PHYSICAL REVIEW D VOLUME 10, NUMBER 5 1 SEPTEMBER 1974

Two-pion correlations in the diffractive excitation model*

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(Received 14 January 1974)

The diffractive excitation model (DEM) is formulated in terms of rapidity, and used to calculate the inclusive rapidity distributions $pp \to \pi^- X$ and $pp \to \pi^-\pi^- X$. Using simplified kinematics, analytic calculations are made for these distributions and the related multiplicity moments. The two-pion correlations in the isotropic-decay model are found to be in agreement with experiment at low energies, but rise too rapidly with energy, lying an order of magnitude above Intersecting-Storage-Rings data. A modified DEM, with longitudinal decay of the diffractive cluster, accounts for the size and weak energy dependence of correlations in the diffractive part of the inelastic cross section. This model is found suitable for use as the diffractive component in a two-component model.

I. INTRODUCTION

The diffractive excitation model (DEM)[1,2] has been most successful[3] in accounting for the inclusive single-particle distributions $pp \to \pi X$ and $pp \to pX$ over a wide range of energies. Recent observations at NAL and at the CERN ISR have confirmed the existence of high-mass diffractive clusters,[4] as conjectured in the model. A further assumption, the isotropic decay of the cluster in its rest frame (the "nova" mechanism), requires testing; on the basis of angular distributions from the decay of low-mass diffractive resonances, it has not been possible to discriminate between isotropic or longitudinal phase-space decays.[5] We find that two-pion correlations are a sensitive measure of cluster decay.

We have formulated the isotropic DEM in terms of rapidity variables, and calculated the detailed rapidity spectra $pp \to \pi^- X$ and $pp \to \pi^-\pi^- X$. Simplified kinematics permit us to perform analytic calculations (Sec. III), which reveal a rapid growth of two-pion correlations with rising energy. For purposes of comparison with experiment, we display (Sec. IV) the rapidity correlations in the forms

$$N(y)=\frac{1}{\sigma}\frac{d\sigma}{dy} \text{ for } pp \to \pi^- X$$

and

$$N(y_1,y_2)=\frac{1}{\sigma}\frac{d\sigma}{dy_1 dy_2} \text{ for } pp \to \pi^-\pi^- X$$

over a range of energies ($s = 62.5$–$4000\ \text{GeV}^2$). We also calculate, at a typical ISR energy ($s = 3000\ \text{GeV}^2$), the correlation difference and ratio

$$C(y_1,y_2)=N(y_1,y_2)-N(y_1)N(y_2)$$

and

$$R(y_1,y_2)=N(y_1,y_2)/N(y_1)N(y_2)\,.$$

This maximal clustering model gives correlations in accord with experiment at BNL-CERN PS (proton synchrotron) energies,[6] but overestimates their magnitude at the ISR. By assuming longitudinal decays for the diffractive clusters, the correlations are reduced. In a simple analytic model (Sec. V), where the Pomeron-proton collision has multiperipheral properties, we find that diffractive excitation contributes to both long- and short-range correlations. Our longitudinal diffractive model is inserted into a two-component model to provide estimates of multiperipheral parameters, in agreement with other thermodynamic and two-component work.[7]

II. THE DIFFRACTIVE EXCITATION MODEL

The DEM for pp collisions may be simply formulated in terms of rapidity. The beam and target protons (a and b, with mass m) are diffractively excited into massive clusters (A and B, respectively), with masses M_A and M_B, and rapidities Y_A and Y_B. The kinematic restrictions at c.m. energy $\sqrt{s}$,

$$\begin{aligned} &M_A+M_B \le \sqrt{s}\,,\\ &M_A \ge m\,, \qquad\qquad (2.1)\\ &M_B \ge m\,, \end{aligned}$$

take the form (with $Y_A \ge 0$ and $Y_B \le 0$)

$$\begin{aligned} &\sinh Y_A+\sinh(-Y_B) \le \sinh(Y_A-Y_B)\,,\\ &\sinh(Y_A-Y_B) \le 2\cosh Y_{\max}\sinh(-Y_B)\,, \qquad (2.2)\\ &\sinh(Y_A-Y_B) \le 2\cosh Y_{\max}\sinh Y_A\,; \end{aligned}$$

here $Y_{\max}$ is the kinematic maximum c.m. rapidity for a proton, given by

$$\exp Y_{\max}=\tfrac{1}{2}[s^{1/2}+(s-4m^2)^{1/2}]. \tag{2.3}$$

Otherwise, the clusters are excited independently, so that

$$\frac{d\sigma}{dY_A dY_B}\approx\frac{1}{\sigma}\frac{d\sigma}{dY_A}\frac{d\sigma}{dY_B}. \tag{2.4}$$

Let us consider the beam cluster (rapidity $Y\geq 0$ and mass M). It has a mass excitation spectrum

$$\frac{d\sigma}{dY}=e^{Y-Y_{\max}} \text{ for } 0\leq Y\leq Y_{\max}, \tag{2.5}$$

and decays (on the average) into n_Y pions, where

$$n_Y=(M-M_T)/E, \tag{2.6}$$

$M_T=m+m_\pi$ = threshold mass for pion production, and E is the average pion energy in the cluster rest frame. Because the cluster masses (M_A and M_B) are related to the rapidities by

$$e^{Y_A}=\frac{s+M_A^2-M_B^2+\Delta^{1/2}(s,M_A^2,M_B^2)}{2\sqrt{s}M_A}$$

and

$$e^{-Y_B}=\frac{s+M_B^2-M_A^2+\Delta^{1/2}(s,M_A^2,M_B^2)}{2\sqrt{s}M_B}, \tag{2.7}$$

we have (for masses $M_A, M_B\ll\sqrt{s}$)

$$e^{Y_A}\sim\frac{\sqrt{s}}{M_A},\quad e^{-Y_B}\sim\frac{\sqrt{s}}{M_B},\quad e^{Y_{\max}}\sim\frac{\sqrt{s}}{m};$$

hence

$$n_Y\sim\frac{m}{E}\left(e^{Y_{\max}-Y}-\frac{M_T}{m}\right) \text{ for large } s.$$

We assume that pions within a cluster are emitted independently and isotropically, with decay distributions (in the cluster frame)

$$\frac{dN}{d^3p}=\frac{e^{-p^2/\lambda^2}}{\pi^{3/2}\lambda^3}, \tag{2.8}$$

so that $\langle p_\perp^2\rangle=\lambda^2$ and $E\approx(\tfrac{3}{2}\lambda^2+m_\pi^2)^{1/2}$. The pion rapidity distribution is obtained by integration over the transverse momentum,

$$\begin{aligned}D(y)\equiv\frac{dN}{dy}&=\int_{m_\perp=m_\pi}^{\infty}\frac{dm_\perp^2}{\lambda^3\pi^{1/2}}m_\perp\cosh y\exp\left(\frac{m_\perp^2}{\lambda^2}\cosh^2 y-\frac{m_\pi^2}{\lambda^2}\right)\\&=\frac{1}{2\cosh^2 y}e^{m_\pi^2/\lambda^2}\left[1-\operatorname{erf}\left(\frac{m_\pi}{\lambda}\cosh y\right)+\frac{2}{\sqrt{\pi}}\frac{m_\pi}{\lambda}\cosh y\exp\left(-\frac{m_\pi^2}{\lambda^2}\cosh^2 y\right)\right],\end{aligned} \tag{2.9}$$

where

$$\operatorname{erf}(x)\equiv\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du.$$

In our calculations we take $E=0.4$ GeV, and so $\lambda=0.306$ GeV; the decay distribution then has rapidity width ≈ 2, and is well approximated by the $m_\pi=0$ distribution

$$D(y)\approx\frac{1}{2\cosh^2 y}. \tag{2.10}$$

For a given number π of pions resulting from the decay of a diffractively excited proton, we determine the number of charged pions $\pi^\pm$ and negative pions π^-. Neglecting all particles other than pions and nucleons, charge conservation gives

$$Q=1=p+\pi^+-\pi^-,$$
$$\pi=\pi^++\pi^-+\pi^0.$$

Assuming $\pi^-=\tfrac{1}{3}$(number of produced pions) $=\tfrac{1}{3}(2\pi^-+\pi^0)$ gives

$$\pi^-=\pi^0,$$

and taking $p=\tfrac{3}{4}$, we finally obtain

$$\begin{aligned}\pi^-&=\tfrac{1}{3}(\pi-\tfrac{1}{4}),\\\pi^\pm&=\tfrac{1}{3}(2\pi+\tfrac{1}{4}).\end{aligned} \tag{2.11}$$

In terms of the cluster excitation spectrum and the decay distribution D, we may write the inclusive one- and two-pion spectra

$$\begin{aligned}\frac{d\sigma}{dy}=&\int_0^{Y_{\max}}dY\frac{d\sigma}{dY}n_Y D(Y-y)\\&+\int_{-Y_{\max}}^0 dY\frac{d\sigma}{dY}n_Y D(Y-y)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}\frac{d\sigma}{dy\,dy'}=&\int_0^{Y_{\max}}dY\frac{d\sigma}{dY}n_Y(n_Y-1)D(Y-y)D(Y-y')+\int_{-Y_{\max}}^0 dY\frac{d\sigma}{dY}n_Y(n_Y-1)D(Y-y)D(Y-y')\\&+\frac{1}{\sigma}\int_0^{Y_{\max}}dY\frac{d\sigma}{dY}n_Y D(Y-y)\int_{-Y_{\max}}^0 dY\frac{d\sigma}{dY}n_Y D(Y-y')+\frac{1}{\sigma}\int_{-Y_{\max}}^0 dY\frac{d\sigma}{dY}n_Y D(Y-y)\int_0^{Y_{\max}}dY\frac{d\sigma}{dY}n_Y D(Y-y'),\end{aligned} \tag{2.13}$$

where we have assumed that two pions from the same cluster are uncorrelated. The diffractive inelastic cross section (normalized so that $\sigma \to 1$ as $s \to \infty$) is

$$\sigma = \int_0^{Y_{\max}} dY \frac{d\sigma}{dY} = \int_0^{Y_{\max}} dY e^{Y - Y_{\max}} = 1 - e^{-Y_{\max}}. \tag{2.14}$$

Various moments of the multiplicity distribution are given by integrating the rapidity distributions:

$$\int_{-\infty}^{\infty} \frac{1}{\sigma} \frac{d\sigma}{dy} dy = \langle n_A \rangle + \langle n_B \rangle = \langle n \rangle, \tag{2.15a}$$

$$\begin{aligned} \int\int_{-\infty}^{\infty} \frac{1}{\sigma} \frac{d\sigma}{dy dy'} dy dy' &= \langle n_A (n_A - 1)\rangle + \langle n_B (n_B - 1)\rangle \\ &\quad + \langle n_A n_B \rangle + \langle n_B n_A \rangle \\ &= \langle n(n-1)\rangle, \end{aligned} \tag{2.15b}$$

where n_A (n_B) denotes the number of pions in cluster A (B).

III. π-π CORRELATIONS IN A SIMPLE MODEL

In order to appreciate detailed numerical calculations of inclusive pion correlations, we study a simple model, defined by cluster multiplicity

$$n_Y = \frac{m}{E}(e^{Y_{\max} - Y} - 1) \tag{3.1}$$

and decay distribution

$$D(y) = \frac{1}{2\cosh^2 y}. \tag{3.2}$$

Then the inclusive pion distribution is given by

$$\begin{aligned} \frac{d\sigma}{dy} &= \frac{m}{E} \int_{Y=0}^{Y_{\max}} dY e^{Y - Y_{\max}} \, (-1 + e^{Y_{\max} - Y}) \left[\frac{1}{2\cosh^2(Y-y)} + \frac{1}{2\cosh^2(Y+y)}\right] \\ &= \frac{m}{E}[1 - e^{-Y_{\max}} + e^{y - Y_{\max}} \arctan(e^{-Y}) - e^{y - Y_{\max}} \arctan(e^{Y_{\max} - y}) \\ &\qquad + e^{-y - Y_{\max}} \arctan(e^{y}) - e^{-y - Y_{\max}} \arctan(e^{Y_{\max} + y})]. \end{aligned} \tag{3.3}$$

At high energies, this distribution has the behavior

$$\frac{d\sigma}{dy} \sim \frac{m}{E}\{1 + e^{-Y_{\max}}[e^{y} \arctan(e^{-y}) + e^{-y} \arctan(e^{y}) - 1 - \pi \cosh y] + O(e^{-2Y_{\max}})\} \tag{3.3$'$}$$

of a "plateau" in rapidity.

The distribution of two pions, both from cluster A, is given by

$$\left(\frac{d\sigma}{dy dy'}\right)_{AA} = \left(\frac{m}{E}\right)^2 \int_0^{Y_{\max}} dy \frac{e^{Y - Y_{\max}} (-1 + e^{Y_{\max} - Y})^2}{2\cosh^2(Y-y) 2\cosh^2(Y-y')}, \tag{3.4}$$

where we assume $\langle n_Y(n_Y - 1)\rangle = \langle n_Y \rangle^2$, as for a Poisson distribution. The substitution $\omega = e^{-Y}$ reveals the asymptotic behavior more clearly:

$$\begin{aligned} \left(\frac{d\sigma}{dy dy'}\right)_{AA} &= 4\left(\frac{m}{E}\right)^2 e^{Y_{\max}} \int_{\omega = e^{-Y_{\max}}}^{1} d\omega \frac{\omega^2 d\omega (\omega - e^{-Y_{\max}})^2}{[\omega^4 + 2\omega^2 \cosh(y-y') + 1]^2} \\ &\underset{s\to\infty}{\simeq} 4\left(\frac{m}{E}\right)^2 \left\{ e^{Y_{\max}} \int_{\omega=0}^{1} \frac{\omega^4 d\omega}{[\omega^4 + 2\omega^2 \cosh(y-y') + 1]^2} - 2\int_{\omega=0}^{1} \frac{\omega^3 d\omega}{[\omega^4 + 2\omega^2 \cosh(y-y') + 1]^2} + O(e^{-Y_{\max}}) \right\}. \end{aligned} \tag{3.4$'$}$$

This distribution is a function of the rapidity difference $|y - y'|$ and not of y and y' separately; at high energies $(d\sigma)_{AA} \propto \sqrt{s}$, resulting in a rapid energy rise of the two-pion correlation. A similar contribution arises from two pions in cluster B:

$$\left(\frac{d\sigma}{dy dy'}\right)_{BB} = \left(\frac{d\sigma}{d(-y)d(-y')}\right)_{AA} = \left(\frac{d\sigma}{dy dy'}\right)_{AA}. \tag{3.5}$$

The distribution of two pions, one from cluster A and the other from cluster B, is

$$\begin{aligned} \left(\frac{d\sigma}{dy dy'}\right)_{AB} &= \frac{1}{\sigma}\left[\frac{m}{E}\int_0^{Y_{\max}} dY \frac{e^{Y - Y_{\max}}(e^{Y_{\max} - Y} - 1)}{2\cosh^2(Y-y)}\right] \\ &\quad \times \left[\frac{m}{E}\int_0^{Y_{\max}} dY \frac{e^{Y - Y_{\max}}(e^{Y_{\max} - Y} - 1)}{2\cosh^2(Y+y')}\right] \\ &= \frac{1}{\sigma}\left(\frac{d\sigma}{dy}\right)_A \left(\frac{d\sigma}{dy'}\right)_B, \end{aligned} \tag{3.6}$$

where

$$\left(\frac{d\sigma}{dy}\right)_A = \frac{m}{E}\left[\frac{1-e^{-Y_{\max}}}{1+e^{-2y}} + e^{y-Y_{\max}}\arctan(e^{-y}) - e^{y-Y_{\max}}\arctan(e^{Y_{\max}-y})\right], \tag{3.7}$$

and

$$\left(\frac{d\sigma}{dy}\right)_B = \left(\frac{d\sigma}{d(-y)}\right)_A . \tag{3.8}$$

This term depends on y and y' independently, as does the other cross term

$$\left(\frac{d\sigma}{dydy'}\right)_{BA} = \frac{1}{\sigma}\left(\frac{d\sigma}{dy}\right)_B \left(\frac{d\sigma}{dy'}\right)_A . \tag{3.9}$$

The total two-pion distribution is the sum

$$\frac{d\sigma}{dydy'} = \left(\frac{d\sigma}{dydy'}\right)_{AA} + \left(\frac{d\sigma}{dydy'}\right)_{BB} + \left(\frac{d\sigma}{dydy'}\right)_{AB} + \left(\frac{d\sigma}{dydy'}\right)_{BA} . \tag{3.10}$$

By integrating the single-particle distribution [Eq. (3.7)] we obtain the average pion multiplicity from cluster A:

$$\begin{aligned}\langle n_A\rangle &= \int_{-\infty}^{\infty}\frac{1}{\sigma}\left(\frac{d\sigma}{dy}\right)_A dy \\ &= \frac{1}{\sigma}\frac{m}{E}\int_0^{Y_{\max}} dY e^{Y-Y_{\max}}(e^{Y_{\max}-Y}-1) \\ &= \frac{m}{E}\frac{1}{\sigma}(Y_{\max}-1+e^{-Y_{\max}}) \\ &= \frac{m}{E}\left(\frac{Y_{\max}}{1-e^{-Y_{\max}}}-1\right) .\end{aligned} \tag{3.11}$$

Similarly, from Eq. (3.4) we obtain the second multiplicity moment

$$\begin{aligned}\langle n_A(n_A-1)\rangle &= \int\int_{-\infty}^{\infty}\frac{1}{\sigma}\frac{d\sigma}{dydy'}dydy' \\ &= \left(\frac{m}{E}\right)^2\frac{1}{\sigma}\int_0^{Y_{\max}} dY(e^{Y_{\max}-Y}-2+e^{Y-Y_{\max}}) \\ &= \left(\frac{m}{E}\right)^2\frac{1}{\sigma}(e^{Y_{\max}}-1-2Y_{\max}+1-e^{-Y_{\max}}) \\ &= \left(\frac{m}{E}\right)^2\left(e^{Y_{\max}}-\frac{2Y_{\max}}{1-e^{-Y_{\max}}}+1\right) .\end{aligned} \tag{3.12}$$

Since $\langle n_A n_B\rangle = \langle n_A\rangle\langle n_B\rangle$ and $n = n_A + n_B$, we obtain the over-all pion multiplicity averages

$$\begin{aligned}\langle n\rangle &= \frac{2m}{E}\left(\frac{Y_{\max}}{1-e^{-Y_{\max}}}-1\right) \sim \frac{m}{E}\ln s , \\ \langle n(n-1)\rangle &= 2\left(\frac{m}{E}\right)^2\left(e^{Y_{\max}}-\frac{2Y_{\max}}{1-e^{-Y_{\max}}}+1\right) + 2\left(\frac{m}{E}\right)^2\left(\frac{Y_{\max}}{1-e^{-Y_{\max}}}-1\right)^2 \\ &\sim 2\left(\frac{m}{E}\right)^2\frac{\sqrt{s}}{m} .\end{aligned} \tag{3.13}$$

We note that the source of the rapid energy dependence of the two-pion distribution arises from the excitation of large cluster masses. *All* pions from a cluster have rapidity near that of the cluster ($|y-Y|\lesssim 1$), independent of cluster mass; however, the number of pions $n_Y \propto M-m$ grows with cluster mass, $M\sim m e^{Y_{\max}-Y}$. Thus pions from more massive clusters have a greater density in rapidity space. This is in contrast to short-range-order models (SROM), where clusters have fixed masses and multiplicities, so that the rapidity density of pions is determined by the (uniform) rapidity density of clusters. Thus, apart from resonant effects between pions, correlations in the DEM and SROM arise from different sources. In the DEM, correlations arise from pions within a cluster; in the SROM, correlations occur between clusters.[8]

IV. DEM PREDICTIONS FOR π-π CORRELATIONS

In order to facilitate comparisons of the DEM with experiment, we have calculated normalized distributions

$$N(y) = \frac{1}{\sigma}\frac{d\sigma}{dy}, \tag{4.1}$$

$$N(y_1, y_2) = \frac{1}{\sigma}\frac{d\sigma}{dy_1 dy_2} \tag{4.2}$$

for $pp\to\pi^- X$ and $pp\to\pi^-\pi^- X$ over a range of energies from $s = 62.5$ GeV^2 (BNL and CERN PS) to $s = 4000$ GeV^2 (top CERN ISR). In Fig. 1, we plot the single π^- distribution; it is in good agreement with bubble chamber data[9] from 12 to 29 GeV/c, NAL data[10] from 100 to 300 GeV/c, and ISR counter data,[11] both in shape and normalization. On the other hand, the two-pion spectrum of Fig. 2, $N(y_1, 0)$, agrees with experiment only at low energies; the DEM predicts much too rapid a rise of the spectrum with increased s.

In short-range-order models, a convenient measure of correlation is either the difference

$$C(y_1, y_2) = N(y_1, y_2) - N(y_1)N(y_2) \tag{4.3}$$

or the ratio

$$R(y_1, y_2) = N(y_1, y_2)/N(y_1)N(y_2) . \tag{4.4}$$

The energy dependence of the spectra for pions in the central region ($y_1 = y_2 = 0$) is shown in Fig. 3; experimentally,[10,11] however, $R(0,0)\approx 1.4$–1.6 from $s = 400$ to 2500 GeV^2, independent of energy and a full order of magnitude below the DEM prediction. In Sec. V, we suggest a modified DEM to remedy the situation.

The DEM prediction for the $\pi^-\pi^-$ rapidity correlation at $s = 3000$ GeV^2 is shown in Figs. 4 (a)–4(d). The spectra were calculated as in Sec. II,

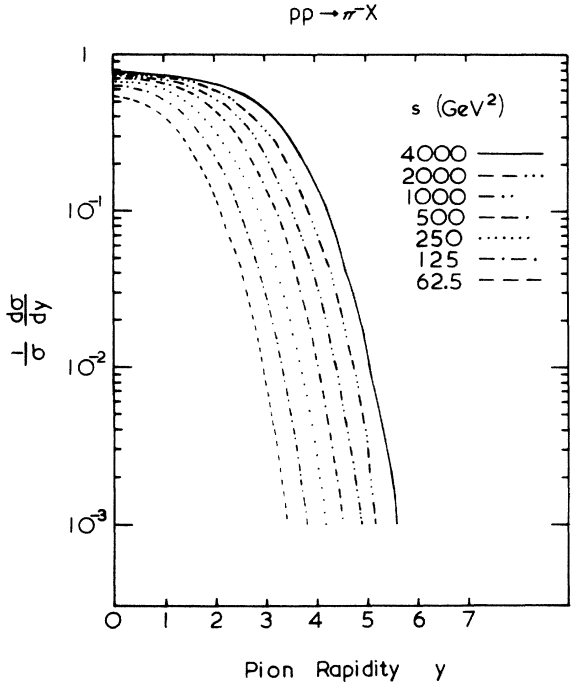

FIG. 1. The DEM prediction for the normalized inclusive rapidity distribution $N(y) = (1/\sigma)d\sigma/dy$ in $pp \rightarrow \pi^- X$ at energies $s = 62.5$–4000 GeV^2.

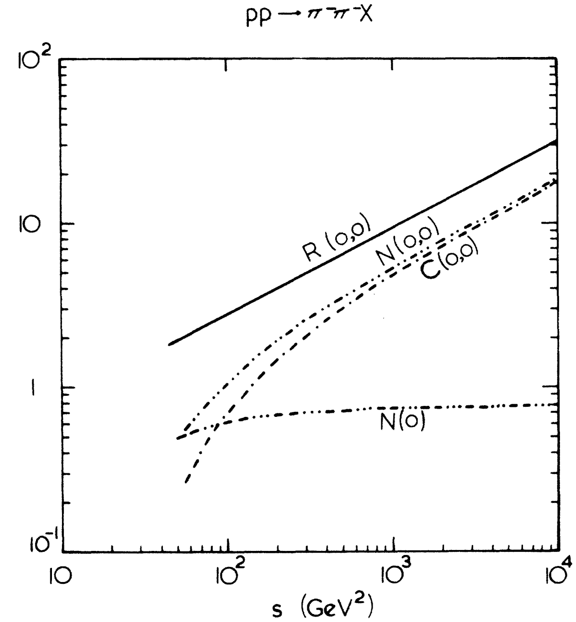

FIG. 3. The energy dependences, for inclusive π^- production at rapidity $y = 0$, of $N(0) = (1/\sigma)(d\sigma/dy)_{y=0}$, $N(0, 0) = (1/\sigma)(d\sigma/dy_1dy_2)_{y_1=y_2=0}$, the correlation difference $C(0, 0) = N(0, 0) - \{N(0)\}^2$, and the correlation ratio $R(0, 0) = N(0, 0)/\{N(0)\}^2$.

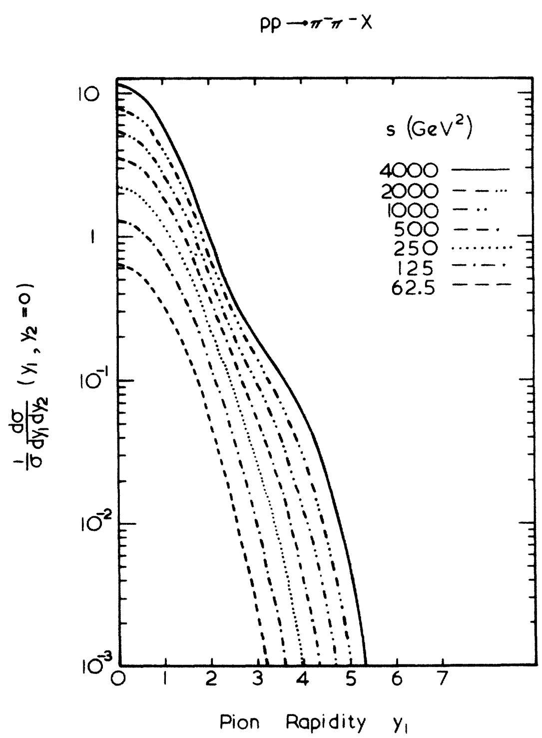

FIG. 2. The DEM two-pion rapidity distribution $pp \rightarrow \pi^-\pi^- X$, $N(y_1, 0) = (1/\sigma)(d\sigma/dy_1dy_2)_{y_2=0}$ at energies $s = 62.5$–4000 GeV^2.

and, in addition, were cut off at the kinematic boundaries by demanding physical values for the missing mass; thus, the shapes of these spectra near the boundaries are not reliable, omitting threshold effect. The main characteristics of the spectra are

(i) a large positive correlation at $y_1 = y_2$, falling off rapidly with a correlation length ≈ 2 and becoming small and negative for large rapidity separations, and

(ii) a drop in the correlation of the $y_1 = y_2$ peak as the rapidity increases into the fragmentation region.

These features are also present in ISR data on charge-charge and charged-γ correlations; the additional long-range positive correlation observed by the Pisa–Stony Brook group[11] is a correlation between the leading proton and a pion in the recoil diffractive cluster.[12] Thus the DEM offers a qualitative explanation of the shape of $\pi\pi$ correlations at the ISR, but, being a maximal-clustering model, overestimates their size.

There are additional difficulties in using the isotropic DEM to explain the full inelastic cross section. The predicted multiplicity distribution $\sigma_k/\sigma \propto k^{-2}$ falls off much less rapidly than the observed distribution. Further, the isotropic DEM permits excitations with mass $M \leqslant \sqrt{s} - m$, whereas

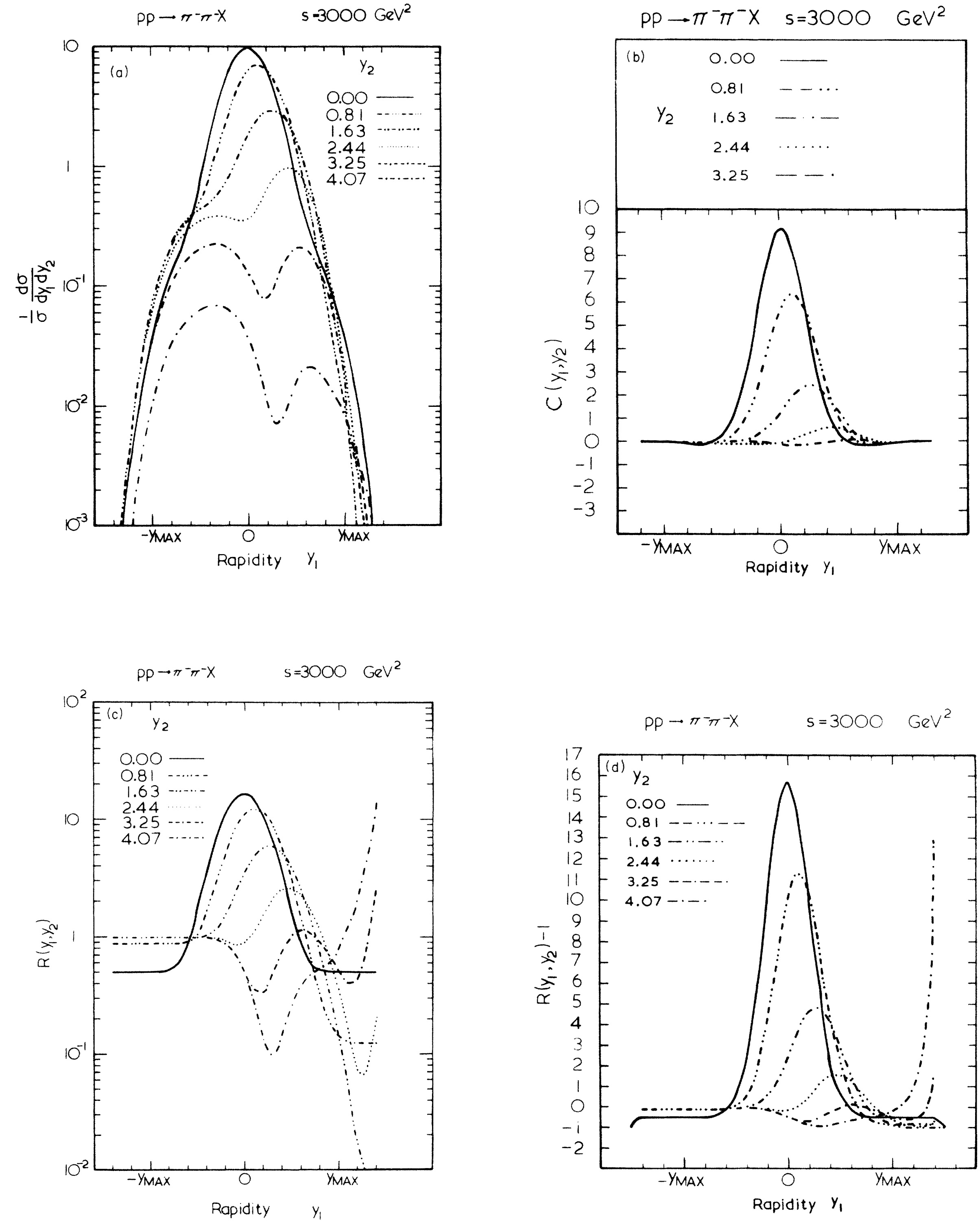

FIG. 4. The DEM prediction for the inclusive rapidity distribution $pp \to \pi^- \pi^- X$ at $s = 3000$ GeV^2 ($Y_{\max} = 4.07$), plotted with the rapidity of the second pion fixed at $y_2 = (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1) Y_{\max}$. (a) Plot of $N(y_1, y_2) = (1/\sigma)\, d\sigma/dy_1 dy_2$ vs y_1. (b) Plot of the correlation difference $C(y_1, y_2) = N(y_1, y_2) - N(y_1)\, N(y_2)$, whose integral gives the Mueller correlation parameter $f_2^- = \int C(y_1, y_2) dy_1 dy_2$. (c) Plot of the correlation ratio $R(y_1, y_2)$. (d) Plot of the correlation ratio $R(y_1, y_2) - 1$, on a linear scale.

the diffractive component is seen in the inclusive proton spectrum $pp \to pX$ only for $|x| \gtrsim 0.9$, implying a mass cutoff at $M^2 \lesssim 0.1\,(\sqrt{s} - m)^2$. Both difficulties may be accommodated by using a momentum-transfer cutoff to describe the coherent diffractive scattering.[3,13] The mass spectrum then becomes

$$\frac{d\sigma}{dM} \propto \frac{e^{Bt_{\min}}}{M^2},$$

where

$t_{\min}$ = the momentum transfer for forward scattering

$$\approx \frac{-m^2\left(\dfrac{M^2 - m^2}{s}\right)^2}{\left(1 - \dfrac{M^2 - m^2}{s}\right)},$$

and B is the diffraction slope (which falls rapidly with rising mass). This mass cutoff suppresses pion production in the central region ($y \approx 0$), so that the isotropic DEM alone cannot explain the pion spectra.[13] Finally, estimates of the diffractive contribution to the inelastic cross section, based on either inclusive proton spectra or NAL multiplicity distributions, assign only 5–8 mb to this process.[4,10]

With these caveats in mind, we now try to construct a diffractive model which must be used in conjunction with a multiperipheral or short-range-order model to explain the entirety of inclusive single- and two-pion spectra.

V. A MODIFIED DIFFRACTIVE MODEL

Recent experimental observations of the inclusive proton spectrum $pp \to pX$ at NAL and the CERN ISR place stringent constraints on diffractive excitations. These data[4,10] suggest that

(a) the invariant proton distribution scales for $|x| > 0.9$ and exhibits a diffractive peak,

(b) the cross section for diffraction excitation is 5–8 mb, and

(c) the average charged-particle multiplicity in a cluster of large mass M grows as $\ln M$ or, in terms of cluster rapidity Y, $n_Y \propto Y_{\max} - Y$.

These observations suggest that the Pomeron-proton collision behaves like a typical two-hadron collision, exciting nonresonant (that is, weakly clustered) states, which contribute to the triple-Pomeron coupling. We expect the decay pions to be aligned longitudinally along the Pp collision axis in the rest frame of the diffractive proton cluster.[14] The resulting angular distribution becomes increasingly anisotropic with increasing cluster mass; this might be associated with the increased spin of the diffractive cluster. [The diffractive spectrum contains ≈3 mb of $N^*(1470)$, $N^*(1520)$, $N^*(1688)$, and $N^*(2190)$ in the spin-parity sequence $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \frac{7}{2}^-$, respectively.]

In triple-Reggeon language, we associate high-mass diffractive excitation with the triple-Pomeron (PPP) coupling (representing diffractive excitation of nonresonant background), rather than with PPR (the diffractive excitation of resonances). The nonscaling mass spectrum of the isotropic DEM is then replaced by the scaling spectrum[12]: $d\sigma/dM \propto M^{-1}$. Also, we expect $n \propto \ln M$ pions distributed uniformly in rapidity. Expressing our modified model in terms of the cluster rapidity Y, we obtain a cluster rapidity distribution

$$\frac{d\sigma}{dY} = \sigma_0 \text{ for } 0 \le Y \le Y_{\max}, \tag{5.1}$$

a decay multiplicity

$$n_Y = \rho(Y_{\max} - Y), \tag{5.2}$$

and a decay distribution

$$D(Y, y) = \begin{cases} \dfrac{1}{2(Y_{\max} - Y)} \text{ for } |y - Y| \le Y_{\max} - Y, \\ 0 \text{ otherwise}, \end{cases} \tag{5.3}$$

where σ_0 and ρ are constants. Thus, our longitudinal cluster contains fewer pions, and these pions wander much further from the central cluster rapidity than those in the isotropic DEM. (Here $|y - Y| \le Y_{\max} - Y$, compared with $|y - Y| \lesssim 1$ for isotropic decay.) Both effects reduce the size of two-pion correlations.

An observant reader will realize that our longitudinal-decay diffractive model bears a distinct resemblance to rapidity formulations of the multiperipheral model.[15] The MPM can be identified with the RRP term of the recoil proton spectrum, and so has cluster rapidity distribution

$$\frac{d\sigma}{dY} = \sigma_1 e^{-2Y} \text{ for } 0 \le Y \le Y_{\max} \tag{5.1$'$}$$

instead of (5.1), favoring more production of pions in the central region, and a smaller rapidity gap between the recoil proton and the decay pions from the cluster. Unlike the isotropic DEM, however, the longitudinal DEM need not produce a noticeable gap between the recoil (leading) proton and the cluster pions in most events.

Confining our attention to single-cluster excitations in the longitudinal DEM, we obtain a total diffractive cross section $\sigma = 2\sigma_0 Y_{\max}$ and pion multiplicity moments

$$\begin{aligned} \langle n \rangle &= \tfrac{1}{2}\rho Y_{\max}, \\ \langle n(n-1) \rangle &= \tfrac{1}{3}\rho^2 Y_{\max}{}^2. \end{aligned} \tag{5.4}$$

It is important to note that the average pion multiplicity of the longitudinal DEM is asymptotically one-half that of the comparable MPM,[16] defined by Eqs. (5.1′), (5.2), and (5.3); this is in accord with experimental observations that diffractive mechanisms contribute chiefly to low-multiplicity final states.

By making an additional assumption on the multiplicity distributions $P_k(Y)$ from a cluster of rapidity Y, we may also calculate the over-all multiplicity distribution σ_k/σ. Taking for the cluster multiplicity distribution the Poisson form

$$P_k(Y)=e^{-n_Y}\frac{(n_Y)^k}{k!}, \tag{5.5}$$

we first determine the multiplicity distribution from a forward cluster:

$$\begin{aligned} Q_k &= \int_0^{Y_{\max}} dY\frac{d\sigma}{dY}P_k(Y)\bigg/\int_0^{Y_{\max}} dY\frac{d\sigma}{dY} \\ &= \frac{1}{Y_{\max}}\int_0^{Y_{\max}} dY\frac{[\rho(Y_{\max}-Y)]^k}{k!} \\ &\qquad\qquad \times\exp[-\rho(Y_{\max}-Y)] \\ &= \frac{1}{\rho Y_{\max}}\left[1-e^{-\rho Y_{\max}}\sum_{l=0}^{k}\frac{(\rho Y_{\max})^l}{l!}\right] \\ &\sim \frac{1}{\rho Y_{\max}} \text{ as } Y_{\max}\to\infty \text{ for finite } k. \end{aligned} \tag{5.6}$$

Thus, the single-cluster multiplicity distribution is essentially flat for low multiplicities at asymptotic energies. This is also the total multiplicity distribution since, upon averaging over both clusters,

$$\frac{\sigma_k}{\sigma}=\frac{\sigma_A}{\sigma}Q_k+\frac{\sigma_B}{\sigma}Q_k=Q_k.$$

Returning to the calculation of inclusive spectra, we begin with the pion rapidity distribution from the forward cluster (A):

$$\begin{aligned}\left(\frac{d\sigma}{dy}\right)_A &= \int_{Y=0}^{Y_{\max}} \tfrac{1}{2}\sigma_0\rho\theta(Y_{\max}-y)\theta(\tfrac{1}{2}Y_{\max}+\tfrac{1}{2}y-Y)dY \\ &= \begin{cases}\tfrac{1}{4}\sigma_0\rho(Y_{\max}+y) & \text{for } |y|\le Y_{\max}, \\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{5.7}$$

and so

$$\left(\frac{d\sigma}{dy}\right)_B = \begin{cases}\tfrac{1}{4}\sigma_0\rho(Y_{\max}-y) & \text{for } |y|\le Y_{\max}, \\ 0 & \text{otherwise.}\end{cases} \tag{5.8}$$

The total pion rapidity spectrum is then

$$\frac{1}{\sigma}\frac{d\sigma}{dy}=\tfrac{1}{4}\rho \text{ for } |y|\le Y_{\max}. \tag{5.9}$$

The spectrum of two pions, both from cluster A, is given by

$$\begin{aligned}\frac{1}{\sigma}\left(\frac{d\sigma}{dy_1dy_2}\right)_{AA} &= \frac{1}{\sigma}\int_0^{Y_{\max}} dY\frac{d\sigma}{dY}\langle n_Y(n_Y-1)\rangle D(Y;y_1)D(Y;y_2) \\ &= \frac{1}{16}\frac{\rho^2}{Y_{\max}}[Y_{\max}+\min(y_1,y_2)] \text{ if } |y_1|\le Y_{\max} \text{ and } |y_2|\le Y_{\max}. \end{aligned} \tag{5.10}$$

By symmetry,

$$\frac{1}{\sigma}\left(\frac{d\sigma}{dy_1dy_2}\right)_{BB}=\frac{1}{16}\frac{\rho^2}{Y_{\max}}[Y_{\max}-\max(y_1,y_2)], \tag{5.11}$$

so that the total two-pion rapidity spectrum is

$$\begin{aligned}\frac{1}{\sigma}\frac{d\sigma}{dy_1dy_2} &= \tfrac{1}{8}\rho^2\left(1-\frac{|y_1-y_2|}{2Y_{\max}}\right) \\ &\sim \tfrac{1}{8}\rho^2 \text{ as } s\to\infty. \end{aligned} \tag{5.12}$$

Consequently, this model contributes to long-range correlations, giving the correlation ratio

$$R(y_1,y_2)=2-\frac{|y_1-y_2|}{Y_{\max}}. \tag{5.13}$$

Our rather idealized model, which assumes complete independence of the pions in a cluster, is easily corrected for the presence of short-range correlations.[7] We simply identify the normalized one- and two-pion distributions within the cluster with those of the equivalent MPM:

$$\langle n_Y\rangle D(Y;y)\equiv N_{Y_{\max}}^{\mathrm{MPM}}(y-Y), \tag{5.14}$$

$$\langle n_Y(n_Y-1)\rangle D(Y;y_1,y_2)\equiv N_{Y_{\max}}^{\mathrm{MPM}}(y_1-Y,y_2-Y). \tag{5.15}$$

The subscript on the normalized MPM distribution is the maximum rapidity for a pion in the rest frame of the multiperipheral cluster; for a cluster with c.m. rapidity Y, this is, of course, $Y_{\max}-Y$. For an MPM with independent emission of β clusters per unit rapidity, each decaying isotropically into ν particles (as in the isotropic DEM), we have

$$N_{Y_{\max}}^{\mathrm{MPM}}(y)\equiv\beta\langle\nu\rangle \text{ for } |y|\le Y_{\max}, \tag{5.16}$$

$$\begin{aligned}C_{Y_{\max}}^{\mathrm{MPM}}(y_1,y_2) &\equiv -N_{Y_{\max}}^{\mathrm{MPM}}(y_1)N_{Y_{\max}}^{\mathrm{MPM}}(y_2)+N_{Y_{\max}}^{\mathrm{MPM}}(y_1,y_2) \\ &\equiv \beta\langle\nu(\nu-1)\rangle I_{Y_{\max}}(y_1,y_2), \end{aligned} \tag{5.17}$$

where the short-range correlations are contained in the integral

$$I_{Y_{\max}}(y_1, y_2) = \int_{-Y_{\max}}^{Y_{\max}} \frac{dY}{2\cosh^2(Y-y_1)2\cosh^2(Y-y_2)}$$
$$\sim \frac{(y_1-y_2)\cosh(y_1-y_2)-\sinh(y_1-y_2)}{\sinh^3(y_1-y_2)} \quad \text{as } s\to\infty. \quad (5.18)$$

For use in two-component models, it is useful to note that

$$I_\infty(0,0)=\tfrac{1}{3}.$$

We may relate the parameters of the MPM to those of the DEM by identifying $\rho \equiv 2\beta\langle\nu\rangle$.

The single-particle spectrum (5.9) is not altered, but the two-pion spectrum (5.12) acquires an additional short-range term, becoming

$$N^D_{Y_{\max}}(y_1,y_2) = \frac{1}{\sigma}\frac{d\sigma}{dy_1 dy_2} = \frac{1}{2}\left(1-\frac{|y_1-y_2|}{2Y_{\max}}\right)[\beta^2\langle\nu\rangle^2 + C^{\rm MPM}_{Y_{\max}}(y_1,y_2)]. \quad (5.19)$$

The corresponding single-particle spectrum being

$$N^D_{Y_{\max}}(y) = \tfrac{1}{2}\beta\langle\nu\rangle \text{ for } |y|\le Y_{\max}, \quad (5.20)$$

we find that the correlation difference for the DEM is

$$C^D(y_1,y_2) = \tfrac{1}{4}\beta^2\langle\nu\rangle^2\left(1-\frac{|y_1-y_2|}{Y_{\max}}\right) + C^{\rm MPM}_{Y_{\max}}(y_1,y_2)\frac{1}{2}\left(1-\frac{|y_1-y_2|}{2Y_{\max}}\right). \quad (5.21)$$

The diffractive model thus contributes to both the long-range and short-range correlations. This is also seen from the multiplicity moments

$$\langle n\rangle_D = \beta\langle\nu\rangle Y_{\max},$$
$$\langle n(n-1)\rangle_D = \beta\langle\nu(\nu-1)\rangle Y_{\max} + \tfrac{4}{3}\beta^2\langle\nu\rangle^2 Y_{\max}, \quad (5.22)$$

where we have used the MPM results for the diffractive cluster

$$\langle n_Y\rangle_{\rm MPM} = 2\beta\langle\nu\rangle(Y_{\max}-Y),$$
$$\langle n_Y(n_Y-1)\rangle_{\rm MPM} - \langle n_Y\rangle_{\rm MPM}{}^2 = 2\beta\langle\nu(\nu-1)\rangle(Y_{\max}-Y) \quad (5.23)$$

obtained by integrating Eqs. (5.16) and (5.17).

We now compute the two-pion correlations in the central region in the two-component model. Our longitudinal DEM gives

$$N^D(0) = \tfrac{1}{2}\beta\langle\nu\rangle,$$
$$C^D(0,0) = \tfrac{1}{4}\beta^2\langle\nu\rangle^2 + \tfrac{1}{2}C^{\rm MPM}_{Y_{\max}}(0,0) \approx \tfrac{1}{4}\beta^2\langle\nu\rangle^2 + \tfrac{1}{6}\beta\langle\nu(\nu-1)\rangle, \quad (5.24)$$

and the MPM gives

$$N^{\rm MPM}(0) = \beta\langle\nu\rangle,$$
$$C^{\rm MPM}(0,0) \approx \tfrac{1}{3}\beta\langle\nu(\nu-1)\rangle. \quad (5.25)$$

Setting α_D and $\alpha_{\rm MPM}$ as the fractions of the inelastic cross section coming from the diffractive and multiperipheral components, we obtain

$$N(0) = \alpha_D N^D(0) + \alpha_{\rm MPM}N^{\rm MPM}(0) = \beta\langle\nu\rangle(\alpha_{\rm MPM}+\tfrac{1}{2}\alpha_D), \quad (5.26)$$

$$C(0,0) = \alpha_D C^D(0,0) + \alpha_{\rm MPM}C^{\rm MPM}(0,0) + \alpha_D\alpha_{\rm MPM}[N^{\rm MPM}(0)-N^D(0)]^2$$
$$= \tfrac{1}{4}\alpha_D(1+\alpha_{\rm MPM})\beta^2\langle\nu\rangle^2 + \tfrac{1}{3}\beta\langle\nu(\nu-1)\rangle(\alpha_{\rm MPM}+\tfrac{1}{2}\alpha_D) \quad (5.27)$$

for the total pion distributions.

These equations form the basis of a phenomenology for analyzing pion-pion correlations, based on measurements of $R(0,0)$, $N(0)$, and α_D. Experimental data obtained at NAL and the CERN ISR for charged-particle correlations[10,11] give the value

$$R(0,0) = 1.6\text{–}1.75.$$

So, using the estimates $R(0,0)=1.65$, $N(0)=1.7$, and $\alpha_D = 0.14$, we calculate for $s\approx 3000$ GeV^2 that the average number of charged pions per (isotropic) cluster is

$$\frac{\langle\nu(\nu-1)\rangle}{\langle\nu\rangle}\approx 2.9$$

and the number of isotropic clusters per unit rapidity is

$$\beta\langle\nu\rangle^2/\langle\nu(\nu-1)\rangle \approx 0.6,$$

results in reasonable agreement with other forms of the two-component model.[17]

Our longitudinal DEM is thus a suitable candidate for the diffractive component of a two-component model,[7] contributing directly to both long- and short-range correlations and, by "interference" with the multiperipheral component, indirectly to long-range correlations. We also predict that all inclusive cross sections contain a diffractive term rising as $\ln s$, consistent with recent ISR measurements, which may be parametrized[17] by

$$\sigma_{\rm total} = 22.4 + 2.4\ln s + \frac{53.78}{\sqrt{s}} - \frac{34.67}{s}\ \text{mb}, \quad (5.28)$$

with s in GeV^2. We conclude that a longitudinal decay for diffractive clusters, rather than the isotropic decay originally conjectured, can account for the limited size and weak energy dependence of observed $\pi\pi$ correlations.

*Work supported by the United Kingdom Science Research Council.

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$$\sigma = \sigma_1[1 - \exp(-2Y_{\max})],$$

$$\langle n\rangle = \rho\left(\frac{Y}{1-\exp(-2Y_{\max})} - \frac{1}{2}\right) \sim \rho\,(Y_{\max} - \tfrac{1}{2}),$$

$$\langle n(n-1)\rangle = \rho^2\left(\frac{Y_{\max}{}^2 - Y_{\max}}{1-\exp(-2Y_{\max})} + \frac{1}{2}\right)$$

$$\sim \rho^2(Y_{\max}{}^2 - Y_{\max} + \tfrac{1}{2}),$$

$$\left(\frac{d\sigma}{dy}\right)_A = \tfrac{1}{4}\sigma_1\rho[1-\exp(-y-Y_{\max})] \quad \text{for } |y| \le Y_{\max},$$

and

$$N(y) = \tfrac{1}{2}\rho\,\frac{1-\exp(-Y_{\max})\cosh y}{1-\exp(-2Y_{\max})}\;.$$

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