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Upper and lower bounds for the sum of $\pi^+ p$ and $\pi^- p$ total cross sections at high energy*

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Upper and lower bounds for weighted integrals of $2\sigma_s = \sigma_{\text{tot}}(\pi^+p) + \sigma_{\text{tot}}(\pi^-p)$ at pion laboratory energies $\omega > 60$ GeV are derived in terms of available experimental data and assumed bounds on the phase of $T_s(\omega)$, the πp crossing-symmetric forward scattering amplitude above 20 GeV. The bounded integrals are of the form $\int_{M_1}^{M}\sigma_s(\omega)K(\omega)d\omega$ with 60 GeV $\leq M_1 < M$, and $K(\omega)\sim\omega^{-\gamma}$ $(1<\gamma<1.72)$ as $\omega\to\infty$. Our analysis also gives an upper bound on δ_1, the minimum of the phase of T_s ($\omega > 60$ GeV).

In anticipation of future experiments at NAL, there has been renewed interest in finding and testing bounds on high-energy cross sections which follow from axiomatic field theory or, more specifically, from forward dispersion relations.

Pham and Truong[1,2] (PT) have recently given upper bounds on the sum of $\sigma_{\text{tot}}(\pi^+p)$ and $\sigma_{\text{tot}}(\pi^-p)$ $[\equiv 2\sigma_s(\omega)]$ for pion laboratory energies ω greater than 60 GeV. Their bounds are of the form

$$\int_{M_1}^{M} d\omega\, \sigma_s(\omega)K(\omega) \leq R\,, \tag{1}$$

where $M > M_1$, and $[M_1, M]$ is any energy interval above 60 GeV. The equality in (1) holds for $M_1 = 60$ GeV, $M\to\infty$. R is calculated from total $\pi^\pm p$ cross-section data up to 60 GeV[3] and data[4] on $\text{Re}T_s(\omega)$, the real part of the forward crossing-symmetric πp amplitude, in the range 8–20 GeV. The function $K(\omega)$ is positive and has the expansion

$$K(\omega) = (A/\omega^2)[1 + (\overline{\omega}/\omega)^2 + \cdots]\,, \tag{2}$$

where $A > 0$, and $\overline{\omega}$ is less than or equal to ω_2,[5] the highest energy (20 GeV) at which $\text{Re}T_s(\omega)$ data are available. If $\sigma_s(\omega)$ satisfies the high-energy Froissart-Martin bound,[6,7] $\sigma_s(\omega) < \text{const}\times\ln^2(\omega/\omega_0)$, the integral in (1) remains well defined if $K(\omega)\sim\omega^{-\gamma}$ $(\gamma > 1)$ times a logarithmic factor. The work of PT then naturally prompts one to investigate the possibility of finding bounds similar to (1) with K functions which fall off more slowly than $1/\omega^2$ with increasing ω. In this note, we use elementary manipulations of forward dispersion relations to show that such bounds can indeed be found, provided one specifies bounds on the phase of T_s for $\omega > \omega_2 = 20$ GeV.

Although we will only derive a particular family of bounds, it will be evident that the techniques involved may be used to derive a variety of other bounds for πp and other scattering amplitudes (provided one assumes the validity of forward dispersion relations).

We normalize $T_s(\omega)$ so that the optical theorem has the form

$$\text{Im}T_s(\omega) = (\omega^2 - \mu^2)^{1/2}\sigma_s(\omega)\,, \tag{3}$$

where μ is the pion mass. In order to simplify the appearance of the ensuing analysis, we will work with the pole-free amplitude

$$T(\omega) = T_s(\omega) - 2\omega_0^2 f^2/(\omega_0^2 - \omega^2)\,, \tag{4}$$

with $\omega_0 = \mu^2/2M$ (M = nucleon mass) and $f^2/4\pi = 0.077 \pm 0.003$.[8,9]

Now consider the amplitude[10]

$$t(\omega) = \frac{(\omega+\mu)^\beta[T(\omega) - T(\mu) - i(\omega^2-\mu^2)^{1/2}\sigma(\mu)]}{\omega^2 - \mu^2}\,, \tag{5}$$

where $0 < \beta < 1$ and $(\omega+\mu)^\beta$ is a real analytic function with a branch cut extending along the real axis from $\omega = -\mu$ to $\omega = -\infty$. With the Froissart-Martin bound on $T(\omega)$,[6,7] $t(\omega\to\infty) = 0$. Thus, $t(\omega)$ satisfies an unsubtracted dispersion relation. For ω real and less than $-\mu$, the dispersion relation may be easily cast into the form

$$\begin{aligned}\frac{(\omega+\mu)^\beta[\text{Re}T(\omega) - T(\mu)]}{\omega^2-\mu^2} &= \frac{1}{\pi}\text{P}\int_\mu^\infty d\omega' \frac{\text{Im}T(\omega') - (\omega'^2-\mu^2)^{1/2}\sigma(\mu)}{\omega'^2-\mu^2}\left[\frac{(\omega'+\mu)^\beta}{\omega'-\omega} + \cos\pi\beta\frac{(\omega'-\mu)^\beta}{\omega'+\omega}\right] \\ &\quad - \frac{1}{\pi}\sin\pi\beta\,\text{P}\int_\mu^\infty \frac{d\omega'(\omega'-\mu)^\beta}{\omega'^2-\mu^2}\frac{\text{Re}T(\omega')-T(\mu)}{\omega'+\omega}\,.\end{aligned} \tag{6}$$

After letting $\omega\to-\mu$ [in which case the left-hand side of (6) vanishes] and using the conventional dispersion relation

$$\mathrm{Re}T(\omega)-T(\mu)=(\omega^2-\mu^2)(2/\pi)\,\mathrm{P}\int_\mu^\infty\frac{\omega' d\omega'}{\omega'^2-\mu^2}\frac{\mathrm{Im}T(\omega')}{\omega'^2-\omega^2} \tag{7}$$

to evaluate $\mathrm{Re}T(\omega)$ in the interval $\mu<\omega<\omega_1$, we may write (6) as

$$\begin{aligned}\int_N^\infty\frac{d\omega\,\mathrm{Im}T(\omega)}{\omega^2-\mu^2}K(\omega,\omega_1,\beta)+\int_{\omega_2}^\infty d\omega\frac{(\omega-\mu)^{\beta-1}}{\omega^2-\mu^2}\sin[\delta(\omega)-\pi\beta]|T(\omega)|\\ =\int_N^\infty d\omega\frac{\sigma(\mu)}{(\omega^2-\mu^2)^{1/2}}K(\omega,\omega_1,\beta)+\int_{\omega_2}^\infty d\omega\frac{(\omega-\mu)^{\beta-1}}{\omega^2-\mu^2}[\cos\pi\beta(\omega^2-\mu^2)^{1/2}\sigma(\mu)-\sin\pi\beta T(\mu)]\\ +\int_{\omega_1}^{\omega_2}d\omega\frac{(\omega-\mu)^{\beta-1}}{\omega^2-\mu^2}\sin\pi\beta[\mathrm{Re}T(\omega)-T(\mu)]\\ -\int_\mu^N d\omega\frac{[\sigma(\omega)-\sigma(\mu)]}{(\omega^2-\mu^2)^{1/2}}[K(\omega,\omega_1,\beta)+\theta(\omega_2-\omega)\cos\pi\beta(\omega-\mu)^{\beta-1}],\end{aligned} \tag{8}$$

where

$$K(\omega,\omega_1,\beta)=(\omega+\mu)^{\beta-1}+\frac{2\omega}{\pi}\sin\pi\beta\,\mathrm{P}\int_\mu^{\omega_1}d\omega'\frac{(\omega'-\mu)^{\beta-1}}{\omega'^2-\omega^2}, \tag{9}$$

$$\theta(x>0)=1,\quad \theta(x<0)=0, \tag{10}$$

and $\delta(\omega)$ is the phase of $T(\omega)$. Since $\mathrm{Im}T(\omega)>0$, we may consider $\delta(\omega)$ to be the interval $(0,\pi)$ radians. It is assumed that experimental data on $\mathrm{Re}T(\omega)$ are available in the interval (ω_1,ω_2) and that total cross-section data are available for $\omega<N$, with $N>\omega_2$, $N\gg\omega_1$.

In deriving bounds, it is important that $K(\omega>N,\omega_1,\beta)>0$. By neglecting μ in (9) and setting $K(N,\omega_1,\beta)>0$, we obtain the approximate condition for this to be true,

$$2\left[1+\frac{\beta}{\beta+2}\left(\frac{\omega_1}{N}\right)^2+\frac{\beta}{\beta+4}\left(\frac{\omega_1}{N}\right)^4+\cdots\right]\frac{\sin\pi\beta}{\pi\beta}\left(\frac{\omega_1}{N}\right)^\beta<1. \tag{11}$$

For $N=60$ GeV, $\omega_1=8$ GeV, and $\omega_2=20$ GeV, we find $K(N,\omega_1,\beta)>0$ for $\beta>0.28$.

We now assume that the phase $\delta(\omega>\omega_2)$ is in the interval (δ_1,δ_2), with $\delta_1<\delta_2$. On the left-hand side of (8), the integrand of the first integral is positive for $\beta>0.28$, and that of the second is positive for $\beta<\delta_1/\pi$ and negative for $\beta>\delta_2/\pi$. The second integral must therefore vanish for at least one value of β in the interval $(\delta_1/\pi,\ \delta_2/\pi)$. Thus we have the following bounds and equality:

$$\begin{aligned}&I^{-1}(\beta)\int_N^\infty d\omega\,K(\omega,\omega_1,\beta)\sigma_s(\omega)/(\omega^2-\mu^2)^{1/2}<R, \quad 0.28<\beta<\delta_1/\pi;\\ &I^{-1}(\beta)\int_N^\infty d\omega\,K(\omega,\omega_1,\beta)\sigma_s(\omega)/(\omega^2-\mu^2)^{1/2}>R, \quad 0.28<\delta_2/\pi<\beta<1;\\ &I^{-1}(\beta)\int_N^\infty d\omega\,K(\omega,\omega_1,\beta)\sigma_s(\omega)/(\omega^2-\mu^2)^{1/2}=R, \quad \beta=\tilde\beta;\end{aligned} \tag{12}$$

where

$$I(\beta)=\int_N^\infty d\omega\,K(\omega,\omega_1,\beta)/(\omega^2-\mu^2)^{1/2}, \tag{13}$$

R is the right-hand side of (8) multiplied by $I^{-1}(\beta)$, and $\tilde\beta$ is a value of β in the interval $(\delta_1/\pi,\ \delta_2/\pi)$. For $\sigma_s(\omega>N)=\text{const}=\sigma$, the left-hand sides of (12) would have the value σ. The first bound in (12) obviously remains valid if the integration interval (N,∞) is replaced by the finite one (M_1,M) with $M>M_1>N$.

$R(\beta)$ is plotted in Fig. 1. In calculating $R(\beta)$, we have used the total cross-section data of Ref. 3 and data on $\mathrm{Re}T_s(8<\omega<20$ GeV) (obtained from

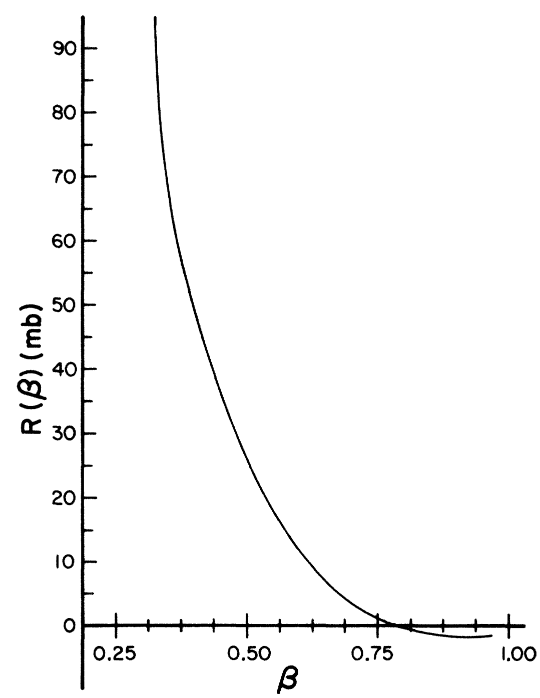

FIG. 1. Numerical evaluation of $R(\beta)$ [≡ right-hand side of Eq. (8)$\times I^{-1}(\beta)$, where $I(\beta)$ is defined by Eq. (13)], with $N=60$ GeV, $\omega_1=8$ GeV, and $\omega_2=20$ GeV.

the Coulomb interference measurements of Foley *et al.*[4] combined with total cross-section data). The estimated uncertainty in $R(\beta)$ is about 4% if we assume, as do PT, that the errors in $\mathrm{Re}T_s(\omega)$ are purely statistical. In preparing the plot of $R(\beta)$ we have used rational-number values for β, in which case $K(\omega, \omega_1, \beta)$ can be expressed analytically.[11]

Since $K(\omega, \omega_1, \beta)$ falls off as $\omega^{\beta-1}$ for large ω, the bounded integrals in (12) are more sensitive to the very-high-energy cross sections than are those of PT. However, the PT bounds are tighter than ours since they become equalities as $M_1 \to N$, $M \to \infty$.

The essential structure of our bounds is easily seen if we use the approximations $(\omega^2 - \mu^2)^{1/2} \approx \omega$ and

$$K(\omega, \omega_1, \beta) \approx \omega^{\beta-1}\left[1 - \frac{2\sin\pi\beta}{\pi\beta}\left(\frac{\omega_1}{\omega}\right)^{\beta}\right], \qquad (14)$$

which are very good for $\omega > N = 60$ GeV and $\omega_1 = 8$ GeV. The first bound of (12) then becomes effectively

$$N\int_N^\infty \frac{d\omega}{\omega^2}\sigma_s(\omega)\left[(1-\beta)\frac{\omega^\beta}{N^\beta}\right]\left[1 - \frac{2\sin\pi\beta}{\pi\beta}\left(\frac{\omega_1}{\omega}\right)^{\beta}\right] \times\left[1 - 2(1-\beta)\frac{\sin\pi\beta}{\pi\beta}\left(\frac{\omega_1}{N}\right)^{\beta}\right]^{-1} < R(\beta). \qquad (15)$$

For comparison purposes, we note that the loose version of the PT bound [Eq. (9) of Ref. 1] corresponds to setting the square brackets in (15) equal to unity and $R(\beta) = 22.2 \pm 1.2$ mb.

Unfortunately, the practical application of the bounds in (12) is complicated by the strong correlation between $\delta(\omega > \omega_2)$ and $\sigma_s(\omega > \omega_2)$. A simple derivation of an approximate formal connection between these quantities has recently been given.[12] We write the ordinary dispersion relation (7) in terms of $q = (\omega^2 - \mu^2)^{1/2}$ and make the substitutions $q/q_0 = e^\xi$, $q'/q_0 = e^{\xi+\eta}$, $T(q = e^\xi) = T(\xi)$, etc. and find

$$\frac{\mathrm{Re}T(\xi) - T(-\infty)}{q_0 e^\xi} = \frac{1}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\eta}{\sinh\eta}\sigma_s(\xi+\eta). \qquad (16)$$

Since $\sigma_s(\xi)$ is limited by the Froissart-Martin bound, $\sigma_s < \text{const} \times \xi^2$, for large ξ, the integral in (16) converges rapidly and we may make a Taylor expansion of $\sigma_s(\xi+\eta)$ about $\eta = 0$. Thus

$$\begin{aligned}\frac{\mathrm{Re}(T(\xi) - T(-\infty))}{q_0 e^\xi} &= \frac{1}{\pi}\sum_{n(\mathrm{odd})}\int_{-\infty}^{\infty}\frac{d\eta}{\sinh\eta}\frac{\eta^n}{n!}\frac{d}{d\xi^n}\sigma_s(\xi)\\ &= \left[\frac{\pi}{2}\frac{d}{d\xi} + \frac{1}{3}\left(\frac{\pi}{2}\frac{d}{d\xi}\right)^3 + \cdots\right]\sigma_s(\xi)\\ &= \tan\left(\frac{\pi}{2}\frac{d}{d\xi}\right)\sigma_s(\xi). \qquad (17)\end{aligned}$$

The result (17) shows explicitly that one cannot generally make assumptions about σ_s and δ independently. If, for example, we follow Jackson[12] and assume (for $\omega > N$) that $\sigma_s(\xi) = a + b\xi + c\xi^2$, then, neglecting $T(-\infty)$, we have from (17)

$$\begin{aligned}\rho(\omega > N) &= \frac{\mathrm{Re}T(\omega)}{\mathrm{Im}T(\omega)}\\ &= \cot\delta(\omega)\\ &= \frac{\pi}{2}\frac{b + 2c\xi}{a + b\xi + c\xi^2}. \qquad (18)\end{aligned}$$

ρ becomes positive and hence $\delta(\omega) < \pi/2$ beyond a certain energy, in agreement with the general theorem of Khuri and Kinoshita.[13] Thus, if the above form for $\sigma_s(\omega > N)$ were used in (12), δ_1 would have to be $< \pi/2$.

One strategy for applying (12) is to assume a form for σ_s with parameters constrained to give, according to (17), a given value of δ_1. If this is done with $\sigma_s(\xi) = a + b\xi + c\xi^2$, we find

$$\cot\delta_1 = \pi c(4ac - b^2)^{-1/2}. \qquad (19)$$

However, if we set $q_0 = (N^2 - \mu^2)^{1/2}$, $N = 60$ GeV, and $a = \sigma(60 \text{ GeV}) = 23.7 \pm 0.4$ mb, the exact form of the PT bound gives a unique value for b. For the cases we have investigated, $0 < \cot\delta_1 < 0.2$, these values of b are all within the bounds implied by the first inequality of (12). For more general forms of $\sigma_s(\omega > 60 \text{ GeV})$, (12) should give constraints beyond those implied by the PT bounds.

When experimental data on $\sigma_s(\omega > 60 \text{ GeV})$ become available, we may use the analysis leading to (12) to obtain an upper bound on δ_1. Let β_0 (>0.28) be such that

$$I^{-1}(\beta_0)\int_{M_1}^{M} d\omega\, K(\omega, \omega_1, \beta_0)\sigma_s(\omega)/(\omega^2 - \mu^2)^{1/2} = R(\beta_0). \qquad (20)$$

Then δ_1 cannot be greater than $\pi\beta_0$. As a simple illustration of this bound, suppose that $\sigma_s(\omega > N) = 22$ mb and hence exactly satisfies the PT bound.[1] Then, from Fig. 1, we see that $\delta_1 < 0.525\pi$. [A direct evaluation of the dispersion relation (7) for this case gives $\delta(\omega > N)$ monotonically decreasing to $\pi/2$ as $\omega \to \infty$.]

In this paper, we have derived bounds on the weighted integrals of $2\sigma_s = \sigma_{\mathrm{tot}}(\pi^+ p) + \sigma_{\mathrm{tot}}(\pi^- p)$ at pion laboratory energies $\omega > 60$ GeV. These bounds are more sensitive to the very-high-energy cross-section behavior than are previously derived bounds.[1,2] Along with these other bounds they may be used directly to test the consistency of $\pi^{\pm}p$ data with forward dispersion relations and to constrain the parameters in theoretical models of high-energy scattering amplitudes.

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Possible non-Regge behavior of electroproduction structure functions*

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The large-ω behavior of deep-inelastic structure functions, e.g., $F_2(\omega, q^2)$, is studied in the framework of asymptotically free field theories. On the basis of certain uniformity assumptions we predict an unbounded growth with ω: slower than any power of ω but faster than any power of $\log\omega$.

The discovery that non-Abelian gauge theories are asymptotically free[1] has attracted a great deal of interest, especially in connection with the search for a field-theoretic explanation of Bjorken scaling. In fact, theories of this class do not quite scale, but they come close in a sense that we shall presently recall. Further development of the subject hinges on the observation of departures from scaling. Does scaling break down in the ways that are characteristic of asymptotically free theories? What is most sharply characteristic of these theories is the large-q^2 behavior of the moments of deep-inelastic structure functions. But it is also natural to consider the implications for the structure functions themselves. Discussion along these lines has been initiated in several recent publications, which deal especially with the threshold region, $\omega \geq 1$.[2] Here we want to focus on the behavior in the limit of large ω.[3]

For definiteness, let us start with the structure function $F_2(\omega, q^2)$ of deep-inelastic electron scattering, where q^2 is minus the invariant momentum transfer squared and $\omega = 2m\nu/q^2$ is the Bjorken scaling variable. The moments of the structure