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Radiative corrections to $K_L \to \mu^+\mu^-$

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Radiative corrections to the unitarity bound for the decay $K_L \to \mu^+\mu^-$ are studied in detail. Unlike the three-particle intermediate state (e.g., the $2\pi\gamma$ state) these corrections tend to enhance the decay amplitude and give a significant contribution ($\sim$ 17% of the theoretical lower bound given by the 2γ state) to the decay rate.

The decay process $K_L \to \mu^+\mu^-$ is of interest for a better understanding of the weak interaction. From this process, which is forbidden in first order by the usual weak-interaction Hamiltonian which does not contain neutral lepton currents, interest arises from possible effects of (a) higher-order weak interaction, (b) existence of neutral lepton currents, and (c) lowest-order weak interaction plus an electromagnetic interaction of order α^2.[1,2]

The approach (c) provides a lower bound for the decay rate, assuming (i) unitarity, (ii) *CPT* invariance, and (iii) time-reversal invariance. A major contribution to the unitarity sum comes from the two-photon intermediate state. Based on the recent experimental value for the branching ratio for the process $K_L \to 2\gamma$, one obtains a lower bound for the process $K_L \to \mu^+\mu^-$ as $\sim 6\times 10^{-9}$.[3,4] Of the two recent experimental results, one puts an upper limit of 1.8×10^{-9} on this branching ratio,[5] which is significantly below the calculated lower limit, while the other[6] gives a value of 10^{-8}.

Regarding the contributions of other intermediate states, we see that the $2\pi\gamma$ state contributes less than 10% of the 2γ state,[4] while the 3π-state contribution is found to be smaller by several orders of magnitude.[7]

Hence if one wishes to improve on the theoretical lower bound for the branching ratio, the higher-order interactions which one must include are the radiative corrections to the 2γ state. In view of the fact that the contribution of three-particle processes is small, these corrections are expected to give a significant contribution. We present an estimate of these radiative corrections.

To estimate the radiative corrections we should consider Figs. 1–4. Of these, Figs. 1(a) and 1(b) are already included in the calculation of the 2γ state, since the coupling constant $f_{K\gamma\gamma}$ in this calculation was obtained from the experimental value for the decay rate of $K_L \to 2\gamma$. The process 1(c) is in fact a three-particle process and is expected to give a very small contribution, while process 1(d) simply contributes to mass renormalization and charge renormalization (as a consequence of field-operator renormalization).

We now present the calculations of the remaining diagrams. The calculations are carried out in the rest frame of K_L.

Propagator correction (Fig. 2). This correction to the 2γ-state absorptive part is written as[8]

$$A_1 = \frac{f_{K\gamma\gamma}}{M} e^2 \int \frac{d^4k}{(2\pi)^4} (-2\pi^2)\, \delta(k^2)\, \delta((P-k)^2)\, \epsilon_{\mu\nu\alpha\beta} k_\alpha P_\beta\, \bar{u}(p_-)\, \gamma_\mu \Sigma_f(q)\, \gamma_\nu\, v(p_+)\ ,$$

where M is the kaon mass, $f_{K\gamma\gamma}$ is a dimensionless coupling constant defined by

$$\mathrm{Rate}(K_L \to 2\gamma) = \frac{|f_{K\gamma\gamma}|^2}{64\pi} M\ ,$$

and

$$\Sigma_f(q) = \frac{\alpha}{2\pi m} \left\{ \frac{1}{2(1-\rho)}\left(1 - \frac{2-3\rho}{1-\rho}\ln\rho\right) - \frac{\not{q}+m}{m}\left[\frac{1}{2\rho(1-\rho)}\left(-2-\rho+\frac{4-8\rho+\rho^2}{1-\rho}\ln\rho\right) + \frac{2}{\rho}\ln\left(\frac{m^2}{\delta^2}\right) - \frac{4}{\rho}\right]\right\},$$

where $\rho = (m^2 - q^2)/m^2$, m is the muon mass, α is the fine-structure constant, and δ is a small rest mass assigned to the photon in order to take care of the infrared divergence and soft-photon emission with photon energy $E < \delta$ and $q = k - p_+$.

On carrying out the k integration, we get after some simplification and expressing the θ integration in terms of ρ (θ being the angle between $\vec{p}_+$

and $\vec{k}$)

$$A_1 = -\frac{f_{K\gamma\gamma}}{8\pi} e^2 \frac{4m^3}{M^3} \frac{\alpha}{1-4m^2/M^2}$$

$$\times \int d\rho \left\{ \frac{\rho - M^2/2m^2}{2(1-\rho)} \left(1 - \frac{2-3\rho}{1-\rho}\ln\rho\right) - \frac{M^2}{2m^2}\left[\frac{1}{2\rho(1-\rho)}\left(-2-\rho+\frac{4-8\rho+\rho^2}{1-\rho}\ln\rho\right) + 4\left(\ln\frac{m}{\delta}-1\right)\frac{1}{\rho}\right]\right\} \bar{u}\gamma_5 v\,.$$

The ρ integration was carried out by introducing[9] a simplification $(1-\rho) \sim -\rho$, whereupon we get

$$A_1 = f_{K\gamma\gamma}\alpha^2(0.32)\bar{u}\gamma_5 v\,.$$

Hence the ratio of A_1 to the Sehgal amplitude[10] is $\sim +1.9\alpha$.

Vertex correction (Fig. 3). This contribution can be written as[8]

$$A_2 = \frac{f_{K\gamma\gamma}}{M} e^2 \int \frac{d^4k}{(2\pi)^4}(-2\pi^2)\delta(k^2)\delta((P-k)^2)\epsilon_{\mu\nu\alpha\beta}k_\alpha P_\beta$$

$$\times \bar{u}(p_-)\gamma_\mu S(q)\Lambda_\nu^f\, v(p_+)\,,$$

where

$$\Lambda_\nu^f = -\frac{\alpha}{2\pi}\int_0^1 dx \int_0^x dy \left[\frac{K_\nu}{a^2} + \gamma_\nu \int_0^1 dz \frac{a^2 - m^2x^2}{m^2x^2 + (a^2 - m^2x^2)z} - 2m^2\gamma_\nu(1 - x - \tfrac{1}{2}x^2)\frac{a^2 - m^2x^2}{a^2m^2x^2}\right],$$

with

$$a^2 = m^2x^2 - (q^2 - m^2)(1-x)(1-y)\,,$$

$$K_\nu = \gamma_\nu C - M_\nu(m - \not{q}) - mx(1-x)\sigma_{\nu\lambda}k_\lambda\,,$$

$$C = (1-x)(1-x+y)(q^2 - m^2)\,,$$

$$M_\nu = \gamma_\nu(1-x^2)m - (p_{+\nu} - q_\nu)(1-x)(1-x+y) + k_\nu(1-x-2y)\,,$$

and

$$\sigma_{\nu\lambda} = \tfrac{1}{2}i(\gamma_\nu\gamma_\lambda - \gamma_\lambda\gamma_\nu).$$

Calculations show that the major contribution comes from the last term (infrared part) of Λ_ν^f. The ratio of A_2 to the Sehgal amplitude is $\sim +2.5\alpha$.

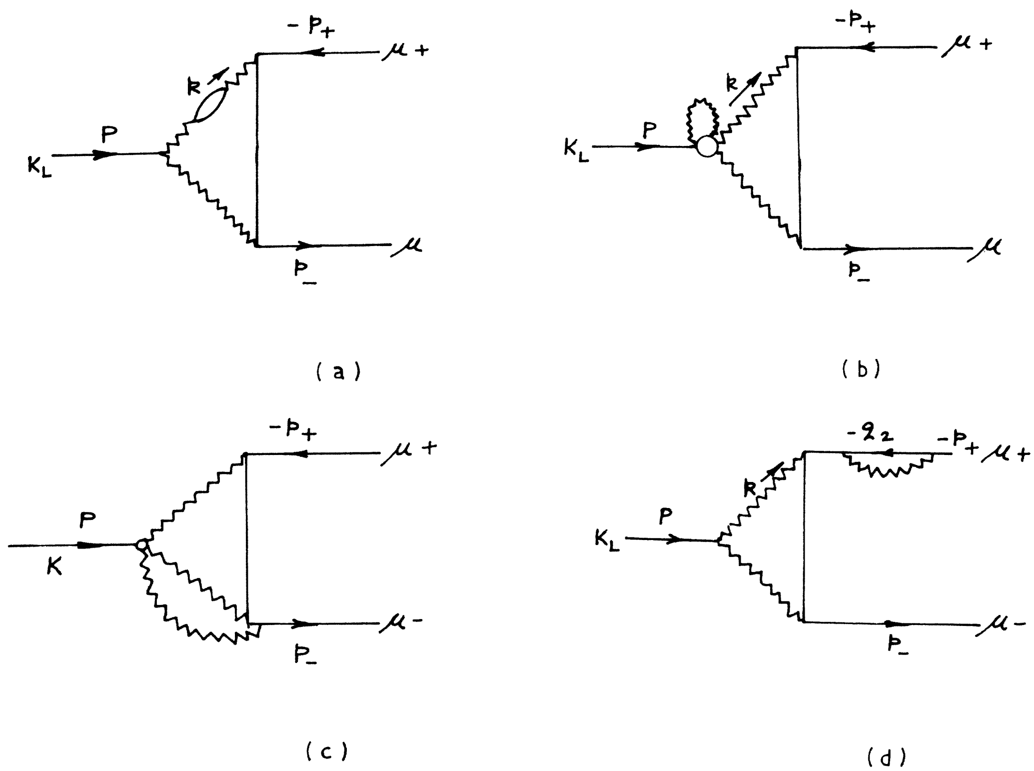

FIG. 1. Diagrams which do not contribute or which give a negligible contribution to the unitarity bound.

Hence for two vertices the total correction is $\sim +5\alpha$.

Box diagram (Fig. 4). For the calculation of the absorptive part of this process, it can be looked upon as coming from (1) the $\mu^+\mu^-$ intermediate state [Fig. 5(a)] and (2) the 2γ intermediate state [Fig. 5(b)].

The absorptive part for Fig. 5(a) is

$$A_3 = -f_{K\mu^+\mu^-}\, e^2 \int \frac{d^4q_2}{(2\pi)^4}(-2\pi^2)\delta(q_2{}^2 - m^2)\delta((P-q_2)^2 - m^2)\, \frac{\bar{u}(p_-)\gamma_\lambda(\not{P} - \not{q}_2 + m)\gamma_5(-\not{q}_2 + m)\gamma_\lambda v(p_+)}{(p_+ - q_2)^2 - \delta^2}$$

$$= f_{K\mu^+\mu^-}\, \alpha\, \frac{4m^2 - 2M^2}{4M^2} \ln \frac{-\delta^2}{m^2 - M^2 - \delta^2}\, \bar{u}\gamma_5 v$$

$$= f_{K\mu^+\mu^-}\, \alpha(5.6)\, \bar{u}\gamma_5 v .$$

Here $f_{K\mu^+\mu^-}\bar{u}\gamma_5 v$ is the real part of the amplitude $K_L \to \mu^+\mu^-$. Using the real part $(K_L \to \mu^+\mu^-)$ of Sehgal[2] (for a cutoff $\Lambda = 1000$ MeV) the ratio of the above absorptive part to the Sehgal amplitude is found to be $\sim +4.6\alpha$.

The absorptive part of the 2γ intermediate state [Fig. 5(b)] can be written as

$$A_4 = i\frac{f_{K\gamma\gamma}}{M} e^4 \int \frac{d^4k\, d^4q_2}{(2\pi)^8}(2\pi^2)\epsilon_{\mu\nu\alpha\beta}\, k_\alpha P_\beta \delta(k^2)\delta((P-k^2))\, \frac{\bar{u}(p_-)\gamma_\delta(\not{P} - \not{q}_2 + m)\gamma_\mu(\not{k} - \not{q}_2 + m)\gamma_\nu(-\not{q}_2 + m)\gamma_\delta v(p_+)}{(q_2{}^2 - m^2)[(P-q_2)^2 - m^2][(p_+ - q_2)^2 - \delta^2]} .$$

After carrying out the k integration and expressing q_2 in terms of P and $q = \frac{1}{2}(q_2 - q_1)$ we get

$$A_4 = \frac{if_{K\gamma\gamma}}{(2\pi)^4 8\pi^2 M}\left\{ \int \frac{d^4q\,\bar{u}(p_-)\gamma_\delta(\frac{1}{2}\not{P} - \not{q} + m)\gamma_\mu(-\frac{1}{2}\not{P} - \not{q} + m)\gamma_\nu y q_\alpha P_\beta}{[(\frac{1}{2}P+q)^2 - m^2][(\frac{1}{2}P-q)^2][(p_+ - \frac{1}{2}P - q)^2 - \delta^2]}\, \epsilon_{\mu\nu\alpha\beta}(-\not{q} - \tfrac{1}{2}\not{P} + m)\gamma_\delta v(p_+) \right.$$

$$\left. + \int \frac{d^4q\,\bar{u}(p_-)\gamma_\delta(\frac{1}{2}\not{P} - \not{q} + m)\gamma_\mu(\frac{1}{2}\not{P}q_\alpha b + \not{q}q_\alpha b + \not{q}q_\alpha c + \gamma_\alpha d)}{[(\frac{1}{2}P+q)^2 - m^2][(\frac{1}{2}P-q)^2 - m^2][(p_+ - \frac{1}{2}P - q)^2 - \delta^2]}\, \gamma_\nu P_\beta \epsilon_{\mu\nu\alpha\beta}(-\not{q} - \tfrac{1}{2}\not{P} + m)\gamma_\delta v(p_+) \right\} ,$$

where y, b, c, and d are constants obtained in the k integration in terms of P and q.[11] Some useful relations involving these are the following:

$$|\vec{q}|^2 y = \frac{\pi}{8}\left(\frac{2}{B} - \frac{A}{B^2}\ln\frac{A+B}{A-B}\right) M |\vec{q}| \qquad (1)$$

and

$$|\vec{q}|^2(b - 3d) = \frac{\pi}{16} M^2 \frac{1}{B} \ln\frac{A+B}{A-B} , \qquad (2)$$

where

$$A = (\tfrac{1}{2}P + q)^2 - m^2 - M(\tfrac{1}{2}P_0 + q_0)$$

and

$$B = M|\vec{q}| .$$

At this stage, to simplify the q integration, we use the Blankenbecler and Sugar approximation.[12] We replace

$$E = \frac{1}{[(\frac{1}{2}P+q)^2 - m^2][(\frac{1}{2}P-q)^2 - m^2]}$$

by

$$E = i2\pi \int \frac{ds'}{s'-s}\, \delta((-\tfrac{1}{2}P' + q)^2 - m^2)\delta((\tfrac{1}{2}P' - q)^2 - m^2) ,$$

where $s' = P'^2$ and $s = P^2$. Evaluating in the c.m. frame, i.e., $P'^2 = P_0'^2$, we get

$$E = i\pi\delta(q_0)/(|\vec{q}|^2 + m^2)^{1/2}(4|\vec{q}|^2 + 4m^2 - M^2) .$$

This gives us after simplification

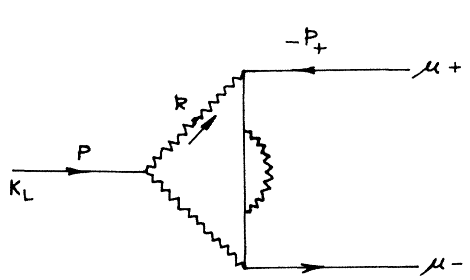

FIG. 2. Lepton propagator correction.

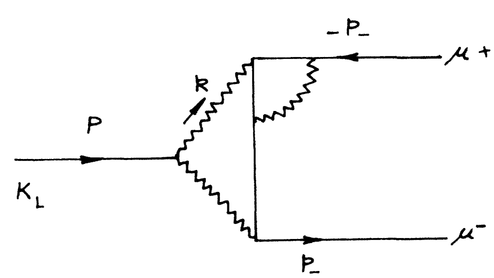

FIG. 3. Vertex correction.

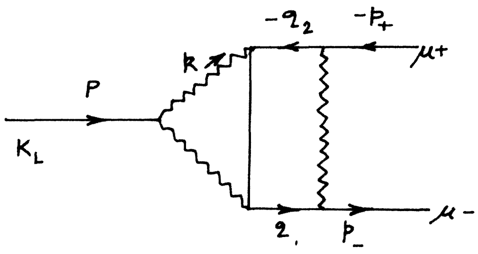

FIG. 4. Box correction.

$$A_4 = \frac{f_{K\gamma\gamma}}{(2\pi)^4 4}\, \frac{2m}{M}\, \bar{u}\, \gamma_5 v \left(\int \frac{\chi_1}{H} \ln \frac{G+H}{G-H} |\vec{q}|^2\, dq - \int \frac{\chi}{H} \ln \frac{G+H}{G-H} |\vec{q}|^2\, dq \right),$$

where

$$G = m^2 - \tfrac{1}{4} M^2 - |\vec{q}|^2 - \delta^2 ,$$

$$H = 2(\tfrac{1}{4} M^2 - m^2)^{1/2} |\vec{q}| ,$$

$$\chi_1 = \frac{\pi}{8}\, \frac{2-(A/B)\ln[(A+B)/(A-B)]}{(|\vec{q}|^2 + m^2)^{1/2}} ,$$

and

$$\chi = \frac{-\tfrac{3}{4} M^2 + m^2 - |\vec{q}|^2}{(|\vec{q}|^2 + m^2)^{1/2} (|\vec{q}|^2 + m^2 - \tfrac{1}{4} M^2)}\, \frac{\pi}{16}\, M^2 \frac{1}{B} \ln \frac{A+B}{A-B} .$$

Calculations show that the contribution of this term is extremely small, the ratio of this term to the Sehgal amplitude being less than 0.1α. Thus the total correction from all the processes is $\sim +11.5\alpha$.

In the above calculations δ was taken of the order of 1 MeV. If we set $\delta \sim 5$ MeV the correction to the Sehgal amplitude turns out to be $\sim +7.5\alpha$.

The correction of $+11.5\alpha$ amounts to about 17% of the lower bound for the decay rate mentioned earlier, while 7.5α amounts to 11%, which is

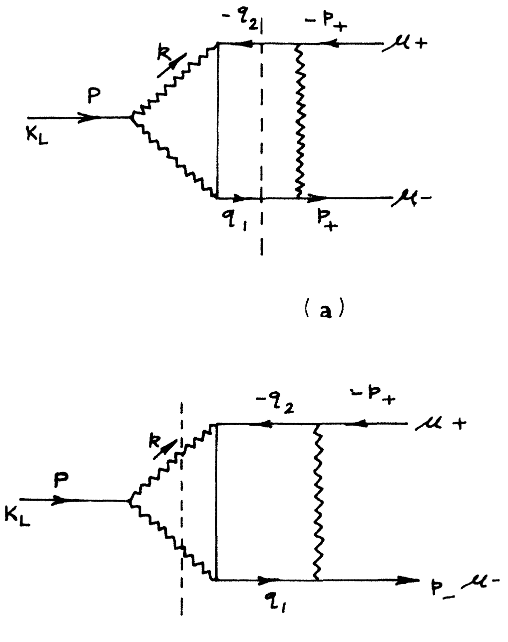

FIG. 5. Two contributions to the absorptive part from the box correction.

quite significant compared with the contribution of the three-particle process (< 10%). It should be emphasized that our calculations include the infrared divergent part of the soft-photon emission.

The thing to be noted here is that these corrections enhance the $K_L \to \mu^+\mu^-$ amplitude. With no definite reliable experimental value for the $K_L \to \mu^+\mu^-$ decay rate available, we really are not in a position to draw any conclusion from these calculations about the existence of neutral lepton currents. All we can say is that these corrections widen the gap between the experimental upper bound of Clark *et al.* and the theoretical lower bound.

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[1]M. A. Baqi Bêg, Phys. Rev. 132, 426 (1963).

[2]L. M. Sehgal, Nuovo Cimento 45, 785 (1966).

[3]L. M. Sehgal, Phys. Rev. 183, 1511 (1969).

[4]B. R. Martin, E. De Rafael, and J. Smith, Phys. Rev. D 2, 179 (1970); M, K. Gaillard, Phys. Lett. 35B, 431 (1971).

[5]A. R. Clark *et al.*, Phys. Rev. Lett. 26, 1667 (1971).

[6]Experiment by a CERN Group (Heidelberg), reported at the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972 (unpublished).

[7]S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, Phys. Rev. D 4, 3495 (1971); N. Christ and T. D. Lee, *ibid.* 4, 209 (1971).

[8]J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), Chap. 9.

[9]The lower limit on ρ is 1.05, however, except for a small region near this point, ρ is large enough so that one can safely replace $(\rho \pm 1)$ by ρ.

[10]We refer to the 2γ-state absorptive part as the Sehgal amplitude. It can be written as $f_{K\gamma\gamma}\alpha(0.168)\bar{u}\gamma_5 v$.

[11]In carrying out the k integration we set

$$\int \frac{d^4k\, k_\alpha \delta(k^2)\delta((P-k)^2)}{q_2^2 + k^2 - 2q_2 \cdot k - m^2} = xP_\alpha + yq_{2\alpha}$$

and

$$\int \frac{d^4k\, k_\alpha k_\lambda \delta(k^2)\delta((P-k)^2)}{q_2^2 + k^2 - 2q_2 \cdot k - m^2} = aP_\alpha P_\lambda + b q_{2\lambda} q_{2\alpha} + c(q_{2\alpha}P_\lambda + q_{2\lambda}P_\alpha) + dg_{\alpha\lambda}.$$

Various relations involving y, a, b, c, and d are obtained in the rest frame of K_L by evaluating the left-hand sides for given α and λ.

[12]R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966). This approximation simplifies $\int d^4q$ to $\int d^3q$ and gives results correct to within 10%.