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Calculation of the weak-decay parameter ratio $f_K/f_\pi f_+(0)$

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Making use of standard smoothness assumptions together with a recent soft-pion theorem due to Mathur and Yang, the weak-decay parameter ratio $f_K/f_\pi f_+(0)$ is successfully calculated.

The application of soft-pion techniques to K_{l3} decay has been regarded as an outstanding problem for many years. The puzzle exists, however, only if experiments confirm a large negative value for the parameter ξ. Such a confirmation would indicate that the K_{l3} decay amplitude could not be extrapolated smoothly off the pion mass shell. In fact, at the present time, the magnitude of ξ is not well known experimentally[1] and cannot be used to test the validity of soft-pion techniques in K_{l3} decay. We shall not be concerned with this parameter theoretically here.

It is the purpose of this note to show that, making use of certain smoothness assumptions, one may predict

$$\frac{f_K}{f_\pi f_+(0)}=1+\frac{2}{3}\frac{m_K{}^2}{m_{K^*}{}^2-m_K{}^2} \tag{1}$$

in the limit of zero pion mass. Numerically, this theoretical prediction is 1.29 compared with the experimental value of (1.27 ± 0.03). Since this latter quantity is well known experimentally (in contrast with ξ) one may conclude that the application of soft-pion techniques to the matrix element describing K_{l3} decay is quite successful.

Initially, we follow the work of Glashow and Weinberg.[2] The Hamiltonian density is assumed to be[2,3]

$$\mathcal{H}=\mathcal{H}_0-\epsilon_0\sigma_0-\epsilon_8\sigma_8\,, \tag{2}$$

where $\mathcal{H}_0$ is SU(3)×SU(3)-symmetric and σ_0 and σ_8 are local scalar fields belonging to the eighteen-dimensional $(3,\bar{3})+(\bar{3},3)$ representation of SU(3) ×SU(3). In order to derive the result, Eq. (1), we begin by defining the three-point functions $\Gamma(k^2,p^2,q^2)$, $f_\pm^\kappa(k^2,p^2,q^2)$:

$$\Delta_\kappa(q^2)\Delta_K(k^2)\Delta_\pi(p^2)\Gamma(k^2,p^2,q^2)=\iint d^4x\,d^4y\,e^{ik\cdot x}e^{-ip\cdot y}\langle 0|T\{\phi_{K^-}(x)\phi_{\pi^0}(y)\phi_{\kappa^+}(0)\}|0\rangle \tag{3}$$

and

$$\Delta_K(k^2)\Delta_\pi(p^2)[f_+^\kappa(k^2,p^2,q^2)(k+p)_\mu+f_-^\kappa(k^2,p^2,q^2)(k-p)_\mu]=\iint d^4x\,d^4y\,e^{ik\cdot x}e^{-ip\cdot y}\langle 0|T\{\phi_{K^-}(x)\phi_{\pi^0}(y)V_\mu^{4+i5}(0)\}|0\rangle. \tag{4}$$

We may define $f_\pm^K(k^2,p^2,q^2)$ and $f_\pm^\pi(k^2,p^2,q^2)$ similarly.

On multiplying Eq. (4) by $iq_\mu=i(k-p)_\mu$, the following relation between the various three-point functions is obtained:

$$f_+^\kappa(k^2,p^2,q^2)(k^2-p^2)+f_-^\kappa(k^2,p^2,q^2)q^2=f_\kappa m_\kappa{}^2\Delta_\kappa(q^2)\Gamma(k^2,p^2,q^2)+\left(\frac{z_\pi}{z_K}\right)^{1/2}(k^2+m_K{}^2)-\left(\frac{z_K}{z_\pi}\right)^{1/2}(p^2+m_\pi{}^2). \tag{5}$$

We have used the current divergence relation

$$\partial_\mu V_\mu^{4+i5}(0)=if_\kappa m_\kappa{}^2\phi_\kappa(0)\,, \tag{6}$$

together with the commutation relations of the $(3,\bar{3})+(\bar{3},3)$ model of chiral symmetry breaking. z_π and z_K are the wave-function renormalization constants.

It is well known[2] that if we impose the smoothness assumption that $\Gamma(k^2,p^2,q^2)$ be no more than a quadratic function of momentum, together with the assumption that $f_+^\kappa(k^2,p^2,0)$ be a constant equal to $f_+^\kappa(0,0,0)$ (or f_+^κ for short) over a certain range of k^2 and p^2, then it follows that $\Delta_\kappa{}^{-1}(q^2)$ is no more than a quadratic function of momentum. Likewise if we impose similar constraints on $f_K^+(0,p^2,q^2)$ and $f_\pi^+(k^2,0,q^2)$, then it follows that $\Delta_K{}^{-1}(k^2)$ and $\Delta_\pi{}^{-1}(p^2)$ are no more than quadratic functions of the momentum. In addition we have the usual results[2]

$$f_+^\kappa=(f_K{}^2+f_\pi{}^2-f_\kappa{}^2)/2f_Kf_\pi\,, \tag{7a}$$

$$f_+^K=(f_K{}^2-f_\pi{}^2-f_\kappa{}^2)/2f_\kappa f_\pi\,, \tag{7b}$$

$$f_+^\pi = (f_K^2 - f_\pi^2 + f_\kappa^2)/2f_K f_\kappa \,. \tag{7c}$$

Setting $k^2 + m_K^2 = 0$, $p^2 + m_\pi^2 = 0$, and $q^2 = 0$ in Eq. (5) we see that

$$f_+^\kappa(m_K^2 - m_\pi^2) = f_\kappa \Gamma(m_K^2, m_\pi^2, 0)\,. \tag{8a}$$

Similarly, we may conclude from equations analogous to Eq. (5) relating $f_\pm^K(k^2, p^2, q^2)$ and $f_\pm^\pi(k^2, p^2, q^2)$ to $\Gamma(k^2, p^2, q^2)$ that

$$f_+^K(m_\kappa^2 - m_\pi^2) = f_K \Gamma(0, m_\pi^2, m_\kappa^2) \tag{8b}$$

and

$$f_+^\pi(m_\kappa^2 - m_K^2) = f_\pi \Gamma(m_K^2, 0, m_\kappa^2)\,. \tag{8c}$$

In order to proceed further we assume K^* dominance of the form factor $f_+^\kappa(k^2, p^2, q^2)$. This assumption is certainly consistent with the q^2 behavior of that form factor.[1] We now have

$$f_+^\kappa(k^2, p^2, q^2)(k+p)_\mu + f_-^\kappa(k^2, p^2, q^2)(k-p)_\mu = F_+^\kappa(k^2, p^2, q^2)\frac{\delta_{\mu\nu} + q_\mu q_\nu/m_{K^*}^2}{1 + q^2/m_{K^*}^2}(k+p)_\nu + F_-^\kappa(k^2, p^2, q^2)(k-p)_\mu\,. \tag{9}$$

From Eq. (9) we see that

$$F_+^\kappa(k^2, p^2, q^2) = f_+^\kappa \tag{10a}$$

and

$$F_-^\kappa(k^2, p^2, q^2) = f_\kappa \Delta_\kappa(q^2)\Gamma(k^2, p^2, m_\kappa^2)\,, \tag{10b}$$

where we have assumed that $F_+^\kappa(k^2, p^2, q^2)$ is independent of q^2.

Making use of Eqs. (7)–(10), it is a simple matter to derive the soft-pion[4] and soft-kaon[5] relations of current algebra,

$$f_+^\kappa(m_K^2, 0, m_K^2) + f_-^\kappa(m_K^2, 0, m_K^2) = f_K/f_\pi \tag{11a}$$

and

$$f_+^\kappa(0, m_\pi^2, m_\pi^2) - f_-^\kappa(0, m_\pi^2, m_\pi^2) = f_\pi/f_K\,. \tag{11b}$$

From Eq. (9) we see that

$$f_-^\kappa(k^2, p^2, q^2) = f_+^\kappa \frac{(k^2 - p^2)}{m_{K^*}^2 + q^2} + f_\kappa \frac{\Gamma(k^2, p^2, m_\kappa^2)}{m_\kappa^2 + q^2}\,. \tag{12}$$

In the soft-pion limit this becomes

$$f_-^\kappa(m_K^2, 0, m_K^2) = -f_+^\kappa \frac{m_K^2}{m_{K^*}^2 - m_K^2} + f_\kappa \frac{\Gamma(m_K^2, 0, m_\kappa^2)}{m_\kappa^2 - m_K^2} = -f_+^\kappa \frac{m_K^2}{m_{K^*}^2 - m_K^2} + \frac{f_\kappa}{f_\pi} f_+^\pi\,, \tag{13}$$

where we have made use of Eq. (8c).

In addition to the soft-pion result Eq. (11a), Mathur and Yang[6] recently derived the following soft-pion relation for the K_{l3} factors:

$$f_+^\kappa(m_K^2, 0, m_K^2) - f_-^\kappa(m_K^2, 0, m_K^2) = \frac{(3f_K^2 - f_\pi^2 + f_\kappa^2)}{2f_K f_\pi}\,. \tag{14}$$

In order to prove this result, it is necessary to make the additional assumption[7] that approximate SU(3)×SU(3) symmetry requires matrix elements to be smooth functions of the symmetry-breaking parameter ϵ_8/ϵ_0, where ϵ_0 and ϵ_8 are the SU(3) ×SU(3)-breaking parameters appearing in the Hamiltonian density Eq. (2).

From Eqs. (11a) and (14) we have

$$2f_-(m_K^2, 0, m_K^2) = (f_\pi^2 - f_\kappa^2 - f_K^2)/2f_K f_\pi = -(f_+^\pi/f_\pi)f_\kappa\,. \tag{15}$$

Comparing Eqs. (13) and (15), we see that

$$\frac{f_+^\pi/f_\pi}{f_+^\kappa/f_\kappa} = \frac{2}{3}\frac{m_K^2}{m_{K^*}^2 - m_K^2}\,. \tag{16}$$

Therefore

$$\frac{2}{3}\frac{m_K^2}{m_{K^*}^2 - m_K^2} = \frac{f_K^2 + f_\kappa^2 - f_\pi^2}{f_K^2 + f_\pi^2 - f_\kappa^2} \tag{17}$$

and

$$1 + \frac{2}{3}\frac{m_K^2}{m_{K^*}^2 - m_K^2} = \frac{2f_K^2}{f_K^2 + f_\pi^2 - f_\kappa^2} = \frac{f_K}{f_\pi f_+^\kappa}\,, \tag{18}$$

where we have made use of Eq. (7a). We have thus proved the assertion made in Eq. (1).

The good agreement between the theoretical prediction Eq. (1) and experiment lends support to the assumptions we have made. In particular it demonstrates that one may successfully apply the usual soft-pion arguments to K_{l3} decay.

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