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Current commutators in the hadronic vacuum and e^+e^- annihilation

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We study the behavior of current commutators in the vacuum within the quark-gluon theory of strong interactions. Especially, the problem of $SU_3 \times SU_3$ symmetry breaking is considered. We derive spectral-function sum rules involving the bare quark masses. Applications to vector-meson dominance, K_{l3} decay, and e^+e^- annihilation are discussed.

I. INTRODUCTION

The indication from inelastic electron and neutrino scattering experiments that scaling may be exactly valid has motivated the extension of current algebra to the light cone in the form of light-cone current algebra.[1] From a relativistic quark-gluon field-theory model one abstracts algebraic results, postulating their exact validity for the real world of hadrons. The abstraction of light-cone commutators differs from that for equal-time commutators in not being true in renormalized perturbation theory. It would be true only if there were an effective cutoff in transverse momentum.

One may envisage a situation in nature where the interaction is soft at high frequencies—where in particular transverse momenta are cut off—and the light-cone algebra of physical currents is isomorphic to the corresponding algebra in free quark theory. At the same time the interaction should be sufficiently strong at low frequencies (long distances) to give permanent binding of quarks and gluons.

As has become apparent in the last few years, the pattern of $SU_3 \times SU_3$ symmetry breaking in nature looks as in quark–vector-gluon theory, where the symmetry is broken by a bare quark mass term $\mathfrak{M}$.[2] The current divergences in particular look like (pseudo) scalar quark densities, multiplied by the bare quark masses.

Hence it seems natural to generalize the light-cone algebra of currents to scalar, pseudoscalar, and tensor densities. The absolute magnitude of the bare-quark masses then enters as a finite measurable entity within the algebraic framework. It appears, for example, as a coefficient in the leading singularity of a commutator involving two current divergences.[1]

It is expected that the bare quark masses are rather small (between ten and a few hundred MeV).[3] Of course, these masses are merely algebraic entities and have nothing to do with the masses of eventually existing real quarks.

It should be noted that the generalization of light-cone current algebra to current divergences is a very strong assumption. As one can easily see, it amounts to specific assumptions about nonleading singularities in current commutators (the leading light-cone singularity is always conserved).

In this paper we concentrate on current products (commutators) in the hadronic vacuum; in other words, we study properties of the c-number part of the current commutator. The high-energy behavior of the latter is not fixed by current or light-cone algebra (connected part). However, there is an indication from the magnitude of the $\pi^0 \to 2\gamma$ decay that the leading singularity of the c-number part has the same form and the same magnitude as in free quark theory.[4,5]

We assume that one can generalize the abstractions from the quark-gluon model to disconnected parts of commutators involving current divergences. Furthermore, we conjecture that not only the leading singularity of a current commutator in the vacuum is given by free quark theory, but also the next to leading singularity, in which the bare quark masses appear as coefficients.

This assumption is definitely wrong in perturbation theory, where it is spoiled by logarithmic corrections. However, it can be proved to be correct if we neglect the gluon propagation and treat the gluon in the tree approximation. We postulate that this assumption is also true in nature, where the interaction is supposed to be soft at high frequencies and where the transverse momenta are bounded.[6]

Using those assumptions we can determine the detailed high-energy behavior of the current spectral functions. In particular it is shown that the e^+e^--annihilation cross section approaches its asymptotic behavior const/s faster than const/s^2.

Specific sum rules for the spectral functions are obtained. Especially, we show that both Weinberg sum rules must be modified. However, one recovers Weinberg's original results if one assumes that the modified sum rules can be saturated by the lowest states.

We derive superconvergence relations for current spectral functions which involve, besides the physical spectral functions, the spectral functions as given by free quark theory. It is suggested that those relations can also be saturated by the lowest states—this amounts to a generalization of duality following the line suggested recently.[7,8] In particular it is shown that the saturation of the sum rule for the isovector spectral function with the ρ state alone leads to a consistent picture. Interesting consequences for the scalar spectral function of the axial-vector current are discussed.

We show further how the squares of the bare quark masses determine a sum rule for the e^+e^--annihilation cross section. The sum rule puts strong restrictions on the high-energy behavior of the latter, since m_q^2 must be positive.

Saturation of a corresponding superconvergence relation by the lowest-lying states suggests that the asymptotic region for e^+e^- annihilation is reached at about $\sqrt{s}\sim 1.3$ GeV:

$$\frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}\simeq 2(\sqrt{s}>1.3 \text{ GeV}), \tag{1.1}$$

provided the electromagnetic current is a pure SU_3 octet.

According to the present experimental data, $R(s)$ seems to rise to rather large values.[9,10] Our analysis suggests that this is *not* due to a late onset of the asymptotic region. Within our model the rise of R can be explained only by the introduction of new hadronic degrees of freedom (charm, color excitation, etc.), or by the presence of new direct quark-lepton interactions without an intermediate photon.[11]

II. LIGHT-CONE SINGULARITIES OF CURRENT PRODUCTS IN THE HADRONIC VACUUM

We abstract the singularity structure of current commutators from a relativistic quark-gluon model with quark statistics.[4,12] The quarks come in three "colors" and transform like (3, 3) under $SU_3 \times SU_3^{\text{color}}$. We assume that the interaction is mediated by a color octet Yang-Mills vector gluon which is neutral with respect to SU_3. The equation of motion for the quark field is written as

$$-i\gamma^\mu[\partial_\mu - igB_\mu(x)]\,q(x)+\mathfrak{M}q(x)=0\,, \tag{2.1}$$

where

$$B^\mu(x)=\lambda^i_{\text{color}}B^i_\mu(x)$$

is the gluon field matrix and $\mathfrak{M}$ represents the quark mass matrix:

$$\mathfrak{M}=\begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s\end{pmatrix}.$$

SU_3 is broken only by the quark mass difference $m_u - m_s$, whereas color symmetry is supposed to be exactly conserved. Physical states, currents, and S-matrix elements are taken to be color singlets such that neither quarks nor gluons have physical particles in their channels.

We postulate that the gluon field is sufficiently smooth at high frequencies in order not to disturb the free-quark light-cone singularities. More precisely, we assume that loops involving virtual gluons are strongly suppressed by the transverse momentum cutoff in such a way that these loops have no effect on the leading singularities of commutators of currents or their divergences. This amounts to treating the gluons as external fields for which the short-distance singularity of the quark two-point function is easily worked out:

$$\langle 0|\bar q(x)q(y)|0\rangle=\frac{i}{4\pi^2}\left[\frac{2z^\mu\gamma_\mu}{z^4}-\frac{im_q}{z^2}+\frac{z^\mu}{z^2}\left(\tfrac{1}{2}m_q^2\gamma_\mu+\frac{ig}{4}\gamma^\alpha\gamma_\mu\gamma^\beta G_{\alpha\beta}\right)+\cdots\right]E(x,y)+\cdots\,. \tag{2.2}$$

In this expression, z stands for $x-y$, $z^2=z^2-i\epsilon z^0$, and

$$G_{\alpha\beta}=\partial_\alpha B_\beta-\partial_\beta B_\alpha-ig[B_\alpha,B_\beta]\,,$$
$$E(x,y)=T_\lambda\exp\left\{-ig\int_0^1 d\lambda\, z^\alpha B_\alpha[(1-\lambda)x+\lambda y]\right\}. \tag{2.3}$$

(T means the ordering of the exponential.) The terms neglected in (2.2) are of order $\ln(z^2)$ and $z^\mu z^\nu/z^2$.

In the expansion (2.2) the bare quark masses m_q ($q\sim u, d, s$, respectively) are finite, real parameters. We assume that there is no u_3 term in the strong Hamiltonian, which amounts to $m_u=m_d$ (this equality is, of course, spoiled after including weak and electromagnetic interactions). SU_3 and $SU_2\times SU_2$ breaking are described by the parameter $f=m_s/m_u$. Exact SU_3 requires $f=1$. In reality f is either approximately 1 ("weak PCAC") or very large (~ 25, strong PCAC).

Using the assumptions outlined above we find for current products in the vacuum

$$\langle 0|F_{i\mu}(x)\,F_{j\nu}(0)|0\rangle = -\partial_\mu\partial_\nu\left(\frac{A_{ij}}{x^4}+\frac{B_{ij}}{x^2}+\cdots\right) + g_{\mu\nu}\left(\frac{8A_{ij}}{x^6}+\frac{C_{ij}}{x^4}+\cdots\right)$$

and

$$\langle 0|F^5_{i\mu}(x)\,F^5_{j\nu}(0)|0\rangle = -\partial_\mu\partial_\nu\left(\frac{\overline{A}_{ij}}{x^4}+\frac{\overline{B}_{ij}}{x^2}+\cdots\right) + g_{\mu\nu}\left(\frac{8\overline{A}_{ij}}{x^6}+\frac{\overline{C}_{ij}}{x^4}+\cdots\right), \tag{2.4}$$

where

$$A_{ij}=\overline{A}_{ij}=\frac{1}{8\pi^4}\delta_{ij}\,,$$

$$B_{ij}=\overline{B}_{ij}=\frac{3}{16\pi^4}\operatorname{tr}\left(\frac{\lambda_i}{2}\,\mathfrak{M}^2\,\frac{\lambda_j}{2}+(\overleftrightarrow{i,j})\right),$$

$$C_{ij}=-\frac{3}{8\pi^4}\operatorname{tr}\left(\left[\mathfrak{M},\frac{\lambda_i}{2}\right]\left[\mathfrak{M},\frac{\lambda_j}{2}\right]\right), \tag{2.5}$$

$$\overline{C}_{ij}=\frac{3}{8\pi^4}\operatorname{tr}\left(\left\{\mathfrak{M},\frac{\lambda_i}{2}\right\}\left\{\mathfrak{M},\frac{\lambda_j}{2}\right\}\right).$$

Analogously, one obtains for the current divergences

$$\partial^\mu F_{i\mu}(x)=\text{“}\,i\,\overline{q}(x)\left[\mathfrak{M},\frac{\lambda_i}{2}\right]q(x)\text{”},$$

$$\partial^\mu F^5_{i\mu}(x)=\text{“}\,i\,\overline{q}(x)\left\{\mathfrak{M},\frac{\lambda_i}{2}\right\}q(x)\text{”},$$

$$i=1,\ldots,8 \tag{2.6}$$

the expansions

$$\langle 0|\partial^\mu F_{i\mu}(x)\;\partial^\nu F_{j\nu}(0)|0\rangle=-8\,\frac{C_{ij}}{x^6}+\frac{D_{ij}}{x^4}+\cdots$$

and

$$\langle 0|\partial^\mu F^5_{i\mu}(x)\;\partial^\nu F^5_{j\nu}(0)|0\rangle=-8\,\frac{\overline{C}_{ij}}{x^6}+\frac{\overline{D}_{ij}}{x^4}+\cdots, \tag{2.7}$$

where

$$D_{ij}=-\frac{3}{4\pi^4}\operatorname{tr}\left(\left[\frac{\lambda_i}{2},\left[\frac{\lambda_j}{2},\mathfrak{M}\right]\right]\mathfrak{M}^3\right),$$

$$\overline{D}_{ij}=-\frac{3}{4\pi^4}\operatorname{tr}\left(\left\{\frac{\lambda_i}{2},\left\{\frac{\lambda_j}{2},\mathfrak{M}\right\}\right\}\mathfrak{M}^3\right). \tag{2.8}$$

Obviously, the coefficients D_{ij} must by symmetric in the indices i,j. This symmetry is not manifest in (2.8), but may easily be checked by direct inspection.

We would like to emphasize that relations (2.6)–(2.8) are not supposed to work for the divergence of the axial-vector baryon current ($i,j=0$). The latter receives an additional anomalous contribution involving the square of the gluon field strength which would spoil the ansatz (2.7).[13]

We note that all our results can be generalized to the case where charm degrees of freedom are present; e.g., one may add a fourth quark u' of charge $\frac{2}{3}$. Here one starts with the group $SU_4 \times SU_3^{\text{color}}$, and SU_4 is broken strongly so that the charmed partners of the known hadrons are shifted to higher energy and are not identified yet.

We also could allow a generalization to the case, where SU_3^c gets excited by strong, weak, and electromagnetic interactions, as, for example, in the Han-Nambu model.[14] There the electromagnetic current has a colored piece:

$$j_\mu=F^{(3,1)}_\mu+\frac{1}{\sqrt{3}}F^{(8,1)}_\mu+F^{(1,3)}_\mu+\frac{1}{\sqrt{3}}F^{(1,8)}_\mu. \tag{2.9}$$

Of course, in the latter case one has to give up the singlet restriction for physical states; color gets excited as a physical degree of freedom.

We emphasize that all these generalizations do not change, but only enlarge relations (2.4)–(2.8). They give the same singularity structure as the usual quark model for products of $SU_3\times SU_3$ currents.

III. CONSEQUENCES FOR CURRENT SPECTRAL FUNCTIONS

We define the spectral functions for vector and axial-vector currents as follows:

$$\frac{1}{2\pi}\int dx\,e^{iqx}\langle 0|F_{i\mu}(x)\,F_{j\nu}(0)|0\rangle = \theta(q_0)\left[(q_\mu q_\nu-g_{\mu\nu}s)\rho^1_{ij}(s)+q_\mu q_\nu\rho^0_{ij}(s)\right],$$

$$\frac{1}{2\pi}\int dx\,e^{iqx}\langle 0|F^5_{i\mu}(x)\,F^5_{j\nu}(0)|0\rangle = \theta(q_0)\left[(q_\mu q_\nu-g_{\mu\nu}s)\overline{\rho}^1_{ij}(s)+q_\mu q_\nu\overline{\rho}^0_{ij}(s)\right],$$

$$s=q^2, \tag{3.1}$$

where

ρ^1, $\overline{\rho}^1$ describes the spin-1 intermediate states,

ρ^0, $\overline{\rho}^0$ describes the spin-0 intermediate states.

(Note: $\rho^0=0$ for conserved currents.) The expansions (2.4) imply the following properties of the spectral functions:

(a) As $s\to\infty$ one finds

$$\rho^1_{ij} \to \pi^2\left(A_{ij} - \frac{C_{ij}}{s} + \cdots\right) ,$$
$$\rho^0_{ij} \to \pi^2\left(\frac{C_{ij}}{s} + \frac{D_{ij}}{s^2} + \cdots\right) , \tag{3.2}$$

and analogously for the spectral functions associated with the axial-vector currents. Note that the combination $\rho^0+\rho^1$ has no $1/s$ term as $s\to\infty$ [such a term would correspond to a noncanonical light-cone singularity not present in the ansatz (2.4)].

(b) The coefficients B_{ij} and $\overline{B}_{ij}$ are not related to the asymptotic behavior of the spectral functions. Instead they determine the integrals

$$\int_0^\infty ds\,(\rho^0_{ij} + \rho^1_{ij} - \pi^2 A_{ij}) = -4\pi^2 B_{ij} ,$$
$$\int_0^\infty ds\,(\overline{\rho}\,^0_{ij} + \overline{\rho}\,^1_{ij} - \pi^2 A_{ij}) = -4\pi^2 B_{ij} , \tag{3.3}$$

which are convergent on account of (3.2).

(c) A further piece of information may be squeezed out of our soft gluon model. It is clear that the difference

$$\Delta_{\mu\nu} = \langle 0|F_{i\mu}(x)\,F_{j\nu}(0)|0\rangle - \langle 0|F^5_{i\mu}(x)\,F^5_{j\nu}(0)|0\rangle \tag{3.4}$$

is smoother at the origin than the individual terms. The corresponding expansion looks qualitatively as follows:

$$\Delta(x) \sim \frac{m^2}{x^4} + \frac{m^4 + g^2\langle G^2\rangle}{x^2} + O(1) . \tag{3.5}$$

In fact, also the term involving $\langle G^2\rangle$ drops out, since for $m=0$ the terms proportional to $1/x^2$ in the vector and axial-vector two-point functions coincide exactly. Furthermore, the free quark model ($g=0$) shows that the terms involving m^2 and m^4 are proportional to $g_{\mu\nu}$. Therefore

$$\Delta_{\mu\nu}(x) \sim g_{\mu\nu}\left(\frac{m^2}{x^4} + \frac{m^4}{x^2}\right) + O(1) . \tag{3.6}$$

This not only implies that the combination $\rho^0 + \rho^1 - \overline{\rho}\,^0 - \overline{\rho}\,^1$ vanishes for $s\to\infty$ faster than $1/s^2$, but in addition demands that the two sum rules

$$\int ds(\rho^0_{ij} + \rho^1_{ij} - \overline{\rho}\,^0_{ij} - \overline{\rho}\,^1_{ij}) = 0 , \tag{3.7}$$

$$\int ds\, s(\rho^0_{ij} + \rho^1_{ij} - \overline{\rho}\,^0_{ij} - \overline{\rho}\,^1_{ij}) = 0 \tag{3.8}$$

be satisfied. (The first integral is the coefficient of a singularity in $\Delta_{\mu\nu}$ proportional to $x_\mu x_\nu/x^6$; the second corresponds to $x_\mu x_\nu/x^4$.) Note that the first sum rule already follows from (3.3). Finally, since the leading singularities up to and including x^{-2} of the term proportional to $g_{\mu\nu}$ are the same as for free quarks, we have the sum rule

$$\int ds\, s(\rho^1_{ij} - \overline{\rho}\,^1_{ij} - \sigma^1_{ij} - \overline{\sigma}\,^1_{ij}) = 0 , \tag{3.9}$$

where σ^1_{ij} and $\overline{\sigma}\,^1_{ij}$ denote the spectral functions of the free quark model which are explicitly determined by the bare quark masses m_u and m_s.

IV. SATURATION OF THE SUM RULES

The two sum rules (3.7) and (3.8) derived in the last section on the basis of the soft gluon model are generalizations of Weinberg's sum rules.[15] We focus on the isovector currents for which isospin symmetry implies

$$\rho^0_{ij} = 0 , \quad \overline{\rho}\,^0_{ij} = \delta_{ij}\,\overline{\rho}\,^0(s) ,$$
$$\rho^1_{ij} = \delta_{ij}\rho(s) , \quad \overline{\rho}\,^1_{ij} = \delta_{ij}\,\overline{\rho}_1(s) ,$$

and the sum rules (3.7) and (3.8) therefore become

$$\int ds(\rho_1 - \overline{\rho}_0 - \overline{\rho}_1) = 0 , \tag{4.1}$$

$$\int ds\, s(\rho_1 - \overline{\rho}_0 - \overline{\rho}_1) = 0 . \tag{4.2}$$

In the $SU_2\times SU_2$ limit ($m_u\to 0$), conservation of the axial-vector current requires[16]

$$\overline{\rho}_0(s) \xrightarrow[m_u\to 0]{} \delta(s) F_\pi{}^2 \tag{4.3}$$

and we therefore get in the $SU_2\times SU_2$ limit

$$\int ds(\rho_1 - \overline{\rho}_1) = F_\pi{}^2 , \quad \int ds\, s(\rho_1 - \overline{\rho}_1) = 0 , \tag{4.4}$$

which is the form originally proposed by Weinberg within the framework of field algebra. In the framework of our assumptions the difference $\rho_1 - \overline{\rho}_1$, however, behaves like $m_u{}^2/s$ as $s\to\infty$ and these sum rules therefore diverge. The tail is absent only in the $SU_2\times SU_2$ limit $m_u=0$. [Note also that the sum rule (3.9) reduces to (4.4) in the limit $m_u\to 0$.[17]]

Clearly the interchange of the integrals with the limit $m_u\to 0$ is not legitimate and we therefore prefer to work with the generalized forms (4.1) and (4.2). In our scheme the quantity $\rho_1 - \overline{\rho}_0 - \overline{\rho}_1$ tends to zero more rapidly than s^{-2}. If we assume, in addition, that the sum rules (4.1) and (4.2) are saturated by the lowest-lying states ρ_1, π, and A_1 we get the relations

$$F_\rho{}^2 - F_\pi{}^2 - F_{A_1}{}^2 \simeq 0 ,$$
$$m_\rho{}^2 F_\rho{}^2 - m_\pi{}^2 F_\pi{}^2 - m_{A_1}{}^2 F_{A_1}{}^2 \simeq 0 . \tag{4.5}$$

If we follow Weinberg and use the "experimental" result

$$F_\rho{}^2 \simeq 2F_\pi{}^2 , \tag{4.6}$$

we obtain the familiar relation

$$m_{A_1} \simeq \sqrt{2}\, m_\rho . \tag{4.7}$$

Thus the assumption that the sum rules (4.1) and (4.2) are saturated by the lowest-lying states allows us to recover Weinberg's original results. Note, however, that the existence of the A_1 is questionable. Thus the result (4.7) cannot be interpreted as a strong justification that our saturation assumption works. Nevertheless, it seems to work to a good approximation, provided the A_1 resonance exists.

Our scheme allows some more specific conclusions. From the previous section one can easily derive the following set of sum rules for the isovector spectral functions:

$$\int ds(\rho_1 - \sigma_1) = 0 , \tag{4.8}$$

$$\int ds(\bar{\rho}_0 + \bar{\rho}_1 - \bar{\sigma}_0 - \bar{\sigma}_1) = 0 , \tag{4.9}$$

$$\int ds\, s(\bar{\rho}_0 - \bar{\sigma}_0) = 0 , \tag{4.10}$$

$$\int ds\, s(\rho_1 - \bar{\rho}_1 - \sigma_1 + \bar{\sigma}_1) = 0 , \tag{4.11}$$

where $\sigma_1, \bar{\sigma}_0, \bar{\sigma}_1$ are the spectral functions of the free-quark model with mass m_u:

$$\sigma_1 = \sigma(s)\left(1 + \frac{2m_u^{\,2}}{s}\right), \quad \bar{\sigma}_0 = \sigma(s)\,\frac{6m_u^{\,2}}{s}, \quad \bar{\sigma}_1 = \sigma(s)\left(1 - \frac{4m_u^{\,2}}{s}\right), \tag{4.12}$$

where

$$\sigma(s) = \frac{1}{8\pi^2}\left(1 - \frac{4m_u^{\,2}}{s}\right)^{1/2} \theta(s - 4m_u^{\,2}). \tag{4.13}$$

(Note $\rho_0 = \sigma_0 = 0$, $\sigma_1 = \bar{\sigma}_0 + \bar{\sigma}_1$.) The quantities ρ_1, $\bar{\rho}_0 + \bar{\rho}_1$, $s(\rho_1 - \bar{\rho}_1)$, and $s\bar{\rho}_0$ therefore not only have to behave asymptotically like the corresponding spectral functions of the free quark model, but moreover have to balance these quantities even in the low-energy region in such a way that the integrals (4.8)–(4.11) vanish.

It is interesting to see what happens if these sum rules saturate not only asymptotically, but already after the lowest-lying contributions. Consider, e.g., the first sum rule which receives a prominent contribution from the ρ. Suppose that there is a mass M_1, above which the spectral function ρ_1 practically coincides with the quark-model function and below which the ρ provides for the dominant contribution. In this case we get

$$F_\rho^{\,2} = \int_0^{M_1^2} ds\, \sigma_1(s) \simeq \frac{1}{8\pi^2} M_1^{\,2} , \tag{4.14}$$

where we have assumed that m_u is much smaller than M_1. This leads to $M_1 \sim 1.3$ GeV which appears to be very reasonable. (Note that one obtains in the Fermi-Dirac quark model $M_1 \sim 2.3$ GeV, which is unacceptably high.)

The second sum rule will of course be saturated at the same value of s if the previous discussion of the difference between the two sum rules is right. Saturating the third sum rule with the pion we get

$$F_\pi^{\,2} m_\pi^{\,2} = \int_0^{M_2^2} ds\, s\bar{\sigma}_0(s) \simeq \frac{3}{4\pi^2} M_2^{\,2} m_u^{\,2} . \tag{4.15}$$

If, as is suggested by considerations concerning the connected part[3] of the current commutator, the mass m_u is of order 20 MeV or less (strong PCAC) we get $M_2 > 2.3$ GeV. Such a large value seems at first sight to be absurd—to assume that the pion is the only state with the quantum numbers of $(\partial^\mu F_\mu^5)$ which contributes significantly to the spectral function $\bar{\rho}_0(s)$ between 0 and $M_2 > 2.3$ GeV certainly amounts to a very bold assumption. The reason for this large value of the cutoff can be understood as follows. If the mass m_u is indeed as small as say 5 MeV, then the spectral function $\bar{\sigma}_0$ of the free quark model is exceedingly small. On the other hand, in the limit $m_u \to 0$, the spectral function $\bar{\rho}_0(s)$ must vanish except for the contribution of the π at $s = 0$. In the free quark model the function $s \cdot \bar{\sigma}_0(s)$ is essentially flat above $s \geq 10 m_u^{\,2}$, but is tiny. In the real world, there is instead a large peak at a small value of s, at $s = m_\pi^{\,2}$. In order to balance this peak in the sum rule we have to integrate the tiny quantity $\bar{\sigma}_0(s)$ out to a very large value of s. In the formal limit $m_u \to 0$ considerations involving the connected part of current commutators suggest that the ratio

$$\frac{m_\pi^{\,2}}{m_u} = M_0 \quad (M_0 \text{ some specific mass}) \tag{4.16}$$

has to be kept fixed in order to get a simple, consistent picture.[3] The cutoff is therefore inversely proportional to m_π: $M_2 \sim (2\pi/\sqrt{3})\, F_\pi M_0 / m_\pi$. In the limit $m_u \to 0$ the pion dominates the spectral function $\bar{\rho}_0(s)$ out to larger and larger values of s—only at very high energies does the asymptotic tail $\bar{\rho}_0(s) \sim m_u^{\,2}/s$ show up.

The same analysis can of course be applied to the spectral functions associated with other SU_3 quantum numbers. For example, the sum rule analogous to (4.10) for the spectral function with the quantum numbers of the η leads to

$$m_\eta{}^2 F_\eta{}^2 \simeq \frac{1}{2\pi^2} M_3{}^2 m_s{}^2, \tag{4.17}$$

provided we can neglect the strange quark mass in calculating $\int_0^{M_3^2} ds\,\sigma_\eta(s)$ which is justified only for $m_s{}^2 << M_3{}^3$.[18]

Taking $F_\eta \simeq F_\pi$ and combining (4.15) and (4.17) one finds in the case of strong PCAC ($m_u << m_s$, $m_\pi{}^2 \sim M_0 m_u$, $2m_K{}^2 \sim M_0 m_s$, $2m_\eta{}^2 \sim \frac{4}{3} M_0 m_s$)

$$M_3 \sim \frac{1}{\sqrt{f}} M_2, \quad f = \frac{m_s}{m_u} .$$

Hence the cutoff mass M_3 is expected to be smaller than M_2.[14]

Applying the sum rule (4.10) to the spectral functions with the quantum numbers of the $\kappa(0^+)$ and the $K(0^-)$, we get

$$m_\kappa{}^2 F_\kappa{}^2 - m_K{}^2 F_K{}^2 \simeq -\frac{3m_u m_s}{\pi^2} M_4{}^2,$$

which implies the upper bound

$$m_\kappa{}^2 F_\kappa{}^2 \leq m_K{}^2 F_K{}^2, \tag{4.18}$$

where the equality sign is valid in the $SU_2 \times SU_2$ limit.

It is interesting to compare this result with the estimates on $m_\kappa{}^2 F_\kappa{}^2$ derived from K_{l3} decay. These estimates are carried out in the framework of the assumption that the integral

$$\Delta(0) = \int ds\, s\rho_\kappa^0(s) \tag{4.19}$$

converges. One can obtain information on $\Delta(0)$ from the assumed $SU_3 \times SU_3$ transformation properties of this quantity.[19] In our framework $\Delta(0)$ is infinite; we nevertheless expect the contribution of the $K\pi$ intermediate states to $\Delta(0)$ to converge—this contribution may be expressed in terms of the scalar K_{l3} form factor which we expect to tend to zero as $t \to \infty$.[3] In the spirit of the above saturation scheme we assume that the $K\pi$ intermediate states provide the dominant contribution to the integral $\Delta(0)$ below the cutoff $s = M_4{}^2$. What we denoted by $m_\kappa{}^2 F_\kappa{}^2$ above should therefore be identified with

$$m_\kappa{}^2 F_\kappa{}^2 = \int_0^\infty ds\, s\rho_{K\pi}(s). \tag{4.20}$$

The recent work[15] on bounds for the K_{l3} form factors can therefore be interpreted as follows. If one demands that

$$m_\kappa{}^2 F_\kappa{}^2 \leq (m_K F_K - m_\pi F_\pi)^2 \tag{4.21}$$

then the K_{l3} data are constrained very strongly to essentially a straight line between $t=0$ and the Callan-Treiman point $t = m_K{}^2$. Our bound (4.18) is slightly more generous than (4.21), but the fact that the new K_{l3} data seem to lie essentially on a straight line through the Callan-Treiman point suggests that the relation (4.18) is indeed in agreement with experiment.[20]

V. APPLICATIONS TO e^+e^- ANNIHILATION

The total cross section for e^+e^- annihilation into hadrons is given to lowest order in electromagnetism by

$$\sigma_{e^+e^- \to \text{hadrons}} = \frac{16\pi^3\alpha^2}{s} \rho_{ee}(s), \tag{5.1}$$

where ρ_{ee} is the spectral function for the electromagnetic current. For comparison we introduce the cross section for the production of a massless lepton pair of charge one

$$\sigma_{e^+e^- \to 1^+1^-} = \frac{4\pi\alpha^2}{3s} \tag{5.2}$$

and define the ratio

$$R(s) = \frac{\sigma_{e^+e^- \to \text{hadrons}}}{\sigma_{e^+e^- \to 1^+1^-}} . \tag{5.3}$$

If the electromagnetic current is a pure SU_3 octet

$$j_\mu = F_{3\mu} + \frac{1}{\sqrt{3}} F_{8\mu}, \quad \rho_{ee} = \rho_{33} + \tfrac{1}{3}\rho_{88}, \tag{5.4}$$

one finds from (2.5) and (3.2)

$$R \xrightarrow[s \to \infty]{} 2 . \tag{5.5}$$

In the case of charm the asymptotic value of R is $\frac{10}{3}$; in the case of the Han-Nambu model it is 4.

According to (3.2) the asymptotic value is approached faster than const/s:

$$s[R(s) - 2] \xrightarrow[s \to \infty]{} 0 . \tag{5.6}$$

Furthermore, one finds from (3.3) an interesting sum rule in which the bare quark masses enter:

$$\int_0^\infty (2-R)ds = 18(\tfrac{4}{9} m_u{}^2 + \tfrac{1}{9} m_d{}^2 + \tfrac{1}{9} m_s{}^2)$$
$$= 2\left(1 + \frac{5}{f^2}\right) m_s{}^2 . \tag{5.7}$$

The most striking consequence of this sum rule comes from the fact that the right-hand side must be positive. This implies in particular that $R(s)$ cannot rise to large values like 5 or 6 and then settle down slowly to its asymptotic value 2 from above.

According to the present experimental data[9,10] $R(s)$ seems to rise to rather large values. Our analysis [in particular the sum rule (5.7)] suggests that this is *not* due to a late onset of the asymptotic region. Within our model the rise of R can be explained only by the introduction of new hadronic degrees of freedom or by the presence of new lepton-quark interactions, i.e., the e^+e^- pair can produce virtual quark-antiquark pairs directly without an intermediate photon, for example, by the exchange of a new boson carrying both leptonic and hadronic quantum numbers.[11]

Both for strong and weak PCAC the right-hand side of (5.7) can be estimated. As an example we adopt the values[3] $m_s \lesssim \frac{1}{2}$ GeV, $f \simeq 1.2$ (weak PCAC), and $f \simeq 25$ (strong PCAC). Then we find

$$\int_0^\infty ds\,(2-R) \lesssim \begin{cases} 2\ \text{GeV}^2 & \text{(weak PCAC)}, \\ 0.5\ \text{GeV}^2 & \text{(strong PCAC)}. \end{cases}$$

Of course, for strong PCAC the integral comes out very small.

Sum rule (5.6) can, of course, be generalized to more extended cases. In general one finds

$$\int_0^\infty ds\Big(\sum_{\text{quarks}} e_q^{\,2} - R(s)\Big) = 6 \sum_{\text{quarks}} e_q^{\,2} m_q^{\,2}, \quad (5.8)$$

where e_q are the formal quark charges and m_q are the formal bare quark masses.[21]

In the Han-Nambu case we can estimate the sum rule as follows, provided there is no substantial SU_3^c breaking:

$$\begin{aligned}\int_0^\infty ds\,(4-R) &= 6(2m_u^{\,2}+m_d^{\,2}+m_s^{\,2}) \\ &= 6\Big(1+\frac{3}{f^2}\Big)m_s^{\,2} \\ &\lesssim \begin{cases} 1.5\ \text{GeV}^2 & \text{(strong PCAC)}, \\ 5\ \text{GeV}^2 & \text{(weak PCAC)}. \end{cases}\end{aligned} \quad (5.9)$$

It is interesting to apply the sum rules (4.8) to e^+e^- annihilation. We define an interpolating function R^{int} as given by free quark theory:

$$R^{\text{int}}(s) = \sum_{\text{quarks}} \theta(s-4m_q^{\,2})e_q^{\,2} \times \frac{s+2m_q^{\,2}}{s}\Big(\frac{s-4m_q^{\,2}}{s}\Big)^{1/2}. \quad (5.10)$$

This function fulfills, of course, also the sum rules (5.7) or (5.8). Hence one has the superconvergence relation

$$\int_0^\infty ds[R(s)-R^{\text{int}}(s)] = 0. \quad (5.11)$$

As in Sec. IV, we now suppose that the integral (5.11) can be saturated already by the lowest-lying vector-meson states. This assumption is evidently related to the generalization of duality as proposed recently.[7,8] Rapid saturation of the sum rule (5.11) implies that there is a relatively low cutoff M^2, above which $R(s)$ practically coincides with $R^{\text{int}}(s)$ and one has

$$\int_0^{M^2} ds[R(s)-R^{\text{int}}(s)] \simeq 0. \quad (5.12)$$

Only $\frac{1}{6}$ of the total e^+e^- annihilation cross section is due to strange quark degrees of freedom; hence it should be a good approximation to take as a cutoff the same value as found for the isovector spectral functions in Sec. IV:

$$\begin{aligned} &M^2 \simeq (1.3\ \text{GeV})^2, \\ &R(s) \simeq R^{\text{int}}(s) \simeq 2 \quad (\sqrt{s} > 1.3\ \text{GeV}). \end{aligned} \quad (5.13)$$

Of course, these considerations make sense only if the electric charge is a pure SU_3 octet.

If new degrees of freedom get excited by electromagnetism (charm, color, etc.), the saturation of (5.11) by low-lying vector mesons can only be valid for the part of R which is due to ordinary SU_3 degrees of freedom. The sum rule (5.11) for the non-SU_3 part of R can certainly *not* be saturated by low-lying ordinary vector-meson states. Hence, if new degrees of freedom get excited, a separation of R into contributions from ordinary SU_3 degrees of freedom and from charm, color, etc., would be necessary to study the question of rapid saturation.

Note added in proof. The sum rules (3.7) and (3.8) were found previously by Morita.[22] We should like to thank him for bringing his paper to our attention.

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[16]We use the normalization $\langle 0|\partial^\mu F^5_{3\mu}|\pi^0\rangle = \langle 0|\partial^\mu F^5_{4-i5,\mu}|K^+\rangle = \sqrt{2}m_K{}^2F_K$, etc. for the coupling constants of the pseudoscalar octet and $\langle 0|F_{3\mu}|\rho^0\rangle = \epsilon_\mu m_\rho F_\rho$ for the vector mesons. The normalization of states is taken as $\langle p'|p\rangle = (2\pi)^3 2p^0\delta^3(\vec{p}'-\vec{p})$.

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[20]An investigation by Broadhurst (private communication) indicates that (4.21) is inconsistent with available information on $\rho_{K\pi}$. Whether or not (4.18) is consistent with the data remains to be seen.

[21]This sum rule can easily be checked in free Dirac theory. The ratio R is given by

$$R = e^2\theta(s-4m^2)\,\frac{2+2m^2}{s}\left(\frac{s-4m^2}{s}\right)^{1/2}$$

and one finds directly

$$\int_0^\infty ds\,(e^2 - R) = 6m^2e^2.$$

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