rings such that each pair yields a peak at the same point as the other pairs. We have not investigated the difficulty of manufacturing a double-ring mass. It is our observation, however, that the mathematical situation producing the flat peak is not especially fragile [indeed we used a rather crude technique in finding Eq. (9)] and that ultrahigh-precision machining is probably not required. More importantly, the mass distribution would have to be verified using, say, γ-ray absorption.

It is clear that the problem of measuring gravitational forces to the precision suggested here has not been solved. One of us (D.R.L.) has been struggling with the problem at the part-per-thousand level for several years. On the other hand, there has never been a thorough study of the physical processes which might set an ultimate limit to the accuracy of a gravitational-force measurement. The gas-thermal-noise problem, as well as the vibration problem, could probably be readily reduced by several orders of magnitude in this laboratory.

Perhaps the progress presented here on the mass separation problem will allow a more vigorous attack on the gravitational-force measurement problem.

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Black hole in a uniform magnetic field*

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Using the fact that a Killing vector in a vacuum spacetime serves as a vector potential for a Maxwell test field, we derive the solution for the electromagnetic field occurring when a stationary, axisymmetric black hole is placed in an originally uniform magnetic field aligned along the symmetry axis of the black hole. It is shown that a black hole in a magnetic field will selectively accrete charges until its charge becomes $Q = 2B_0 J$, where B_0 is the strength of the magnetic field and J is the angular momentum of the black hole. As a by-product of the analysis given here, we prove that the gyromagnetic ratio of a slightly charged, stationary, axisymmetric black hole (not assumed to be Kerr) must have the value $g = 2$.

I. INTRODUCTION

From the results on black-hole uniqueness which have been proved during the last several years (particularly the theorems of Israel,[1] Carter,[2] Hawking,[3] and Robinson[4]), it is now well established that an isolated black hole cannot have an electromagnetic field unless it is endowed with a net electric charge. Thus, an isolated black hole can participate in electromagnetic effects only if there is a mechanism for charging it up. However, if the black hole is not isolated, electromagnetic fields produced by external sources (e.g., plasma accreting onto the black hole) may be present.

Since a collapsed object can have a very strong effect on an electromagnetic field,[5] it is of interest to determine the effects that occur when a black hole is placed in an external electromagnetic field. In particular, one can imagine that if a black hole were placed in even a weak magnetic field (e.g., the galactic magnetic field) it might "pull in" the lines of force and create a strong magnetic field in its neighborhood.

In this paper, we give the solution for the electromagnetic field occurring when a stationary, axisymmetric black hole is placed in an originally uniform magnetic (test) field of strength B_0 aligned along the axis of symmetry of the black hole. The vector potential for this solution is constructed from the Killing vectors of the black-hole spacetime, and no assumption is made concerning the specific form of the metric of the black hole (i.e., the discussion applies to an arbitrary stationary, axisymmetric, vacuum black hole). In Sec. III we give the explicit form of the electromagnetic field tensor for a Kerr black hole in a uniform magnetic field. (There is strong evidence[1-3] for believing that the Kerr black hole is the most general stationary, axisymmetric, vacuum black hole.) No significant field enhancement effects are found. The rotational effects of the black hole do, however, produce electric fields near the black hole as the field invariant $\vec{E}\cdot\vec{B}$ is nonzero.

In Sec. IV we show that a black hole in a uniform magnetic field will preferentially accrete charges of one sign until the charge of the black hole becomes $Q=2B_0J$ (in geometrized units $G=c=1$). This provides one with a mechanism for charging up a black hole. However, for reasonable magnetic fields the charge-to-mass ratio produced by this mechanism will be very small.

In Sec. II, we review the fact that a Killing vector in a vacuum spacetime generates a solution of Maxwell's equations and we discuss the properties of the solutions generated by the timelike and axial Killing vectors in an asymptotically flat spacetime. The results on a black hole in a uniform magnetic field are presented in Secs. III and IV. Finally, as a by-product of the discussion of Sec. II, we prove in Sec. V that the gyromagnetic ratio of any slightly charged stationary, axisymmetric black hole must have the value 2.

II. KILLING VECTORS AND MAXWELL FIELDS

A Killing vector ξ^μ is an infinitesimal generator of an isometry. It satisfies the equation

$$0=\mathcal{L}_\xi g_{\mu\nu}=\xi_{\mu;\nu}+\xi_{\nu;\mu}\,, \tag{2.1}$$

where the semicolon denotes covariant derivative and $\mathcal{L}$ denotes Lie derivative. It is well known[6] that a Killing vector in a vacuum spacetime generates a solution of Maxwell's equations in that spacetime. Namely, if we set

$$F_{\mu\nu}=\xi_{\nu;\mu}-\xi_{\mu;\nu}=-2\xi_{\mu;\nu} \tag{2.2}$$

then $F_{\mu\nu}$ satisfies the source-free Maxwell's equations

$$F^{\mu\nu}{}_{;\nu}=-2\xi^{\mu;\nu}{}_{;\nu}=0\,. \tag{2.3}$$

To prove Eq. (2.3) we use the equation which defines the Riemann curvature tensor,

$$\xi_{\mu;\nu;\sigma}-\xi_{\mu;\sigma;\nu}=-\xi^\lambda R_{\lambda\mu\nu\sigma}\,. \tag{2.4}$$

Permuting the indices μ, ν, σ cyclically, adding, and using Eq. (2.1) and the symmetries of the Riemann tensor, we find that all Killing vectors satisfy

$$\xi_{\mu;\nu;\sigma}=\xi^\lambda R_{\lambda\sigma\mu\nu}\,. \tag{2.5}$$

Contracting the indices ν and σ, we obtain

$$\xi^{\mu;\nu}{}_{;\nu}=\xi^\lambda R_{\lambda\nu}{}^{\mu\nu}=R^\mu{}_\lambda\xi^\lambda\,. \tag{2.6}$$

However, for a vacuum spacetime $R_{\mu\nu}$ vanishes, by virtue of the Einstein field equations, and thus we obtain Eq. (2.3).

In flat space there are ten independent Killing vectors. The electromagnetic field generated by the four translation Killing vectors vanishes. The three rotational Killing vectors generate uniform magnetic fields, and the three boost Killing vectors generate uniform electric fields.

In the balance of this paper we shall be concerned with the properties of the electromagnetic test fields generated by the time translation and axial Killing vectors in asymptotically flat spacetimes which possess both these symmetries. We shall denote the timelike Killing vector $\partial/\partial t$ by η^μ and the axial Killing vector $\partial/\partial\varphi$ by ψ^μ. The dual 1-forms will be written $\eta=\eta_\mu dx^\mu$ and $\psi=\psi_\mu dx^\mu$.

Consider the test field F_ψ generated by the axial Killing vector,

$$F_\psi=d\psi\,, \tag{2.7}$$

where we use the differential-forms notation $F=\frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ and d denotes exterior derivative. Then F_ψ is a stationary, axisymmetric solution, i.e.,

$$\mathcal{L}_\psi F_\psi=\mathcal{L}_\eta F_\psi=0\,. \tag{2.8}$$

[Equation (2.8) follows from the fact that the Lie derivative and exterior derivative commute, and $\mathcal{L}_\psi\psi=0$ (trivial), and $\mathcal{L}_\eta\psi=0$ since η and ψ commute.] Furthermore, at large distances F_ψ asymptotically becomes a uniform magnetic field since $d\psi$ approaches its flat-space value as the spacetime becomes flat. The magnetic monopole moment associated with F_ψ vanishes,

$$4\pi\mathcal{P}_\psi = \int F_\psi = \int d\psi = 0\,, \tag{2.9}$$

where the integral is taken over a topological 2-sphere. However, the charge of F_ψ is given by

$$4\pi q_\psi = \int {}^\star F_\psi = \int {}^\star d\psi\,, \tag{2.10}$$

where $\star$ denotes dual. But the right side is an expression for the angular momentum of the spacetime,[7]

$$\int {}^\star d\psi = 16\pi J\,. \tag{2.11}$$

Thus, *the axial Killing vector generates a stationary, axisymmetric test electromagnetic field which asymptotically approaches a uniform magnetic field, has no magnetic monopole moment, and has charge =4J.*

Similarly, the electromagnetic field generated by the timelike Killing vector,

$$F_\eta = d\eta\,, \tag{2.12}$$

is also a stationary, axisymmetric test field. F_η vanishes asymptotically at large distances as the spacetime becomes flat. The magnetic monopole moment of F_η vanishes, and the charge is

$$4\pi q_\eta = \int {}^\star F_\eta = \int {}^\star d\eta\,. \tag{2.13}$$

But the quantity on the right-hand side of Eq. (2.13) is an expression for the mass of the original vacuum spacetime,[7]

$$\int {}^\star d\eta = -8\pi m\,. \tag{2.14}$$

Thus, *the timelike Killing vector generates a stationary, axisymmetric test field which vanishes asymptotically, has no magnetic monopole moment, and has charge =−2m.*

We recall now the following theorem on electromagnetic perturbations of black holes which was proved independently by Ipser[8] and myself[9] for a Kerr black hole and generalized by Carter[2] for an arbitrary stationary, axisymmetric, vacuum black hole:

Theorem. Let F be a Maxwell test field on a stationary, axisymmetric, vacuum black-hole spacetime. Suppose F satisfies the following properties: (1) F is stationary and axisymmetric. (2) F is nonsingular in the exterior region and on the horizon of the black hole. (3) F vanishes asymptotically at large distances from the black hole. (4) F has no charge or magnetic monopole moment. Then $F=0$.

This theorem shows that there can be at most one perturbation of a stationary, axisymmetric, vacuum black hole which corresponds to adding a charge Q to the black hole, i.e., all the higher multipole moments of an asymptotically vanishing electromagnetic test field around a black hole are uniquely determined by Q. (This conclusion follows from the theorem since the difference of any two test fields of charge Q around a black hole will satisfy the hypothesis of the theorem and hence will vanish.) But the previous discussion shows that for any stationary black hole there always exists a well behaved perturbation which adds a charge Q to the black hole, namely the solution $(-Q/2m)F_\eta$. Thus *$(-Q/2m)F_\eta$ is the (unique) solution which adds a charge Q to the black hole.* It is easily verified that for the case of a Kerr black hole, $(-Q/2m)F_\eta$ is precisely the Kerr-Newman (charged Kerr) test field.

III. BLACK HOLE IN A UNIFORM MAGNETIC FIELD

We are interested in obtaining the solution for the electromagnetic test field F which occurs when a stationary, axisymmetric black hole is placed in an originally uniform magnetic field of strength B_0 aligned along the symmetry axis of the black hole. On physical grounds it is clear that F must satisfy the following properties:

(I) F must be stationary and axisymmetric.

(II) F must be nonsingular throughout the exterior region and on the horizon of the black hole.

(III) At large distances from the black hole, F must asymptotically approach a uniform magnetic field of strength B_0.

(IV) The charge and magnetic monopole moment of F must vanish.

We now claim that properties (I)–(IV) uniquely determine an electromagnetic field F. Namely, suppose F and F' satisfy properties (I)–(IV). Then the difference field $\tilde{F}=F-F'$ satisfies the hypothesis of the theorem quoted at the end of the previous section, so $\tilde{F}=0$. Thus, $F=F'$ and properties (I)–(IV) uniquely determine F.

Thus, in order to obtain the solution for a black hole in a uniform magnetic field we need only write down a field tensor F which satisfies (I)–(IV). But from the results of the previous section, it is easily verified that the following field tensor indeed satisfies these properties:

$$F = \tfrac{1}{2}B_0\left(d\psi + \frac{2J}{m}d\eta\right), \tag{3.1}$$

where J and m are the angular momentum and mass of the black-hole spacetime, ψ is the axial Killing vector, and η is the timelike Killing vector. Thus, Eq. (3.1) gives the solution for a black hole in a uniform magnetic field.

We give the explicit expression for F for the case of a Kerr black hole,

$$ds^2 = -(1 - 2mr/\Sigma)dt^2 - (4mar\sin^2\theta/\Sigma)dt\,d\varphi + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \qquad (3.2)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad (3.3)$$

$$\Delta = r^2 + a^2 - 2mr. \qquad (3.4)$$

Using the orthonormal tetrad

$$\omega^0 = \left(\frac{\Delta}{\Sigma}\right)^{1/2}(dt - a\sin^2\theta\,d\varphi), \qquad (3.5)$$

$$\omega^1 = \left(\frac{\Sigma}{\Delta}\right)^{1/2}dr, \qquad (3.6)$$

$$\omega^2 = \Sigma^{1/2}d\theta, \qquad (3.7)$$

$$\omega^3 = \frac{\sin\theta}{\Sigma^{1/2}}[(r^2+a^2)d\varphi - a\,dt], \qquad (3.8)$$

Eq. (3.1) becomes

$$F = B_0\left[\frac{ar\sin^2\theta}{\Sigma} - \frac{ma(r^2 - a^2\cos^2\theta)(1+\cos^2\theta)}{\Sigma^2}\right]\omega^1\wedge\omega^0 + B_0\frac{\Delta^{1/2}r\sin\theta}{\Sigma}\omega^1\wedge\omega^3 + B_0\frac{\Delta^{1/2}a\sin\theta\cos\theta}{\Sigma}\omega^2\wedge\omega^0 + \frac{B_0\cos\theta}{\Sigma}\left[r^2 + a^2 - \frac{2mra^2(1+\cos^2\theta)}{\Sigma}\right]\omega^2\wedge\omega^3. \qquad (3.9)$$

From this expression it is easily seen that the strength of the field as measured by the field invariant

$$B^2 - E^2 = \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} \qquad (3.10)$$

is not enhanced near the black hole. However, if the black hole is rotating ($a \neq 0$) the field invariant

$$\vec{E}\cdot\vec{B} = \tfrac{1}{4}\,{}^\star F_{\mu\nu}F^{\mu\nu} \qquad (3.11)$$

is nonzero. Thus, in the rotating case all observers in the vicinity of the black hole see an electric field.

The analogous expressions for the electromagnetic field occurring when a black hole is placed in a uniform electric field can be obtained by performing a duality rotation on Eqs. (3.1) and (3.9).

IV. CHARGE ACCRETION

From the expression, Eq. (3.9), for F, it is easily seen that a positively charged particle on the symmetry axis of a Kerr black hole will be pulled into the black hole, while a negatively charged particle will be repelled. (The opposite is true if the magnetic field and spin of the black hole are antiparallel.) Thus, a black hole in a magnetic field surrounded by an ionized interstellar medium will selectively accrete charged particles and build up a net electric charge. Analogous effects for a conducting sphere in flat space have been found by Ruffini and Treves.[10] The magnitude of the accreted charge in the present case can be found by the following injection-energy argument (see Carter[2] for a general discussion of injection energy):

The energy of a particle in a stationary spacetime with a stationary electromagnetic field is given by

$$E = -\mathcal{P}_\mu\eta^\mu, \qquad (4.1)$$

where, as before, η^μ is the timelike Killing vector, and the 4-momentum $\mathcal{P}_\mu$ is given by

$$\mathcal{P}_\mu = mu_\mu - eA_\mu, \qquad (4.2)$$

where u_μ is the 4-velocity and A_μ is the vector potential of the electromagnetic field. In order to keep the discussion simple, we will consider injection of particles along the axis of symmetry. If we lower a charged particle down the axis of symmetry into a black hole, the change in the electrostatic energy of the particle is given by

$$\epsilon = E_{\text{final}} - E_{\text{initial}} = eA_\mu\eta^\mu|_{\text{horizon}} - eA_\mu\eta^\mu|_\infty. \qquad (4.3)$$

If ϵ is negative, it will be energetically favorable for the black hole to accrete particles with this charge; if ϵ is positive, the black hole will accrete particles of opposite charge. In either case, the black hole will selectively accrete charges until A_μ is changed sufficiently that the electrostatic injection energy ϵ is reduced to zero.

Now, the solution for a black hole with charge Q in a uniform magnetic field is simply the sum of the uncharged solution, Eq. (3.1), and the charge perturbation $(-Q/2m)F_\eta$ discussed at the end of Sec. II. The vector potential for this solution is given by

$$A_\mu = \tfrac{1}{2}B_0\left(\psi_\mu + \frac{2J}{m}\eta_\mu\right) - \frac{Q}{2m}\eta_\mu. \qquad (4.4)$$

Using the fact that $\psi_\mu\eta^\mu = 0$ on the symmetry axis, $\eta_\mu\eta^\mu \to -1$ at infinity, and $\eta_\mu\eta^\mu = 0$ on the horizon on the symmetry axis, it is easily seen that for the electromagnetic field, Eq. (4.4), the electrostatic injection energy ϵ is given by

$$\epsilon = e\left(\frac{Q}{2m} - \frac{B_0 J}{m}\right). \qquad (4.5)$$

We have derived Eq. (4.5) only for injection along the symmetry axis, but Carter[2] has shown that the electrostatic injection energy is constant over the black hole, and thus Eq. (4.5) is valid for general injection of particles. As discussed above, a black hole in a uniform magnetic field will accrete charge until the value of the charge Q is such that $\epsilon = 0$. Thus, a black hole in a uniform magnetic field will charge up to a value

$$Q = 2B_0 J\,. \qquad (4.6)$$

Note that Eq. (4.6) holds for an arbitrary stationary, axisymmetric black hole, i.e., we have not assumed that the black hole is a Kerr black hole anywhere in the derivation.

A Kerr black hole must satisfy $J \leq m^2$. Hence, the charge-to-mass ratio of a Kerr black hole in a uniform magnetic field is

$$\frac{Q}{m} \leq 2B_0 m = 1.7\times 10^{-20}\left(\frac{m}{m_{\odot}}\right) B_0(\text{gauss})\,, \qquad (4.7)$$

where in the second equality we have converted m and B_0 from geometrized units to solar-mass units and gauss. From Eq. (4.7) we see that a solar-mass black hole sitting in interstellar space should selectively accrete charges as a result of the galactic magnetic field $B_0 \sim 10^{-4}$–10^{-5} gauss until its charge-to-mass ratio in geometrized units is $\lesssim 10^{-24}$. If a magnetized plasma surrounds the black hole, B_0 and hence the charge-to-mass ratio can be much larger, but for astrophysically reasonable black hole masses and magnetic fields the charge-to-mass ratio is always much less than unity.

V. GYROMAGNETIC RATIO OF A SLIGHTLY CHARGED BLACK HOLE

In this section we will use the results of Sec. II to prove that the gyromagnetic ratio of any slightly charged, stationary, axisymmetric, vacuum black hole must have the value $g = 2$. Here the g factor is defined by the equation

$$\mu = g\frac{QJ}{2m}, \qquad (5.1)$$

where μ, Q, J, and m are, respectively, the magnetic dipole moment, charge, angular momentum, and mass of the black hole. The basic argument used below to prove $g = 2$ is due to Geroch.

Even in the stationary case, there is considerable ambiguity involved in the definition of multipole moments. Basically, one has freedom in deciding what part of a field constitutes a bona fide l-pole moment and what part should be regarded as curvature corrections to multipole moments of order lower than l. Thus, the definition of the static gravitational multipole moments given by Clarke and Sciama[11] does not agree with that of Geroch.[12] However, this ambiguity does not arise for the lowest multipole moments Q, μ, m, and J and thus the g factor, Eq. (5.1), is unambiguously defined for any stationary, asymptotically flat spacetime.

To prove that $g = 2$ for a slightly charged black hole, we note that the angular momentum of a stationary, asymptotically flat spacetime is determined from the asymptotic behavior of the twist ω of the timelike Killing vector η^μ in precisely the same manner as the magnetic dipole moment is determined from the magnetic scalar potential ϕ of the electromagnetic field $F_{\mu\nu}$. Here the twist ω is defined by the equation

$$\nabla_\mu \omega = \epsilon_{\mu\nu\lambda\rho}\eta^\nu \eta^{\lambda;\rho}\,, \qquad (5.2)$$

while the magnetic scalar potential ϕ is defined by

$$\nabla_\mu \phi = \epsilon_{\mu\nu\lambda\rho}\eta^\nu F^{\lambda\rho}\,. \qquad (5.3)$$

But we proved in Sec. II that the electromagnetic field of a slightly charged stationary, axisymmetric black hole is given by

$$F^{\lambda\rho} = -\frac{Q}{2m}(F_\eta)^{\lambda\rho} = \frac{Q}{m}\eta^{\lambda;\rho}\,. \qquad (5.4)$$

Substituting this in Eq. (5.3) we find

$$\phi = \frac{Q}{m}\omega\,, \qquad (5.5)$$

and hence

$$\mu = \frac{Q}{m}J\,. \qquad (5.6)$$

Comparison of Eq. (5.6) with (5.1) establishes $g = 2$ for an arbitrary, slightly charged, stationary, axisymmetric black hole. The fact that $g = 2$ for a charged Kerr black hole is well known.[13]

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Conservation laws and variational principles in metric theories of gravity*

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Conservation of energy, momentum, and angular momentum in metric theories of gravity is studied extensively both in Lagrangian formulations (using generalized Bianchi identities) and in the post-Newtonian limit of general metric theories. Our most important results are the following: (i) The matter response equations $T^{\mu\nu}{}_{;\nu}=0$ of any Lagrangian-based, generally covariant metric theory (LBGCM theory) are a consequence of the gravitational-field equations if and only if the theory contains no absolute variables. (ii) Almost all LBGCM theories possess conservation laws of the form $\Theta_\mu{}^\nu{}_{,\nu}=0$ (where $\Theta_\mu{}^\nu$ reduces to $T_\mu{}^\nu$ in the absence of gravity). (iii) $\Theta_\mu{}^\nu$ is always expressible in terms of a superpotential, $\Theta_\mu{}^\nu=\Lambda_\mu{}^{[\nu\alpha]}{}_{,\alpha}$. If the superpotential $\Lambda_\mu{}^{[\nu\alpha]}$ can be expressed in terms of asymptotic values of field quantities, then the conserved integral $P_\mu=\int\Theta_\mu{}^\nu d^3\Sigma_\nu$ can be measured by experiments confined to the asymptotically flat region outside the source. (iv) In the Will-Nordtvedt ten-parameter post-Newtonian (PPN) formalism there exists a conserved P_μ if and only if the parameters obey five specific constraints; two additional constraints are needed for the existence of a conserved angular momentum $J_{\mu\nu}$. (This modifies and extends a previous result due to Will.) (v) We conjecture that for metric theories of gravity, the conservation of energy-momentum is equivalent to the existence of a Lagrangian formulation; and using the PPN formalism, we prove the post-Newtonian limit of this conjecture. (vi) We present "stress-energy-momentum complexes" $\Theta_\mu{}^\nu$ for all currently viable metric theories known to us.

I. INTRODUCTION AND SUMMARY

The variational principle is an elegant and compelling foundation upon which fundamental theories are formulated and leads to physically useful conservation laws. In fact, most complete and self-consistent theories of gravity are derivable from variational principles—i.e., are "Lagrangian based." In this paper, a member of a series[1-4] of papers which discuss general properties of gravitation theories beyond the parametrized post-Newtonian (PPN) formalism,[5] we concentrate on conservation laws and consequences of variational principles in metric theories of gravity. It would be very helpful for the reader to have read Ref. 2 (hereafter referred to as TLL) for definitions of the terms and concepts used in this paper.[6]

First our discussion focuses on the identities and conservation laws that follow from a variational principle. We demonstrate that for the case when all fields present in the action are varied (when there are no absolute variables), the resulting Euler-Lagrange equations contain redundancies, i.e., identities. As a result of the specific form of these identities, we prove that the matter response equation $T_\mu{}^\nu{}_{;\nu}=0$ is a consequence of the gravitational-field equations if and only if no absolute variables are present. We also prove