lication by Robinson[10] has proved the following theorem: Stationary, axially symmetric, charged black holes form discrete, continuous families, each depending on at most four parameters, of which only one—the Kerr-Newman family—contains members with zero angular momentum. Thus the theorem proved in the present paper is a special case of the formally more general result due to Robinson.

We thank Professor J. T. Cushing, Professor W. R. Johnson, and Professor W. D. McGlinn for helpful conversations.

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[6] Equations (2a) and (2b) of the text have been written using the same notation as Ref. 5. One remembers that $\mathcal{E}$ and Φ are independent of the azimuthal angle ϕ and that the Laplacian as well as the scalar products appearing in (2a), (2b) are defined with respect to the Euclidean metric. A^* denotes the complex conjugate of A.

[7] The utility of transformation (3a) has been emphasized by Ernst [Phys. Rev. 167, 1175 (1968)], but that of (3b) seems to have gone unnoticed.

[8] It is shown in Ref. 5 that the metric of Ref. 3 can be written in the form of Eq. (6) of the text.

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PHYSICAL REVIEW D VOLUME 10, NUMBER 6 15 SEPTEMBER 1974

Producing an ultrahigh-uniformity gravitational field*

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(Received 9 April 1974)

We present a new "double-ring" source mass distribution which produces a gravitational field which varies by less than one part in one hundred billion for a 0.025% change in the position of the test point. Such a source mass distribution removes a major obstacle to making ultrahigh-precision measurements of the gravitational field in the laboratory: the formidable problem of precisely locating the test mass. The double-ring source mass would allow the gravitational inverse square law to be tested in the laboratory using current position-measuring techniques to an accuracy one thousand times better than the limits now given by perihelion data.

Over the years there has been relatively little interest in the Cavendish-type experiment for investigating the gravitational interaction. There are two major reasons for this circumstance. Measuring the gravitational force to better than a few parts per ten thousand in the laboratory is very difficult; but even if one succeeds in this task, the problem of relating the measurement to theory to better than a few parts in ten thousand has generally been regarded as hopeless because of the difficulty in measuring the position of the test mass. In any sort of Cavendish experiment the test mass must be very delicately suspended, and available optical techniques can only determine its position to about 0.01 mm. If, as has generally been the case,[1,2] spherical or cylindrical source masses are used, then the typical mass separation of 10 cm gives an error of two parts per ten thousand. It is evident that the problem of accurately determining the test-mass position must be circumvented before it is worthwhile to attempt to measure gravitational forces to any greater precision. In all likelihood this problem has greatly retarded the development of gravitational-force measuring techniques.

Faller and Koldewyn[3] have taken an ingenious approach to this problem by employing a ring for the attracting mass. At the "Helmholtz" point on the axis of the ring the gravitational field reaches a maximum, and hence there is a point in the field where the first derivative is zero and the field is uniform to a certain extent.

We have attempted to improve the uniformity and spatial extent of the uniform region by using a two-ring geometry. We have met with impressive success, and find that it is possible to flatten the gravitational field to such an extent that laboratory examination of the gravitational inverse square law may become competitive in sensitivity with the perihelion data.

The magnitude of the gravitational force at a point z on the axis of the thin rings is given by

$$F_L = \frac{GM_L z}{(\rho_L^2 + z^2)^{3/2}}\,, \quad F_S = \frac{GM_S(z-z')}{[\rho_S^2 + (z-z')^2]^{3/2}}\,,$$

where F_L is the force due to the large ring and F_S that due to the small ring and the other symbols are defined by Fig. 1. We are interested in the total gravitational force field, F_T:

$$F_T = F_L + F_S\,.$$

The following definitions allow the problem to be approached in a convenient manner:

$$f_T \equiv \frac{\rho_L^2 F_T}{GM_L}\,, \quad f_L \equiv \frac{\rho_L^2 F_L}{GM_L}\,,$$

$$f_S \equiv \frac{\rho_L^2 F_S}{GM_L}\,, \quad \delta \equiv \frac{z}{\rho_L}\,,$$

$$\eta \equiv \frac{\rho_S}{\rho_L}\,, \quad \lambda \equiv \frac{z'}{\rho_L}\,, \quad \xi \equiv \frac{M_S}{M_L}\,.$$

We may now work with the dimensionless total force f_T, where

$$f_T = f_L + f_S\,, \tag{1}$$

$$f_L \equiv \frac{\delta}{(1+\delta^2)^{3/2}}\,, \quad f_S \equiv \frac{\xi(\delta-\lambda)}{[\eta^2 + (\delta-\lambda)^2]^{3/2}}\,. \tag{2}$$

We use the Taylor expansion about the point δ_0 to systematically reduce the dependence of $f_T(\delta)$ on δ:

$$f_T(\delta) = f_T(\delta_0) + \frac{df_T}{d\delta}\bigg|_{\delta_0}(\delta-\delta_0) + \frac{d^2 f_T}{d\delta^2}\bigg|_{\delta_0}\frac{(\delta-\delta_0)^2}{2!} + \frac{d^3 f_T}{d\delta^3}\bigg|_{\delta_0}\frac{(\delta-\delta_0)^3}{3!} + \frac{d^4 f_T}{d\delta^4}\bigg|_{\delta_0}\frac{(\delta-\delta_0)^4}{4!} + \cdots\,.$$

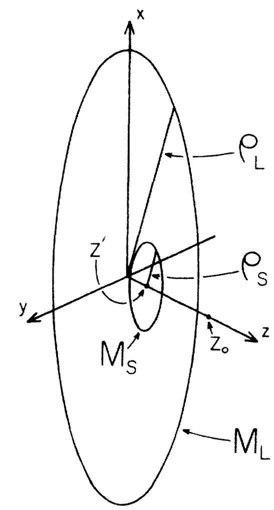

FIG. 1. The two thin rings are perpendicular to the z axis (coming out of the paper). The center of the large ring is at the origin $z=0$. The center of the small ring is at z'. M_S and M_L are the ring masses.

We have a sufficient number of free parameters to require that

$$0 = \frac{df_T}{d\delta}\bigg|_{\delta_0} = \frac{df_L}{d\delta}\bigg|_{\delta_0} + \frac{df_S}{d\delta}\bigg|_{\delta_0}\,, \tag{3}$$

$$0 = \frac{d^2 f_T}{d\delta^2}\bigg|_{\delta_0} = \frac{d^2 f_L}{d\delta^2}\bigg|_{\delta_0} + \frac{d^2 f_S}{d\delta^2}\bigg|_{\delta_0}\,, \tag{4}$$

$$0 = \frac{d^3 f_T}{d\delta^3}\bigg|_{\delta_0} = \frac{d^3 f_L}{d\delta^3}\bigg|_{\delta_0} + \frac{d^3 f_S}{d\delta^3}\bigg|_{\delta_0}\,. \tag{5}$$

This requirement clearly leaves

$$(\delta-\delta_0)^4$$

as the leading δ-dependent term, which, of course, is very small for

$$|\delta-\delta_0| \ll 1.$$

Using Eq. (2) in Eqs. (3), (4), and (5), we can use the equality resulting from Eq. (3) in those resulting from Eqs. (4) and (5) to eliminate ξ and to find the following simultaneous equations for δ_0, λ, and η:

$$0 = g_1(\delta_0, \lambda, \eta) \equiv \frac{6(\delta_0-\lambda)^3 - 9(\delta_0-\lambda)\eta^2}{[\eta^2 + (\delta_0-\lambda)^2][\eta^2 - 2(\delta_0-\lambda)^2]} - \frac{6\delta_0^3 - 9\delta_0}{(1+\delta_0^2)(1-2\delta_0^2)}\,,$$
$$0 = g_2(\delta_0, \lambda, \eta) \equiv \frac{-24(\delta_0-\lambda)^4 + 72(\delta_0-\lambda)^2\eta^2 - 9\eta^4}{[\eta^2 + (\delta_0-\lambda)^2]^2[\eta^2 - 2(\delta_0-\lambda)^2]} - \frac{-24\delta_0^4 + 72\delta_0^2 - 9}{(1+\delta_0^2)^2(1-2\delta_0^2)}\,. \tag{6}$$

We have proceeded to find the solution to these equations by first assuming a particular value for δ_0 and then numerically plotting $g_1(\delta_0, \lambda, \eta)$ in the λ, η plane using curves of "iso-$g_1(\delta_0, \lambda, \eta)$." The same was done for $g_2(\delta_0, \lambda, \eta)$. The approximate root curves ($g_1 = 0$, $g_2 = 0$) in the λ, η plane for g_1 and g_2 can be obtained in this manner and the approximate region of intersection found. The region of intersection of the root curves is then numerically explored in detail to find the simultaneous solutions to Eqs. (6).

In this manner we have obtained one solution to the problem[4]:

$$\begin{aligned}\delta_0 &= 0.4014\ ,\\ \eta &= 0.19000\ ,\\ \lambda &= 0.072717\ ,\\ \xi &= 0.0204454\ .\end{aligned}\tag{7}$$

The force field, f_T, obtained using these values for the parameters is displayed in Fig. 2, where it is compared to the fields due to a single ring[5] and fields due to a sphere,[6] both taken to be centered at the center of the large ring. It is clear from Fig. 2 that the double-ring configuration produces a substantially more uniform field than the single ring.

We have also numerically investigated the field off of the axis due to the double-ring configuration. It turns out that the field uniformity going off of the axis is very nearly the same as the uniformity along the axis.

It is interesting to examine the limit to which the inverse square law may be tested which is imposed by the test-mass location problem using the above double-ring configuration. If we consider the historic inverse-square-law failure of the form[7]

$$r^{-2+\phi}\ ,\tag{8}$$

then data from the perihelion of Mercury yield the limit[8]

$$\phi < 3.7\times 10^{-9}\ .\tag{9}$$

A most direct procedure for examining the inverse square law in the laboratory is to measure the force due to a small, close double-ring source mass and compare it to that due to a larger double-ring source mass much farther away. Characterizing the distance from the test mass to the small double-ring source mass as r_1, and that to the large double-ring mass as r_2, we can show from Eq. (8) that the fractional discrepancy in the comparison of the measured and calculated forces, $\Delta F/F$, produced by an inverse-square-law failure is given by

$$\frac{\Delta F}{F} \simeq \phi \ln \frac{r_2}{r_1}\ .\tag{10}$$

Now reasonable values for the relative sizes of the masses suggest that $r_2/r_1 \approx 6$ and hence

$$\phi \simeq \frac{1}{1.8}\,\frac{\Delta F}{F}\ .\tag{11}$$

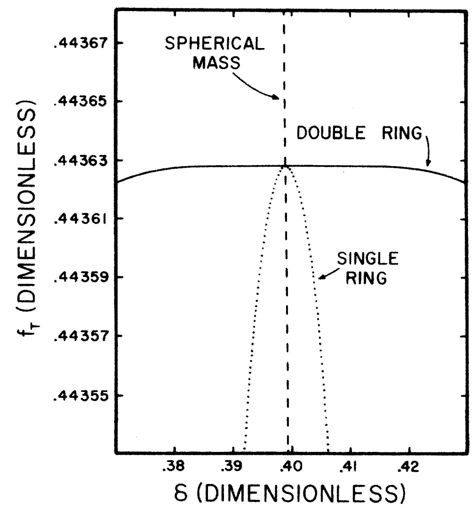

FIG. 2. The gravitational-field magnitudes are plotted for a sphere, a single ring, and a double-ring configuration with the parameters as specified in Eq. (7).

Now the force discrepancy $\Delta F/F$ is determined by a comparison of the measured and calculated forces, and the accuracy of the calculated forces is determined by the accuracy of the measured position of the test mass. Current laboratory techniques allow the position of the test mass to be determined to an accuracy of 0.01 mm, and if the mass separation (for the small double-ring mass) is about 5 cm, then the fractional accuracy of the test-mass position is 2×10^{-4}. If we take a closer look at f_T near the peak we find

$$\begin{aligned}\delta &= 0.4012, & f_T &= 0.443628309979\ ,\\ \delta &= 0.4013, & f_T &= 0.443628309984\ ,\\ \delta &= 0.4014, & f_T &= 0.443628309987\ ,\\ \delta &= 0.4015, & f_T &= 0.443628309985\ ,\\ \delta &= 0.4016, & f_T &= 0.443628309978\ .\end{aligned}\tag{12}$$

Hence we find, for the above error in position, an error in the calculated force of

$$\frac{\Delta f_T}{f_T} \simeq 0.6\times 10^{-11}\ .\tag{13}$$

Clearly, the limit in the accuracy of $\Delta F/F$ which is imposed by the test-mass position measurement when a double-ring source mass is used is just $\Delta f_T/f_T$, and thus ϕ could be determined to

$$\phi \simeq 3.3\times 10^{-12}\ ,\tag{14}$$

which is about 10^3 times smaller than the limits given in Eq. (9) from perihelion data.

Up to this point we have used mathematically thin rings. It is clear, however, that a finite mass distribution can be built up out of pairs of such rings such that each pair yields a peak at the same point as the other pairs. We have not investigated the difficulty of manufacturing a double-ring mass. It is our observation, however, that the mathematical situation producing the flat peak is not especially fragile [indeed we used a rather crude technique in finding Eq. (9)] and that ultrahigh-precision machining is probably not required. More importantly, the mass distribution would have to be verified using, say, γ-ray absorption.

It is clear that the problem of measuring gravitational forces to the precision suggested here has not been solved. One of us (D.R.L.) has been struggling with the problem at the part-per-thousand level for several years. On the other hand, there has never been a thorough study of the physical processes which might set an ultimate limit to the accuracy of a gravitational-force measurement. The gas-thermal-noise problem, as well as the vibration problem, could probably be readily reduced by several orders of magnitude in this laboratory.

Perhaps the progress presented here on the mass separation problem will allow a more vigorous attack on the gravitational-force measurement problem.

We gratefully thank Professor E. Forsman and Professor R. Gibbs for a careful reading of this manuscript.

*This research was supported in part by a grant by the Research Corporation.

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[4]There are, of course, infinitely many solutions. Cursory inspection suggests that there are probably no solutions for $\delta_0 > 0.65$.

[5]The center of the single ring is taken at $\delta = 0$, and its radius chosen to give the peak at $\delta = 0.4000$. Its mass was chosen to give the same field magnitude as the double rings.

[6]The center of the sphere is taken at $\delta = 0$, and the mass chosen to give the same field magnitude as the double rings.

[7]D. R. Long, Phys. Rev. D 9, 850 (1974). It is pointed out in this paper that assuming this sort of failure is excessively naive, and that problems with the inverse square law may be encountered at far lower precision than we discuss in the text.

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PHYSICAL REVIEW D VOLUME 10, NUMBER 6 15 SEPTEMBER 1974

Black hole in a uniform magnetic field*

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(Received 22 April 1974)

Using the fact that a Killing vector in a vacuum spacetime serves as a vector potential for a Maxwell test field, we derive the solution for the electromagnetic field occurring when a stationary, axisymmetric black hole is placed in an originally uniform magnetic field aligned along the symmetry axis of the black hole. It is shown that a black hole in a magnetic field will selectively accrete charges until its charge becomes $Q = 2B_0 J$, where B_0 is the strength of the magnetic field and J is the angular momentum of the black hole. As a by-product of the analysis given here, we prove that the gyromagnetic ratio of a slightly charged, stationary, axisymmetric black hole (not assumed to be Kerr) must have the value $g = 2$.

I. INTRODUCTION

From the results on black-hole uniqueness which have been proved during the last several years (particularly the theorems of Israel,[1] Carter,[2] Hawking,[3] and Robinson[4]), it is now well established that an isolated black hole cannot have an electromagnetic field unless it is endowed with a net electric charge. Thus, an isolated black hole can participate in electromagnetic effects only if there is a mechanism for charging it up. However, if the black hole is not isolated, electromagnetic fields produced by external sources (e.g., plasma accreting onto the black hole) may be present.