4.16), and (4.28) can be combined to obtain the master formula for the power radiated by a particle in a highly relativistic orbit about a Kerr black hole:

$$P_{\rm out}^{(s)} = \left(\frac{f_s\mu}{M}\right)^2 A(r_\gamma)\sum_{m=0}^{\infty}\left[(s!)^2\sum_{k\ \rm even}\left(\frac{4m}{\epsilon\pi m_{\rm crit}}\right)^{1-s}\epsilon^{1/2}\frac{k!}{[(k/2)!]^2 2^k}e^{-\pi\epsilon/2} + s^2\sum_{k\ \rm odd}\left(\frac{4m}{\epsilon\pi m_{\rm crit}}\right)^{1-s}2k\epsilon^{-1/2}\frac{(k-1)!}{\{[(k-1)/2]!\}^2 2^{k-1}}e^{-\pi\epsilon/2}\right], \tag{5.1}$$

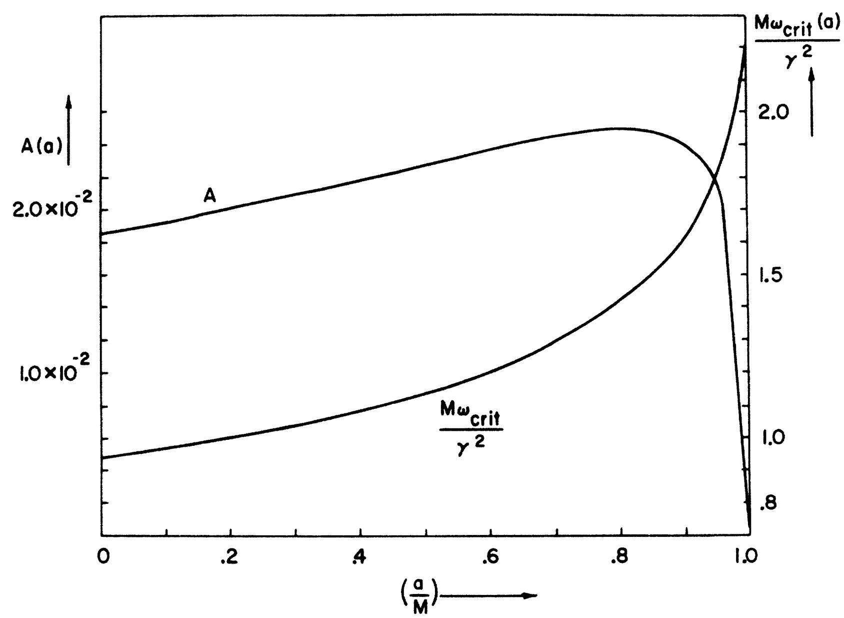

FIG. 3. Shown as a function of the Kerr parameter a/M are the cutoff frequency and an amplitude factor [see Eqs. (5.2) and (5.3)] which characterize the GSR spectra. Notice that the amplitude, proportional to $[1-(a/M)^2]$ as $a\to M$, dies off rapidly for a $>0.95M$.

where $f_s = (f, q/\mu, 1)$, $\epsilon = 1 + 2k + (4/\pi)m/m_{\text{crit}}$. The most important quantities in these formulas are

$$A(r_\gamma) = \frac{2(r_\gamma - M)(3Mr_\gamma)^{1/2}M^2}{\pi^{1/2}r_\gamma{}^2(r_\gamma + 3M)^2} \tag{5.2}$$

and

$$\omega_{\text{crit}}(r_\gamma) = \omega_0 m_{\text{crit}}(r_\gamma) = \frac{4\sqrt{3}\,\gamma^2}{\pi r_\gamma}, \tag{5.3}$$

which are plotted in Fig. 3. When one neglects all but the first term in the k = even series, which accounts for 96% of the total power, Eq. (5.1) reduces to

$$P^{(s)}_{\text{out}} = (s!)^2\left(\frac{f_s\mu}{M}\right)^2 A(r_\gamma) \times \sum_{m=0}^{\infty}\left(1 + \frac{4m}{\pi m_{\text{crit}}}\right)^{s-1/2} \times \left(\frac{4m}{\pi m_{\text{crit}}}\right)^{1-s} e^{-\pi/2 - 2m/m_{\text{crit}}}. \tag{5.4}$$

Hence the power spectra are

$$\frac{dP^{(s)}_{\text{out}}}{d\omega} = (s!)^2\left(\frac{f_s\mu}{M}\right)^2 \bar{A}(r_\gamma)\left(1 + \frac{4m}{\pi m_{\text{crit}}}\right)^{s-1/2} \times \left(\frac{4m}{\pi m_{\text{crit}}}\right)^{1-s} e^{-\pi/2 - 2m/m_{\text{crit}}}, \tag{5.5}$$

with the high- and low-frequency limits

$$\frac{dP^{(s)}_{\text{out}}}{d\omega} = (s!)^2\bar{A}(r_\gamma)\left(1 + \frac{4m}{\pi m_{\text{crit}}}\right)^{1-s} e^{-\pi/2}, \quad \omega \ll \omega_{\text{crit}} \tag{5.6a}$$

$$\frac{dP^{(s)}_{\text{out}}}{d\omega} = (s!)^2\bar{A}(r_\gamma)\left(\frac{4m}{\pi m_{\text{crit}}}\right)^{1/2} e^{-\pi/2 - 2m/m_{\text{crit}}}, \quad \omega \gg \omega_{\text{crit}}. \tag{5.6b}$$

Here, one has defined

$$\bar{A}(r_\gamma) = \omega_0{}^{-1}A(r_\gamma) = \frac{r_\gamma{}^{1/2}(r_\gamma + 3M)}{2M^{1/2}}A(r_\gamma). \tag{5.7}$$

One of the important features that characterize these power spectra is the exponential cutoff at frequencies higher than $\omega_{\text{crit}} \sim M^{-1}\gamma^2$, a result qualitatively similar to the $\omega_{\text{crit}} \sim r_0{}^{-1}\gamma^3$ cutoff that occurs in flat-space electromagnetic synchrotron radiation. The γ^3 dependence of the cutoff frequency for the flat-space case can be understood with the aid of Fig. 4 which illustrates the synchrotron emission from an accelerated particle to a distant observer on the orbital plane. Since radiation from a relativistic particle is beamed into a narrow forward cone of half-width γ^{-1}, our distant observer detects only the radiation emitted while the particle is in the small arc of the orbit $\Delta\varphi_{AB} \sim \gamma^{-1}$. (Conversely, the radiation emitted at any point along the orbit, i.e., a flash of radiation, is seen only by equatorial observers within an angular spread $\Delta\varphi \sim \gamma^{-1}$.) As shown in Fig. 4, the first of the synchrotron radiation arrives at a fixed distant observer at the time

$$t_1 = (t_{AB})_{\text{rad}} + (t_{B\infty})_{\text{rad}}, \tag{5.8}$$

and the last of the pulse is seen at

$$t_2 = (t_{AB})_{\text{part}} + (t_{B\infty})_{\text{rad}}. \tag{5.9}$$

Hence, the total time interval that an observer measures radiation is

$$\Delta t = t_2 - t_1 \sim r_0\Delta\varphi(v^{-1} - 1) \sim \tfrac{1}{2}r_0\gamma^{-3} \tag{5.10}$$

since $\Delta\varphi \sim \gamma^{-1}$. By the properties of Fourier transforms, it follows that the only frequencies at which there is a significant amount of radiation are

$$\omega \leq \omega_{\text{crit}} \sim (\Delta t)^{-1} \sim r_0{}^{-1}\gamma^3, \tag{5.11}$$

in agreement with the known cutoff frequency.

When combined with $\omega_{\text{crit}} \sim M^{-1}\gamma^2$, the above analysis can be turned around to provide us with a qualitative picture of GSR. From the cutoff frequency and the Fourier transform argument, we find that radiation is received in bursts of duration

$$\Delta t \sim \omega_{\text{crit}}{}^{-1} \sim M\gamma^{-2}. \tag{5.12}$$

Now consider Fig. 5. The first of the observed radiation is emitted when the particle is at point 0 because radiation emitted into the inner half of the forward cone is captured by the black hole. The time it takes the waves from point 0 to reach the observer is

$$t_1 = (t_{0B})_{\text{rad}} + (t_{B\infty})_{\text{rad}}. \tag{5.13}$$

The last of the radiation to reach the observer is sent at the angle $\Delta\varphi$ (point B) and arrives at

$$t_2 = (t_{0B})_{\text{part}} + (t_{B\infty})_{\text{rad}}. \tag{5.14}$$

By using the preceding three equations, one finds

$$\Delta t \sim M\gamma^{-2} \sim M\Delta\varphi(v^{-1} - 1) \sim M\Delta\varphi\gamma^{-2} \tag{5.15}$$

or

$$\Delta\varphi \sim 1. \tag{5.16}$$

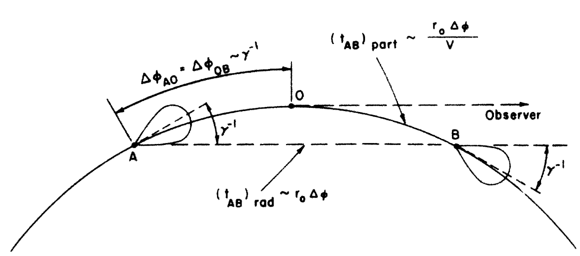

FIG. 4. Radiation from an accelerated particle. Shown is a small arc of the particle's circular orbit and the emitted radiation, which is beamed into a narrow forward cone with half-width γ^{-1}. Choose the particular observer at infinity that is in the direction tangent to the orbit at point 0. Owing to the beaming of radiation, this observer only sees radiation that is emitted between points A and B, where $\Delta\phi_{A0} = \Delta\phi \sim \gamma^{-1}$.

Equation (5.16) has an important consequence: The radiation is not azimuthally beamed in the sense that a single flash of radiation is sprayed out over an angle $\Delta\varphi \sim 1$. (By comparison, only observers within the angle $\Delta\varphi \sim \gamma^{-1}$ of the forward direction detect a flash from an accelerated particle.) Yet, the GSR received by each observer arrives in periodic ($T = 1/2\pi\omega_0$) pulses of short duration $\Delta t \sim M\gamma^{-2}$, each pulse consisting of radiation emanating from a finite arc of the circular orbit. This is analogous to the accelerated orbit "rotating searchlight" effect in which energy pulses lasting $\Delta t \sim r_0\gamma^{-3}$ are observed at periodic intervals.

The above conclusions as to the observed azimuthal distribution of GSR are in quantitative disagreement with the work of Hughes.[35] In this earlier beaming calculation, the value of $\Delta\varphi$ is found by determining the difference in arc length Δl of null goedesics from points 0 and B (see Fig. 5) to a fixed distant radius. The quantity Δl can be expressed as a difference between two elliptic integrals, which, when evaluated by Hughes, gave $\Delta\varphi \sim \gamma$ and, consequently, $\Delta t \sim M\gamma^{-1}$. In addition to finding a pulse length longer than that in Eq. (5.12), Hughes concludes from the $\Delta\varphi \sim \gamma$ result that a large number ($\sim\gamma$) of images of a single event are seen by a distant observer. The angular spreading $\Delta\varphi \sim 1$ found in the preceding discussion, on the other hand, does not lead to strong multiple images. In view of these discrepancies, either the evaluation of the elliptical integral difference is in error or the Fourier transform argument given above is invalid.

The second possibility, that $\Delta t \sim \omega_{\text{crit}}^{-1}$ is not true, can be ruled out by explicitly carrying out the indicated sums in Eq. (2.19) to find the observed scalar field amplitude. It is a straightforward exercise to verify that (in the Schwarzschild limit)

$$\Phi \simeq \frac{-if\mu}{(3\pi)^{1/2}\gamma} \sum_{m=-\infty}^{\infty} \frac{e^{im\psi}}{r} \frac{m}{|m|^{5/4}} \frac{e^{-\pi\epsilon/4}}{\epsilon^{1/4}} \quad (5.17)$$

near infinity. Here we have defined $\psi = \varphi - \omega_0(t - r^*)$. By approximating $\epsilon^{1/4} \simeq 1$ and converting the sum on m to an integral, one finds

$$\Phi \simeq \frac{2f\mu\Gamma(\frac{3}{4})e^{-\pi\epsilon/4}}{\sqrt{3\pi}\,\gamma r} m_{\text{crit}}^{3/4} \frac{(\psi m_{\text{crit}})}{(1+\psi^2 m_{\text{crit}}^2)^{7/8}}, \quad (5.18)$$

so the half-width $\psi_{1/2}$ is given by

$$\psi_{1/2} \sim m_{\text{crit}}^{-1/2}. \quad (5.19)$$

Hence, for an observer at constant r and φ, Eq. (5.12) and the subsequent discussion follow.

It is clear by comparing Figs. 4 and 5 that a relativistic particle on a geodesic travels a considerable distance while emitting radiation to the same distant observer. In this respect, GSR is more analogous to linear acceleration than circular motion in flat-space electromagnetic theory. Indeed, the power radiated by a linearly accelerated charge is

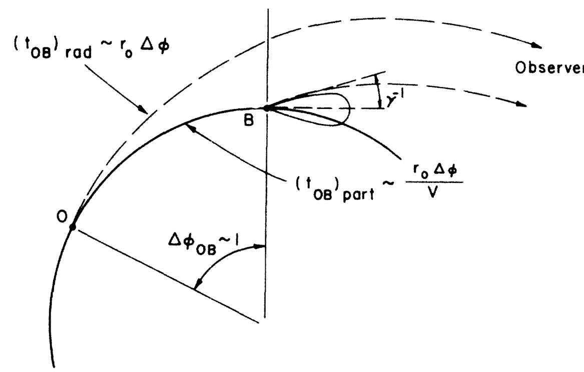

FIG. 5. Radiation emitted by a geodesic particle. The radiation emitted at point 0 follows a null geodesic of the background geometry and reaches a particular observer at infinity. Photons emitted into the inner half of the forward cone before the particle is at point 0 cannot reach this observer, for they are captured by the black hole. Let B be the point on the orbit where photons on the outer edge of the forward cone reach the given observer. Then, by construction, only radiation emitted between points 0 and B is seen by the observer. In the text, the properties of Fourier transforms and the cutoff frequency $\omega_{\text{crit}} \sim \omega_0\gamma^2$ are used to argue that $\Delta\phi_{0B} \sim 1$.

$$P = \tfrac{2}{3}\gamma^2 q^2 a^2, \quad (5.20)$$

which is more like the GSR result

$$P_{\text{out}}^{(s)} \sim \gamma^2\left(\frac{f_s\mu}{M}\right)^2 \quad (5.21)$$

than the synchrotron emission in flat space:

$$P \sim \gamma^2\left(\frac{q\gamma}{r_0}\right)^2 . \quad (5.22)$$

Since Fig. 5 qualitatively differs little from a schematic diagram of radiation from a relativistic particle on a radial geodesic (linear acceleration), one might speculate that $P_{\text{out}} \sim \gamma^2(f_s\mu/M)^2$ is a general feature of relativistic test-particle motion about a black hole. However, analyses[36,37] of gravitational brehmsstrahlung from a relativistic particle reveal

$$W_{\text{out}} \sim \gamma^3\left(\frac{\mu^2}{M}\right)\left(\frac{M}{b}\right)^3 . \quad (5.23\text{a})$$

Compare this with the energy

$$W_{\text{GSR}} \sim P_{\text{out}}\,\omega_0^{-1} \sim \gamma^2\left(\frac{\mu^2}{M}\right) \quad (5.23\text{b})$$

radiated in one revolution of a synchrotron orbit with $r_0 \simeq r_\gamma \ll b$ and impact parameter $b_{GSR} \simeq 3\sqrt{3}M$ (for $a=0$). One sees, surprisingly, that when $\gamma > (b/M)^3$ more energy is predicted from the distant encounter $b/M \gg 1$ than for a close encounter $b/M \simeq b_{GSR}/M \simeq 3\sqrt{3}$.

A second critical feature of the GSR power formulas is the spin dependence of the spectra.[17] Chitre and Price[34] have attributed the dependence to a breakdown of a geometrical optics analysis of the radial wave equation. In fact, we have shown the WKB approximation to be valid in the radial direction, and have used a geometrical optics approach to derive the power spectra. Rather, the spin dependence of the spectra is a property of the coupling of a transverse radiation field to a relativistic particle. A comparison of Eqs. (4.6) and (4.20b) with (2.25) gives

$$\frac{dP^{(1)}_{\text{out}}}{d\omega} = |u\cdot e|^2 \frac{dP^{(0)}_{\text{out}}}{d\omega} \tag{5.24a}$$

and

$$\frac{dP^{(2)}_{\text{out}}}{d\omega} = |u\otimes u : \underline{e}|^2 \frac{dP^{(0)}_{\text{out}}}{d\omega}\ . \tag{5.24b}$$

As can be seen from Eqs. (4.8) and (4.23), the above inner products are

$$|u\cdot e|^2 \sim (u^{\hat\varphi})^2 \left(\frac{k_T}{k_{\hat\xi}}\right)^2 \tag{5.25a}$$

and

$$|u\otimes u : \underline{e}|^2 \sim (u^{\hat\varphi})^4 \left(\frac{k_T}{k_{\hat\xi}}\right)^4 , \tag{5.25b}$$

where k_T denotes the radiation momentum (measured by a Bardeen observer) in directions transverse to the particle motion. We have $u^{\hat\varphi} \sim \gamma \sim m_{\text{crit}}^{1/2}$ from the geodesic equations and $k_T/k_{\hat\xi} \sim \sin\psi$, with ψ the angle between the direction of motion of the particle and a typical emitted photon. For GSR, the half-width of the radiation beam is $\Delta\vartheta \sim |m|^{-1/2}$ below the cutoff frequency, so

$$\frac{k_T}{k_{\hat\xi}} \sim \sin\psi \sim \psi \sim |m|^{-1/2} . \tag{5.26}$$

Consequently, Eqs. (5.22) become

$$|u\cdot e|^2 \sim \left|\frac{m_{\text{crit}}}{m}\right| \tag{5.27a}$$

and

$$|u\otimes u : \underline{e}|^2 \sim \left|\frac{m_{\text{crit}}}{m}\right|^2 , \tag{5.27b}$$

and the power spectra are related by

$$\frac{dP^{(s)}_{\text{out}}}{d\omega} \sim \left|\frac{m}{m_{\text{crit}}}\right|^{-s} \frac{dP^{(0)}_{\text{out}}}{d\omega} \tag{5.28}$$

below the cutoff frequency.

It is interesting to note that the spin dependence of the radiation spectra from relativistic, circling accelerated body follows from similar considerations. For the accelerated case, one has $m_{\text{crit}} \sim \gamma^3$ and $\Delta\vartheta \sim |m|^{-1/3}$ at frequencies $\omega < \omega_{\text{crit}}$, so $|u| \sim m_{\text{crit}}^{1/3}$ and $k_T/|k| \sim |m|^{-1/3}$. Hence, below the cutoff the power spectra are

$$\frac{dP^{(s)}}{d\omega} = \left|\frac{m}{m_{\text{crit}}}\right|^{-2s/3} \frac{dP^{(0)}}{d\omega} , \tag{5.29}$$

which is in agreement with other computations.[38-40]

A final feature of importance in Eq. (5.1) is the vanishing of the power radiated, proportional to $r_\gamma - M$, as the Kerr parameter a approaches M. A similar factor appears in formulas[41] describing scalar radiation from particles on stable orbits in the extreme Kerr geometry, so the vanishing of the radiated power may be a general feature of test-particle motion near the horizon of an extreme Kerr black hole. Gravitational red shifts do not explain the $r_\gamma - M$ factor and seem to be unimportant even when $a = M$. The frequency of radiation measured by a Bardeen observer at the source is related to the frequency measured at infinity by

$$\nu_B = \left(\frac{B}{\rho^2\Delta}\right)^{1/2} \nu_\infty(1 - b\,\omega_B) , \tag{5.30}$$

where b is the impact parameter of the radiation. For radiation in a null circular orbit, which roughly describes the photons emitted by a relativistic circling particle, b is given by Eq. (A5) with $r_0 = r_\gamma$. Substituting this value of b into Eq. (5.27), one finds

$$M(1 - b\,\omega_B) \sim \Delta^{1/2} , \tag{5.31}$$

so that ν_B/ν_∞ remains finite even as $a \to M$. Other possible explanations for the $r_\gamma - M$ factor are being sought.

APPENDIX A: GEODESIC CIRCULAR ORBITS

Geodesic motion in the Kerr geometry has been studied by a number of authors[11,42,43] since Carter[44] first reduced the problem to first-order equations with four constants of motion. Here we confine our analysis to circular orbits on the equator and, following Bardeen, Press, and Teukolsky,[11] we derive analytic formulas describing these orbits. Finally, the circular orbit relations are specialized to the cases of interest for the GSR calculations.

When the particle motion is restricted to the $\theta = \pi/2$ plane, the geodesic equations take the form

$$\left(\frac{dr}{d\lambda}\right)^2 + W(r) = 0 , \tag{A1a}$$

$$r^2 \frac{d\varphi}{d\lambda} = aP\Delta^{-1} - (aE - L_z), \quad \text{(A1b)}$$

$$r^2 \frac{dt}{d\lambda} = (r^2 + a^2)P\Delta^{-1} - a(aE - L_z), \quad \text{(A1c)}$$

where

$$P = E(r^2 + a^2) - a L_z \quad \text{(A2)}$$

and

$$W(r) = -r^{-4}\{P^2 - \Delta[\mu^2 r^2 + (aE - L_z)^2]\} \quad \text{(A3)}$$

is the radial potential. In the above, $E = \gamma\mu = -p_t$ is the conserved energy at infinity, $L_z = p_\varphi$ is the conserved angular momentum, μ is the particle rest mass, and $\tau = \mu\lambda$ is the test particle's proper time.

For circular motion at radius r_0, one has $W(r_0) = dW(r_0)/dr = 0$. Applying these two conditions to the definition of $W(r)$, the conserved energy and angular momentum can be found as functions of M, a, and r_0:

$$\gamma^2 = \left(\frac{E}{\mu}\right)^2 = \frac{(r_0^{3/2} - 2Mr_0^{1/2} \pm aM^{1/2})^2}{r_0^{3/2}(r_0^{3/2} - 3Mr_0^{1/2} \pm 2aM^{1/2})}, \quad \text{(A4)}$$

$$\bar{b} \equiv \frac{L_z}{E} = \frac{\pm M^{1/2} r_0^2 - a r_0^{3/2}}{r_0^{3/2} - 2Mr_0^{1/2} \pm aM^{1/2}}. \quad \text{(A5)}$$

The upper (lower) sign holds for prograde (retrograde) orbits. Inserting these equations for the constants of motion into Eqs. (A1b) and (A1c), one obtains

$$\left(\frac{d\varphi}{d\tau}\right)^2 = Mr_0^{-3/2}(r_0^{3/2} - 3Mr_0 \pm 2M^{1/2}a)^{-1}, \quad \text{(A6)}$$

$$\left(\frac{dt}{d\tau}\right)^2 = (r_0^{3/2} \pm aM^{1/2})^2 r_0^{-3/2} \times (r_0^{3/2} - 3Mr_0 \pm 2M^{1/2}a)^{-1}, \quad \text{(A7)}$$

which together yield the orbital frequency

$$\omega_0 = \frac{d\varphi}{dt} = \frac{\pm M^{1/2}}{r_0^{3/2} \pm aM^{1/2}}. \quad \text{(A8)}$$

The above relations describe in the $\gamma \to \infty$ limit circular photon geodesics at a radius $r_0 \equiv r_\gamma$ satisfying

$$r_\gamma^{3/2} - 3Mr_\gamma^{1/2} \pm 2M^{1/2}a = 0. \quad \text{(A9)}$$

In addition, the last stable orbit $r_0 = r_{ls}$ for a given value of a can be found by imposing the additional condition $d^2W(r_{ls})/dr^2 = 0$. The formula

$$3a = \pm r_{ls}^{1/2} M^{1/2}\left[4 - \left(\frac{3r_{ls}}{M} - 2\right)^{1/2}\right], \quad \text{(A10)}$$

first obtained by Bardeen,[6] then follows. All orbits at radii $r_0 > r_{ls}$ satisfy $d^2W(r_0)/dr^2 > 0$, so by Eq. (A1a) they are stable with respect to radial perturbations.

Important for the study of GSR are particle orbits close to the null circular orbit radius, i.e.,

$$r_0 = (1 + \delta)r_\gamma, \quad \text{(A11)}$$

with $\delta \ll 1$. From Eqs. (A1c) and (A7), we find for prograde energetic ($\gamma \gg 1$) orbits that

$$\gamma^2 \simeq \frac{r_\gamma - M}{6r_\gamma\delta} \quad \text{(A12)}$$

and

$$\left(\frac{dt}{d\tau}\right)^2 \simeq \frac{(r_\gamma + 3M)^2}{6r_\gamma(r_\gamma - M)\delta}, \quad \text{(A13)}$$

so energetic orbits require $\delta \ll r_\gamma/M - 1$, a stronger condition than just $\delta \ll 1$. In the extreme Kerr ($a = M$) limit, Eqs. (A9) and (A10) indicate that even stable orbits satisfy $\delta \ll 1$, for when $a \cong M$, $r_{ls} \cong r_\gamma \cong M$. To clarify the $a = M$ situation, define $\alpha^2 = 1 - (a/M)^2$ and consider the $\alpha \ll 1$ limit. The horizon r_+ is exactly at $r_+ = M(1 + \alpha)$, while

$$r_\gamma = M\left[1 + \frac{2}{\sqrt{3}}\alpha + O(\alpha^2)\right], \quad \text{(A14)}$$

$$r_{ls} = M[1 + (2\alpha^2)^{1/3} + O(\alpha^{4/3})]. \quad \text{(A15)}$$

With the aid of equations (A4) and (A15), γ can be shown to remain small for all stable orbits even as $a \to M$. For the last stable orbit, for example, $\gamma^2 \to \frac{1}{3}$, so particles executing stable circular motion are never highly energetic.

APPENDIX B: SPHEROIDAL HARMONICS

Upon separating the scalar wave equation in the Kerr background, one obtains the angular differential equation

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS_{ml}}{d\theta}\right) + \cos^2\theta\left(\frac{m^2}{\sin^2\theta} - a^2\omega^2\right)S_{ml} = QS_{ml}. \quad \text{(B1)}$$

Q is the separation constant introduced in Sec. II and $S_{ml}(-a^2\omega^2, \cos\theta)$, satisfying (B1), is an oblate spheroidal angular function. When the Meixner-Schäfke[45] normalization

$$\int_{-1}^{1}[S_{ml}(-a^2\omega^2, \eta)]^2 d\eta = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} \quad \text{(B2)}$$

is used, it is convenient to define the spheroidal harmonics

$$Z_l^m(\theta, \varphi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} S_{ml}(-a^2\omega^2, \cos\theta)e^{im\varphi}, \quad \text{(B3)}$$

for the Z_l^m's form an orthonormal basis for the angular functions:

$$\int Z_l^m(\Omega)^* Z_{l'}^{m'}(\Omega)d\Omega = \delta_{mm'}\delta_{ll'} \; . \tag{B4}$$

For radiation in high-frequency modes $M\omega \gg 1$ with impact parameter $b \equiv m/\omega = \omega_0^{-1} \sim M$, analytic expressions for Q, $Z_l^m(\pi/2, 0)$, and $dZ_l^m(\pi/2,0)/d\theta$ are obtainable since the integers l and m both must be large. It is useful to define a latitudinal angle $\vartheta = \theta - \pi/2$ and a new angular function

$$S_{ml}(\theta) = \frac{T_{ml}(\vartheta)}{\cos\vartheta} \; . \tag{B5}$$

With these definitions, (B1) becomes

$$-\frac{d^2 T_{ml}}{d\vartheta^2} + \sin^2\vartheta\left(\frac{m^2 - \frac{1}{4}}{\cos^2\vartheta} - a^2\omega^2\right) T_{ml} = (Q + \tfrac{1}{2})T_{ml} \, , \tag{B6}$$

which may be solved as an effective potential problem for the $m \gg 1$ case considered here. The classical turning points are at $\vartheta_{\rm tp} = \pm O(m^{-1/2})$, and, since T_{ml} is exponentially damped beyond $\vartheta_{\rm tp}$, the trigonometric functions in (B6) can be expanded about $\vartheta = 0$ to give

$$-\frac{d^2 T_{ml}}{d\vartheta^2} + m^2(1 - a^2 b^{-2})\vartheta^2 T_{ml} = [Q + O(m^0)]\, T_{ml} \, . \tag{B7}$$

Now define a new angular coordinate

$$\xi = m^{1/2}(1 - a^2 b^{-2})^{1/4}\vartheta \, , \tag{B8}$$

so (B7) can be written as

$$\frac{d^2 T_{ml}}{d\theta^2} + \left[\frac{Q}{m(1 - a^2 b^{-2})^{1/2}} - \xi^2\right] T_{ml} = 0 \; . \tag{B9}$$

We see, from (B5), that the angular function T_{ml} vanishes as $\vartheta \to \pm\pi/2$ or $\xi \to \pm\infty$. The solution to (B9) with these boundary conditions is the harmonic oscillator function

$$T_{ml} = C_{mk} H_k(\xi) e^{-\xi^2/2} \, , \tag{B10}$$

where $k \equiv l - m$ and

$$Q = m(1 - a^2 b^{-2})^{1/2}(2k + 1) + O(m^0) \tag{B11}$$

are the eigenvalues. $H_k(\xi)$ is the Hermite polynomial of order k, and C_{mk} is the normalization constant

$$|C_{mk}|^2 = \frac{2m^{1/2}(1 - a^2 b^{-2})^{1/4}(2m + k)!}{2^k (k!)^2 \pi^{1/2}(2m + 2k + 1)} \; . \tag{B12}$$

To obtain this value for the constant, one used the orthogonality properties of the Hermite functions and demanded that the normalization in (B2) be satisfied.

With the constant coefficient determined, (B10) can be evaluated at $\xi = 0$ to find the value of the spheroidal function on the plane. Specifically, we obtain

$$S_{ml}(\vartheta = 0) = \begin{cases} (1 - a^2 b^{-2})^{1/8}\left[\dfrac{(-)^{k/2}\, 2^{1/2} m^{1/4}(2m + k)!^{1/2}}{(k/2)!\, 2^{k/2}\pi^{1/4}(2m + 2k + 1)^{1/2}}\right], & k = \text{even} \\ 0, \quad k = \text{odd} \, . \end{cases} \tag{B13}$$

Since the only dependence on the Kerr parameter a is in the first term of the k = even equation, (B13) becomes

$$S_{ml}(\vartheta = 0) = (1 - a^2 b^{-2})^{1/8} P_l^m(\vartheta = 0) \quad , \tag{B14}$$

for the spheroidal functions satisfy $S_{ml}(\eta) = P_l^m(\eta)$ when $a = 0$. As a check of the accuracy of this result, one can compare the bracketed quantity in (B13) to the actual value of $P_l^m(\vartheta = 0)$ and find that they agree to terms of order m^{-1}.

From (B3) and (B14), it follows that the spheroidal harmonics, evaluated on the $\theta = \pi/2$ plane, are

$$\begin{aligned} |Z_l^m(\pi/2, 0)|^2 &= (1 - a^2 b^{-2})^{1/4} |Y_l^m(\pi/2, 0)|^2, \quad k = \text{even} \\ &= (1 - a^2 b^{-2})^{1/4}\frac{k!\, m^{1/2}}{2\pi^{3/2}[(k/2)!]^2 2^k} \, , \end{aligned} \tag{B15}$$

where Stirling's approximation has been used to evaluate $|Y_l^m|^2$ at the large values of l, m considered here.

An analogous approach can be taken to find $dZ_l^m(\pi/2, 0)/d\theta$. First, note that

$$\left.\frac{dT_{ml}}{d\theta}\right|_{\theta = \pi/2} = \frac{d\xi}{d\theta}\left.\frac{dT_{ml}}{d\xi}\right|_{\xi = 0} , \tag{B16}$$

and then use the properties of Hermite functions to obtain

$$\begin{aligned} \left.\frac{dT_{ml}}{d\theta}\right|_{\theta = \pi/2} &= m^{1/2}(1 - a^2 b^{-2})^{1/4}\frac{C_{mk}}{C_{m+1,k-1}}\, 2k H_{k-1}(0) C_{m+1,k-1} \\ &= 2m^{1/2}k(1 - a^2 b^{-2})^{1/4}\frac{C_{mk}}{C_{m+1,k-1}}\, T_{m+1,l}(\theta = \pi/2) \, . \end{aligned} \tag{B17}$$

Finally, we evaluate the ratio of the coefficients and take advantage of (B3) to rewrite the above equation in terms of spheroidal harmonics:

$$\left|\frac{dZ_l^m(\pi/2, 0)}{d\theta}\right|^2 = 2mk(1 - a^2 b^{-2})^{1/2}|Z_l^{m+1}(\pi/2, 0)|^2 \, . \tag{B18}$$

The form of this result could have been anticipated by looking at (B7). There we see that the "energy" of the Z_l^m mode is

$$E = Q = m(1 - a^2 b^{-2})^{1/2}(2k+1) \quad \text{(B19)}$$

on the equatorial plane. For large values of m, WKB techniques could have been employed to find

$$\frac{dZ_l^m}{d\theta} \sim E^{1/2} Z_l^m \,. \quad \text{(B20)}$$

However, because of the symmetry of Z_l^m about the $\theta = \pi/2$ plane, either the spheroidal harmonic or its first derivative vanishes on the equator. Rather, (B20) takes the form

$$\frac{dZ_l^m}{d\theta} \sim E^{1/2} Z_l^{m+1} \,, \quad \text{(B21)}$$

which is the same as (B18).

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