PHYSICAL REVIEW D VOLUME 10, NUMBER 5 1 SEPTEMBER 1974

Coherent states and particle production. II. Isotopic spin*

J. C. Botke, D. J. Scalapino, and R. L. Sugar
Department of Physics, University of California, Santa Barbara, California 93106
(Received 18 October 1973)

The coherent-state representation is used to discuss the production of isovector pions. Techniques are developed for constructing states of definite isotopic spin and G parity, and a variety of models are discussed. Particular attention is paid to correlations between charged and neutral pions which appear to play an important role in the understanding of the underlying dynamics.

I. INTRODUCTION

The large number of pions produced in high-energy hadron-hadron collisions suggests that it is useful to study the scattering operator and the pion field density operator in the coherent-state representation. We have previously used this representation to discuss models for the production of spinless, isoscalar pions.[1] In that work we also pointed out how our ideas could be generalized to include internal symmetries. Here we develop in detail the techniques which are necessary to discuss the production of isovector pions. We consider a variety of models and obtain expressions for their generating functions. Particular attention is paid to correlations between charged and neutral pions, which play an important role in determining the isospin structure of the underlying dynamics.

While we do not expect that the pion field produced in a high-energy collision will be well represented by a single coherent state, we have suggested in I that coherent states can provide a useful basis for expanding the operators of interest. In fact, using these states we have been able to construct simple parameterizations for the scattering operator and the pion field density operator which describe a wide class of pion distributions ranging from Poisson to Gaussian.[1,2]

In order to construct coherent states of physical pions, it is convenient to introduce creation and annihilation operators which are vectors in isospin space. The creation operators will be written in the form

$$\vec{a}^\dagger(q)=(a_1^\dagger(q),\, a_2^\dagger(q),\, a_3^\dagger(q))\ . \tag{1}$$

The four-momentum q will be written in terms of the rapidity y and the transverse momentum $\vec{q}_\perp$, and the creation and annihilation operators will be normalized so that their commutation relations take on the Lorentz-invariant form

$$[a_i(y,\vec{q}_\perp),\, a_j^\dagger(y',\vec{q}_\perp')]=\delta_{ij}\,\delta(y-y')\,\delta^2(\vec{q}_\perp-\vec{q}_\perp')\ . \tag{2}$$

The physical pions are created by the standard linear combinations

$$a_\pm^\dagger(q)=\mp 2^{-1/2}[a_1^\dagger(q)\pm i a_2^\dagger(q)]\ ,\qquad a_0^\dagger(q)=a_3^\dagger(q)\ . \tag{3}$$

The coherent states are by definition the eigenstates of the annihilation operator

$$\vec{a}(q)|\vec{\Pi}\rangle=\vec{\Pi}(q)|\vec{\Pi}\rangle\ , \tag{4}$$

where the components of $\vec{\Pi}(q)$ are arbitrary complex functions of q. $|\vec{\Pi}\rangle$ can be written in the form

$$|\vec{\Pi}\rangle=\exp\left[-\tfrac{1}{2}\int dq\,|\vec{\Pi}(q)|^2\right]\exp\left[\int dq\,\vec{\Pi}(q)\cdot\vec{a}^\dagger(q)\right]|0\rangle\ . \tag{5}$$

Here $|0\rangle$ is the vacuum state defined by $\vec{a}(q)|0\rangle=0$, and $dq\equiv dy\,d^2q_\perp$ is the invariant phase-space volume element. The simple coherent state given by Eq. (5) is not an eigenstate of total charge, total isotopic spin, or G parity. However, states with definite internal quantum numbers can be generated from the coherent states in a variety of ways.

Let us first consider making a direct projection on a single coherent state. We denote by $U(\alpha,\beta,\gamma)$ the unitary operator which produces a rotation in isospin space through the Euler angles α, β, and γ. Then,

$$U(\alpha,\beta,\gamma)|\vec{\Pi}\rangle=|R^{-1}(\alpha,\beta,\gamma)\vec{\Pi}\rangle\ , \tag{6}$$

where $R(\alpha,\beta,\gamma)$ is the usual three-dimensional rotation matrix. The projection onto the state of total isotopic spin I and z component M is given by

$$|\vec{\Pi};I,M\rangle=\sum_\lambda\int d\Omega_{\alpha\beta\gamma}D^I_{M,\lambda}(\alpha,\beta,\gamma)^*\times|R^{-1}(\alpha,\beta,\gamma)\vec{\Pi}\rangle\ , \tag{7}$$

where $D^I(\alpha,\beta,\gamma)$ is the Wigner rotation matrix and $d\Omega_{\alpha\beta\gamma}\equiv d\alpha\,d\cos\beta\,d\gamma$. $|\vec{\Pi};I,M\rangle$ is not a state of definite G parity, but since

$$G|\vec{\Pi}\rangle=|-\vec{\Pi}\rangle \tag{8}$$

one can always construct such states by taking appropriate linear combinations of $|\vec{\Pi}; I,M\rangle$ and $|-\vec{\Pi}; I,M\rangle$. One can of course employ the same technique on a general pion state by expanding it in the coherent-state representation.

An alternative approach, which was outlined in I, is to use functional integration techniques to construct superpositions of coherent states with definite internal quantum numbers. Let us write

$$|\psi\rangle = \int \delta\vec{\Pi}\, P(\vec{\Pi})|\vec{\Pi}\rangle\,, \tag{9}$$

where $\int\delta\vec{\Pi}$ indicates an integral over all functional forms for $\vec{\Pi}(q)$ subject to a proscribed set of boundary conditions. If the weight function $P(\vec{\Pi})$ is rotationally invariant, then the isospin and G parity of the state $|\psi\rangle$ are determined by the boundary conditions on $\vec{\Pi}$. For example, if these boundary conditions are invariant under rotation and reflection then a state of zero isotopic spin and even G parity is produced. More general states can be generated by modifying the boundary conditions.

In Sec. II we study the properties of states of the form given in Eqs. (7) and (9). Particular attention is given to two limiting cases: the "global," in which the isospins of pions over the whole rapidity range are coupled to give a definite value of I, and the "local," in which the isospins of only those pions which are close together in rapidity space are coupled. The global state is of the type that one would obtain in the uncorrelated-jet model.[3] It is an example of a model with long-range correlations. The local state is similar to what one would obtain the multiperipheral model,[4] and contains only short-range correlations.[5]

In the models discussed in Sec. II it is assumed that few particles other than pions are produced, and that these other particles have little effect on the final state of the pions. A different approach is discussed in Sec. III. If a high-energy hadron-hadron collision involves a large number of underlying degrees of freedom which couple strongly to the pion field, then a statistical theory in which a pion field density matrix is introduced appears to be useful.[2] Because of the overcompleteness of the coherent states, the density matrix can always be written in the diagonal form[1,6]

$$\rho = \int \delta\vec{\Pi}\,|\vec{\Pi}\rangle \frac{e^{-F(\vec{\Pi})}}{Z}\langle\vec{\Pi}|\,. \tag{10}$$

Here Z is a normalization factor and $F(\vec{\Pi})$ is a functional which characterizes the pion field. In general, $F(\vec{\Pi})$ can be singular; however, a wide class of possible pion distributions can be described by a simple, nonsingular parameterization. In order to discuss the production of pions in a state of definite isospin we again apply projection techniques. Choosing $F(\vec{\Pi})$ to be a scalar functional of $\vec{\Pi}$, the projection of the density matrix ρ onto the manifold with total isospin I and z component M is given by

$$\rho_{IM} = \frac{1}{Z_{I,M}} \sum_{\lambda} \int d\Omega_{\alpha\beta\gamma} D^{I}_{M\lambda}(\alpha,\beta,\gamma)^{*} \times \int \delta\vec{\Pi}\,|R^{-1}(\alpha,\beta,\gamma)\vec{\Pi}\rangle e^{-F(\vec{\Pi})}\langle\vec{\Pi}|\,. \tag{11}$$

Here Z_{IM} normalizes $\mathrm{tr}\rho_{IM}$ to unity. It is possible to extend this procedure to construct density matrices which describe a phase-coherent linear combination of several isospin manifolds.

In Sec. III we study the isospin structure of a statistical model of pion production whose density matrix has the form given in Eq. (11). The results are qualitatively similar to the short-range correlation models of Sec. II.

A particularly useful probe of the isospin structure of all of these models is provided by the study of correlations among charged and neutral pions. Two quantities for which experimental data exist are the charge-neutral correlation moment,

$$f_{c0} = \langle n_0 n_c\rangle - \langle n_0\rangle\langle n_c\rangle\,, \tag{12}$$

and the average number of π_0's produced for a given number of charged pions, $\langle n_0(n_c)\rangle$. We shall concentrate our attention on them. In all models that we have studied and in any model with only short-range correlations, $\langle n_0(n_c)\rangle$ can be written in the form

$$\langle n_0(n_c)\rangle = An_c + BY + C + O(n_c/Y)\,, \tag{13}$$

where A, B, and C approach constants independent of n_c at high energies. Y is the rapidity difference between the incident particles, $Y\approx \ln s$. It will be helpful to keep in mind that experimentally[7] f_{c0} and A are both positive. A is increasing at low energies, but may well be approaching a constant limit at high energies. B, on the other hand, appears to be small.

II. ISOSPIN-ZERO STATES

In this section we shall consider a class of models in which the produced pions are described by a single state vector, $|\psi\rangle$, of the type described in Eqs. (7) and (9). In these models we can write the semi-inclusive cross section for the production of a given number of charged and neutral pions in the form

$$\sigma(n_+, n_-, n_0)/\sigma \equiv P(n_+, n_-, n_0) = (n_+!\, n_-!\, n_0!)^{-1} \int dq_1 \cdots dq_{n_+}\, dp_1 \cdots dp_{n_-}\, dr_1 \cdots dr_{n_0} \times |\langle 0| a_+(q_1) \cdots a_+(q_{n_+})\, a_-(p_1) \cdots a_-(p_{n_-})\, a_0(r_1) \cdots a_0(r_{n_0}) |\psi\rangle|^2 / \langle\psi|\psi\rangle . \tag{14}$$

Here σ is the total cross section and $n_\pm, n_0$ denote the number of $\pi^\pm$, π^0 produced. We imagine that the energy-momentum-conservation δ functions have been evaluated in integrating over the phase space of particles other than pions.[1] These δ functions of course place constraints on the momenta of the pions, the most important being that in the laboratory system the rapidities of the pions are restricted to the approximate range $0 < y < Y$, where $Y \approx \ln s$ is the rapidity of the incident particle.

The quantities in which we shall be interested can be most easily obtained from the generating function

$$\Omega(z_+, z_-, z_0) = \sum_{n_\pm, n_0} z_+^{n_+} z_-^{n_-} z_0^{n_0} P(n_+, n_-, n_0) . \tag{15}$$

Since any pion state $|\psi\rangle$ can be expressed in the form of Eq. (9) we see from Eqs. (4), (5), and (14) that

$$\Omega(z_+, z_-, z_0) = I(z_+, z_-, z_0)/I(1, 1, 1), \tag{16}$$

where

$$I(z_+, z_-, z_0) = \int \delta\vec{\Pi}\, \delta\vec{\Pi}'\, P(\vec{\Pi}) P(\vec{\Pi}')^* \exp\left\{ -\int dq \left[\tfrac{1}{2}|\vec{\Pi}(q)|^2 + \tfrac{1}{2}|\vec{\Pi}'(q)|^2 - \vec{\Pi}'^*(q)\cdot Z \cdot \vec{\Pi}(q) \right] \right\} , \tag{17}$$

and the matrix Z is given by

$$Z = \begin{bmatrix} \tfrac{1}{2}(z_+ + z_-) & 0 & \frac{1}{2i}(z_+ - z_-) \\ 0 & z_0 & 0 \\ \frac{1}{2i}(z_- - z_+) & 0 & \tfrac{1}{2}(z_+ + z_-) \end{bmatrix} . \tag{18}$$

If one neglects the production of particles other than pions, then in p-p and π-p collisions $|\psi\rangle$ can be at most a linear combination of states of isospin 0, 1, and 2. In order to present our techniques in the simplest possible setting we shall limit ourselves to the study of $I = 0$ states. In this case all partial cross sections except those with $n_+ = n_- \equiv \frac{1}{2} n_c$ vanish, and one sees from Eq. (15) that it is possible to set $z_+ = z_- \equiv z_c$ without loss of any information. Z then becomes a diagonal matrix. Although the restriction to $I = 0$ states does simplify the calculations, there is no real difficulty in including states with higher values of isospin.[3]

As a first example let us consider a state obtained by taking the isospin-zero projection of a single coherent state. In the uncorrelated-jet model[3] one expects the isospins of all pions, regardless of their momenta, to be coupled to form the total isospin. This can be accomplished by considering coherent states with $\vec{\Pi}(q)$ of the form

$$\vec{\Pi}(q) = \Pi(q)\, \hat{e} , \tag{19}$$

where $\hat{e}$ is a unit vector independent of q. Taking the z axis along $\hat{e}$, one sees from Eq. (7) that the required state is

$$|\Pi; 0\rangle = \frac{1}{2\pi} \int d\Omega_{\alpha\beta\gamma} |\Pi R^{-1}(\alpha, \beta, \gamma)\, \hat{e}\rangle = \int d\Omega_{\hat{n}} |\Pi \hat{n}\rangle , \tag{20}$$

where

$$\hat{n} = \left(\frac{-\sin\beta}{\sqrt{2}} e^{-i\alpha}, \cos\beta, \frac{\sin\beta}{\sqrt{2}} e^{i\alpha} \right) \tag{21}$$

and $d\Omega_{\hat{n}} = d\alpha\, d\cos\beta$. We shall refer to the model obtained from $|\Pi; 0\rangle$ as global. In order for this model to have any chance of fitting the experimental data $\Pi(q)$ must fall off rapidly with $\vec{q}^{\,2}$ and be a smooth function of y in the central region, $0 < y < Y$, going to zero for y outside this interval. As a result,

$$c = \int dq |\Pi(q)|^2 \tag{22}$$

will grow linearly with Y.

The generating function for the global model is given by Eq. (16) with

$$I(z_c, z_0) = \int d\Omega_{\hat{n}}\, d\Omega_{\hat{n}'}\, e^{c\hat{n}\cdot z \cdot \hat{n}'} . \tag{23}$$

With a slight amount of algebra one easily finds that

$$\langle n\rangle = \left(\frac{\partial}{\partial z_c} + \frac{\partial}{\partial z_0} \right) I(1, 1) \Big/ I(1, 1) = c \coth(c) - 1 \simeq c - 1 = 3\langle n_0\rangle = \tfrac{3}{2}\langle n_c\rangle , \tag{24}$$

$$f_{c0}=\langle n_c n_0\rangle-\langle n_c\rangle\langle n_0\rangle \simeq -\tfrac{2}{45}\left[2\langle n\rangle^2+3\langle n\rangle-3\right], \tag{25}$$

and

$$\langle n_0(n_c)\rangle=\frac{\partial}{\partial z_0}\frac{\partial^{n_c}}{\partial z_c^{n_c}}I(0,1)\Big/\frac{\partial^{n_c}}{\partial z_c^{n_c}}I(0,1) \simeq\langle n\rangle-(n_c+1). \tag{26}$$

Notice that Eqs. (25) and (26) are in disagreement with present trends in the data which indicate that both f_{c0} and $\partial\langle n_0(n_c)\rangle/\partial n_c$ are positive at high energies.[7] One can generalize this model by introducing suitably weighted functional integrals over $\vec{\Pi}(q)$ and by including states of higher isospin. However, it does not appear to be possible to arrange for $\partial\langle n_0(n_c)\rangle/\partial n_c$ to be positive. The reason is that in this type of model the probability of producing n particles is sharply peaked around $\langle n\rangle$ and is essentially independent of the ratio of charged to neutral particles. As a result, an increase in the number of charged particles always leads to a decrease in the number of neutrals.

Let us now turn to a model which has only short-range correlations. For simplicity we neglect the dependence on the transverse momenta and write

$$|\psi\rangle=\int\delta\vec{\Pi}\exp\left\{-\int_0^Y dy\left[c\,\frac{\partial\vec{\Pi}(y)}{\partial y}\cdot\frac{\partial\vec{\Pi}(y)}{\partial y}+V(\Pi(y))\right]\right\}|\vec{\Pi}\rangle. \tag{27}$$

Here $\int\delta\vec{\Pi}$ indicates a functional integral over all *real* fields $\vec{\Pi}(y)$. The reason for requiring $\vec{\Pi}(y)$ to be real will become apparent below. We shall take $\vec{\Pi}(y)$ to be free on the boundaries $y=0, Y$. Since these boundary conditions are invariant under rotation and reflection, $|\psi\rangle$ will be a state of isospin zero and positive G parity provided the effective potential, V, is a function of $\vec{\Pi}^2(y)$ only. The term involving $\partial\vec{\Pi}/\partial y$ introduces short-range correlations in rapidity. The utility of this parameterization of $|\psi\rangle$ arises from the fact that it is possible to find simple forms for $V(\vec{\Pi}^2)$ which describe a wide class of pion states ranging from incoherent to highly coherent.

In order to understand the content of this state, consider the probability amplitude for finding n pions with rapidities $y_1,\ldots,y_n$ and isospin indices $i_1,\ldots,i_n$:

$$A(y_1,i_1;\ldots;y_n,i_n)=\langle 0|a_{i_1}(y_1)\cdots a_{i_n}(y_n)|\psi\rangle\langle\psi|\psi\rangle^{-1/2}$$
$$=\int\delta\vec{\Pi}\,\Pi_{i_1}(y_1)\cdots\Pi_{i_n}(y_n)\exp\left\{-\int_0^Y dy\left[c\,\frac{\partial\vec{\Pi}}{\partial y}\cdot\frac{\partial\vec{\Pi}}{\partial y}+V(\vec{\Pi}^2)+\tfrac{1}{2}\Pi^2\right]\right\}\langle\psi|\psi\rangle^{-1/2}, \tag{28}$$

where the factor $\langle\psi|\psi\rangle^{-1/2}$ has been introduced because our state is not normalized. This functional integral is most easily performed by first finding the energy levels of the Schrödinger equation[1,2]

$$\left[-\frac{1}{4c}\nabla_\Pi^2+V(\vec{\Pi}^2)+\tfrac{1}{2}\vec{\Pi}^2\right]\psi_l(\vec{\Pi})=\epsilon_l\,\psi_l(\vec{\Pi}), \tag{29}$$

where

$$\nabla_\Pi^2\equiv\frac{\partial^2}{\partial\Pi_1^2}+\frac{\partial^2}{\partial\Pi_2^2}+\frac{\partial^2}{\partial\Pi_3^2}.$$

Because of the rotational invariance of the effective Hamiltonian, the ψ_l will be states of definite isotopic spin. Following the techniques of Ref. 2, one finds that

$$A(y_1,i_1;\ldots;y_n,i_n)=\sum_{l_1,\ldots,l_{n+1}}G_{l_1}\exp(-y_1\epsilon_{l_1})g^{i_1}_{l_1,l_2}\exp[-(y_2-y_1)\epsilon_{l_2}]\cdots$$
$$\times\exp[-(Y-y_n)\epsilon_{l_{n-1}}]\,G_{l_{n+1}}\langle\psi|\psi\rangle^{-1/2}, \tag{30}$$

where

$$G_l=\int d^3\Pi\,\psi_l(\vec{\Pi}) \tag{31}$$

and

$$g^i_{l,l'}=\int d^3\Pi\,\psi_l(\vec{\Pi})\,\Pi_i\,\psi_{l'}(\vec{\Pi}). \tag{32}$$

Equation (30), of course, has just the form that one would obtain in a one-dimensional multi-Regge model. Notice that if we had allowed $\vec{\Pi}(y)$ to be complex, the imaginary part of $\vec{\Pi}(y)$ would have given rise to trajectories with negative residues. One sees directly from Eq. (31) that because of our choice of free boundary conditions, no isotopic spin is transferred to the pion field at the boundaries $y=0, Y$. Clearly, by generalizing the boundary conditions one can allow for such a transfer.

As always, the generating function is given by Eqs. (16) and (17). To compute it one must solve for the energy levels of the Hamiltonian

$$H=-\frac{1}{4c}(\nabla_\Pi{}^2+\nabla_{\Pi'}{}^2)+V(\vec{\Pi}^2)+V(\vec{\Pi}'^2)+\tfrac{1}{2}(\vec{\Pi}^2+\vec{\Pi}'^2)-\vec{\Pi}\cdot Z\cdot\vec{\Pi}'. \qquad (33)$$

Working to leading power in $s\approx e^Y$ one finds

$$\Omega(z_c,z_0)=\frac{\exp\{-[\epsilon_0(z_c,z_0)-\epsilon_0(1,1)]Y\}G_0{}^2(z_c,z_0)}{G_0{}^2(1,1)}, \qquad (34)$$

with

$$G_0(z_c,z_0)=\int d^3\Pi\, d^3\Pi'\,\psi_0(\vec{\Pi},\vec{\Pi}';z_c,z_0). \qquad (35)$$

Here $\epsilon_0(z_c,z_0)$ and $\psi_0(\vec{\Pi},\vec{\Pi}';z_c,z_0)$ are the ground-state energy level and wave function.

As a first example, take $V(\vec{\Pi}^2)=a\vec{\Pi}^2$, which gives rise to a Gaussian distribution of pions. In this case H is the Hamiltonian for a pair of coupled three-dimensional oscillators. After making the change of variables

$$\vec{\Pi}^{\pm}=2^{-1/2}(\vec{\Pi}\pm\vec{\Pi}'), \qquad (36)$$

one easily finds that

$$\epsilon_0(z_c,z_0)=\tfrac{1}{2}\sum_{i=1}^{3}\{[(a+\tfrac{1}{2}+\tfrac{1}{2}z_i)/c]^{1/2}+[(a+\tfrac{1}{2}-\tfrac{1}{2}z_i)/c]^{1/2}\} \qquad (37)$$

and

$$\psi_0=(2c^{1/2}/\pi)^{3/2}\prod_{i=1}^{3}[(a+\tfrac{1}{2})^2-\tfrac{1}{4}z_i^2]^{1/8}\times\exp\{-(\Pi_i^+)^2[(a+\tfrac{1}{2}+\tfrac{1}{2}z_i)c]^{1/2}\}\times\exp\{-(\Pi_i^-)^2[(a+\tfrac{1}{2}-\tfrac{1}{2}z_i)c]^{1/2}\}, \qquad (38)$$

where $z_1=z_2=z_c$, $z_3=z_0$. Since H does not contain any coupling between the charged and neutral fields, the generating functions can be written in product form, $\Omega(z_c,z_0)=\Omega_c(z_c)\Omega_0(z_0)$. As a result, $f_{c0}=0$ and $\langle n_0(n_c)\rangle$ is independent of n_c.

In order to introduce correlations between charged and neutral pions, one must include terms in $V(\vec{\Pi}^2)$ which couple the charged and neutral components of $\vec{\Pi}$. The simplest generalization of $V(\vec{\Pi}^2)$ would be to add a term of the form $b(\vec{\Pi}^2)^2$. However, since the energy levels of the anharmonic oscillator are not analytic functions of b at $b=0$, it is more convenient to consider a potential which grows less rapidly for large values of $\vec{\Pi}^2$. As an example, let us take

$$V(\vec{\Pi}^2)=a\vec{\Pi}^2+b(\vec{\Pi}^2)^2[1+d(\vec{\Pi}^2)^2]^{-1}. \qquad (39)$$

For small values of b one can use first-order perturbation theory to calculate the shift in the ground state energy from the harmonic oscillator value given in Eq. (37). Taking d to be small also, one has

$$\begin{aligned}\Delta\epsilon_0(z_c,z_0)&=b\int d^3\Pi\, d^3\Pi'|\psi_0|^2\{(\vec{\Pi}^2)^2[1+d(\vec{\Pi}^2)^2]^{-1}+(\vec{\Pi}'^2)^2[1+d(\vec{\Pi}'^2)^2]^{-1}\}\\&\simeq b\int d^3\Pi\, d^3\Pi'|\psi_0|^2[(\vec{\Pi}^2)^2+(\vec{\Pi}'^2)^2]\\&=\tfrac{1}{4}b\left[\sum_{i=1}^{3}(\alpha_i^++\alpha_i^-)^2+\tfrac{1}{2}\sum_{i,j=1}^{3}(\alpha_i^+\alpha_j^++\alpha_i^-\alpha_j^-+2\alpha_i^+\alpha_j^-)\right],\end{aligned} \qquad (40)$$

where ψ_0 is given in Eq. (38) and

$$\alpha_i^{\pm}=[c(a+\tfrac{1}{2}\pm\tfrac{1}{2}z_i)]^{-1/2}. \qquad (41)$$

Working to leading order in Y one then finds

$$f_{c0}=-(b/32c)[a^{-3/2}-(a+1)^{-3/2}]^2Y, \qquad (42)$$

and

$$\frac{\partial\langle n_0(n_c)\rangle}{\partial n_c}=-bc^{-1/2}\tfrac{3}{8}(a+\tfrac{1}{2})^{-1}\times[a^{-3/2}-(a+1)^{-3/2}]. \qquad (43)$$

The point we wish to emphasize is the correlation between the signs of f_{c0} and $\partial\langle n_0(n_c)\rangle/\partial n_c$. If one wishes to have both be positive, as is indicated by the data, then b must be negative. In other words the coupling between the charged and neutral components of the pion field must correspond to an attractive term in the effective Hamiltonian.

One can obviously generalize this model by considering more sophisticated forms for the effective Hamiltonian; however, our primary purpose here is to illustrate techniques rather than to present a finished model. As a result, we have not attempted to make a detailed fit to the data.

III. THE DENSITY MATRIX

In the previous section we considered models in which the pions came off in a definite final state, so the pion field density matrix was separable.[1] We now consider a different class of models. In computing an inclusive or semi-inclusive cross section for pion production one must integrate over the coordinate of all other produced particles. If they are strongly coupled to the pion field, then the pion density matrix will not be even approximately separable. Under these circumstances a statistical approach, in which the pion density matrix is taken to be diagonal in the coherent-state representation, is likely to be most useful.[1,2]

The inclusive cross section for the production of n pions with rapidities $y_1, \ldots, y_n$ and isospin indices $i_1, \ldots, i_n$ can be written[1]

$$\frac{1}{\sigma}\frac{d^n\sigma_{i_1\cdots i_n}}{dy_1\cdots dy_n} = \mathrm{tr}[\rho_{IM}\, a^\dagger_{i_1}(y_1)\cdots a^\dagger_{i_n}(y_n)\, a_{i_1}(y_1)\cdots a_{i_n}(y_n)]\,. \tag{44}$$

Using the density matrix given in Eq. (11), for $I=0$, Eq. (44) takes the form

$$\frac{1}{\sigma}\frac{d^n\sigma_{i_1\cdots i_n}}{dy_1\cdots dy_n} = \int d\Omega_{\alpha\beta\gamma}\int\delta\vec{\Pi}\exp\left[-\left(F-\int_0^Y dy\,\vec{\Pi}^*(R^{-1}-1)\vec{\Pi}\right)\right][\vec{\Pi}^*_{i_1}(y_1)(R^{-1}\vec{\Pi}(y_1))_{i_1}\cdots\vec{\Pi}^*_{i_n}(y_n)(R^{-1}\vec{\Pi}(y_n))_{i_n}]\,. \tag{45}$$

As in the short-range correlation models of Sec. II, we have neglected the transverse momentum.

Constructing a generating function from the inclusive cross sections in the usual way leads for large Y to the form

$$\Omega(z_c, z_0) = \frac{J(z_c, z_0)}{J(1,1)}\,, \tag{46}$$

where $J(z_c, z_0)$ is given by

$$J(z_c, z_0) = \int d\Omega_{\alpha\beta\gamma}\int\delta\vec{\Pi}\exp\left[-\left(F-\int_0^Y dy\,\vec{\Pi}^*(ZR^{-1}-1)\vec{\Pi}\right)\right]. \tag{47}$$

The matrix Z appearing in the exponential is just that previously defined in Sec. II, Eq. (18), with $z_+=z_-=z_c$. In the spherical basis, the inverse of the three-dimensional rotation matrix is

$$R^{-1} = \begin{bmatrix} \frac{1}{2}(1+\cos\beta)e^{i(\alpha+\gamma)} & \frac{\sin\beta}{\sqrt{2}}e^{i\gamma} & \frac{1}{2}(1-\cos\beta)e^{i(\gamma-\alpha)} \\ \frac{-\sin\beta}{\sqrt{2}}e^{i\alpha} & \cos\beta & \frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} \\ \frac{1}{2}(1-\cos\beta)e^{-i(\gamma-\alpha)} & \frac{-\sin\beta}{\sqrt{2}}e^{-i\gamma} & \frac{1}{2}(1+\cos\beta)e^{-i(\alpha+\gamma)} \end{bmatrix}. \tag{48}$$

Just as in Sec. II, the functional integration in Eq. (47) to leading order in Y becomes

$$J(z_c, z_0) = \int d\Omega_{\alpha\beta\gamma}\exp[-\epsilon_0(z_c, z_0;\alpha,\beta,\gamma)Y] \times G_0^2(z_c, z_0;\alpha,\beta,\gamma)\,. \tag{49}$$

As an example of this formalism, let us again consider the harmonic oscillator model for which

$$F(\Pi) = \int_0^Y dy\left(a\vec{\Pi}^*\cdot\vec{\Pi} + c\,\frac{d\vec{\Pi}^*}{dy}\cdot\frac{d\vec{\Pi}}{dy}\right), \tag{50}$$

and choose for the complex vector field the boundary condition[2]

$$\left.\frac{\partial\vec{\Pi}}{\partial y}\right|_{0,Y} = 0\,. \tag{51}$$

For f_{c0}, the parameters z_c and z_0 are near unity. In this case, ϵ_0 can be most simply determined as the perturbed ground-state eigenvalue of the zero-order Hamiltonian

$$H_0 = -\frac{1}{4c}\nabla_\Pi{}^2 + (a+1)\vec{\Pi}^*\cdot\vec{\Pi} - \vec{\Pi}^*\cdot Z\cdot\vec{\Pi} \tag{52}$$

with the perturbation

$$V = \vec{\Pi}^*\cdot Z(1-R^{-1})\cdot\vec{\Pi}\,. \tag{53}$$

The ground-state energy of H_0 is

$$\epsilon_0^{(0)}=[(a+1-z_0)/c]^{1/2}+2[(a+1-z_c)/c]^{1/2}. \quad (54)$$

The shift in the ground-state energy due to V is, in first order,

$$\Delta\epsilon_0^{(1)}=\int d^6\Pi\,|\psi_0|^2\vec{\Pi}^*\cdot Z(1-R^{-1})\cdot\vec{\Pi}$$
$$=\frac{z_0(1-\cos\beta)}{2[c(a+1-z_0)]^{1/2}}$$
$$+\frac{z_c[2-(1+\cos\beta)\cos(\alpha+\gamma)]}{2[c(a+1-z_c)]^{1/2}}. \quad (55)$$

Here ψ_0 is the ground-state wave function. In view of Eq. (49), for large Y, the regions of $d\Omega_{\alpha\beta\gamma}$ phase space in which the ground-state energy is lowest will dominate. From Eq. (55) we see that this corresponds to $\beta^2 \lesssim 1/Y$ and $\alpha^2+\gamma^2\lesssim 1/Y$. In this region

$$\Delta\epsilon_0^{(1)}\simeq\frac{1}{4}\,\frac{z_0\beta^2}{[c(a+1-z_0)]^{1/2}}+\frac{1}{4}\,\frac{z_c[\beta^2+2(\alpha+\gamma)^2]}{[c(a+1-z_c)]^{1/2}}. \quad (56)$$

Because of R^{-1}, which connects Π_0 with $\Pi_\pm$, there is also a $\sin^2\beta\sim\beta^2$ contribution in second-order perturbation theory. This is simply calculated, giving

$$\Delta\epsilon_0^{(2)}=\frac{z_0 z_c\beta^2}{2}\times\frac{[c(a+1-z_0)]^{-1/2}[c(a+1-z_c)]^{-1/2}}{[(a+1-z_0)/c]^{1/2}+[(a+1-z_c)/c]^{1/2}}. \quad (57)$$

All other contributions to the ground-state energy are of higher order in the angular variables.

Combining these three contributions, the angular integral for J reduces to

$$J(z_c,z_0)=\exp[-\epsilon_0^{(0)}(z_c,z_0)Y]\,\frac{H(z_c,z_0)}{Y^{3/2}}, \quad (58)$$

where $H(z_c,z_0)$ is independent of Y. Therefore, it follows directly from the separability of $\epsilon_0^{(0)}$ that

$$f_{c0}=\frac{\partial}{\partial z_0}\frac{\partial}{\partial z_c}\ln\Omega\bigg|_{z_0=z_c=1}=\text{const}, \quad (59)$$

where the constant is independent of Y.

In order to obtain a contribution to f_{c0} which varies linearly with Y it is necessary, as in Sec. II, to couple the components of the charged and neutral pion fields. Using the interaction discussed in Sec. II [last part of Eq. (39)] one has for small values of b and d a ground-state energy shift given by

$$\int d^6\Pi\,|\psi_0|^2\frac{b(\vec{\Pi}^2)^2}{1+d(\vec{\Pi}^2)^2}\cong\frac{b}{c}\,[(a+1-z_0)^{-1/2}(a+1-z_c)^{-1/2}+\tfrac{1}{2}(a+1-z_0)^{-1}+\tfrac{3}{2}(a+1-z_c)^{-1}]. \quad (60)$$

The first term gives a nonseparable contribution to the coefficient of Y appearing in the exponential of J. Carrying out the differentiation of Eq. (59) then gives

$$f_{c0}=-\frac{bY}{4ca^3}+O(\text{const}). \quad (61)$$

Just as discussed in Sec. II, in order to obtain a positive Y slope for f_{c0} it is necessary to choose $b<0$.

Turning next to the evaluation of $\langle n_0(n_c)\rangle$, one needs the generating function for $z_0\sim 1$ and $z_c\sim 0$. In this case, for the harmonic model, the appropriate unperturbed Hamiltonian is

$$H_0=-\frac{1}{4c}\nabla_{\vec{\Pi}}^2+(a+1)\vec{\Pi}^2-\vec{\Pi}^*\cdot ZR^{-1}(\alpha,0,\gamma)\cdot\vec{\Pi}, \quad (62)$$

and the perturbation is

$$V=\vec{\Pi}^*\cdot Z[R^{-1}(\alpha,0,\gamma)-R^{-1}(\alpha,\beta,\gamma)]\cdot\vec{\Pi}. \quad (63)$$

Proceeding as before, one finds that the zero-order energy is

$$\epsilon_0^{(0)}=\left(\frac{a+1-z_0}{c}\right)^{1/2}+\left(\frac{a+1-z_c e^{i\alpha'}}{c}\right)^{1/2}+\left(\frac{a+1-z_c e^{-i\alpha'}}{c}\right)^{1/2}, \quad (64)$$

where $\alpha'\equiv\alpha+\gamma$. For small z_c this can be expanded as

$$\epsilon_0^{(0)}\simeq\left(\frac{a+1-z_0}{c}\right)^{1/2}+2\left(\frac{a+1}{c}\right)^{1/2}-\frac{z_c\cos\alpha'}{[(a+1)c]^{1/2}}. \quad (65)$$

Computing to order β^2 involves again both first- and second-order perturbation theory. We find

$$\Delta\epsilon_0^{(1)}\cong\frac{\beta^2}{4c^{1/2}}\left[\frac{z_0}{(a+1-z_0)^{1/2}}+2\text{Re}\left(\frac{z_c e^{i\alpha'}}{(a+1-z_c e^{i\alpha'})^{1/2}}\right)\right] \quad (66)$$

and

$$\Delta\epsilon_0^{(2)} = \frac{1}{2}\,\frac{\beta^2 z_0 z_c}{c^{3/2}}\,\mathrm{Re}\left[\frac{e^{i\alpha'}(a+1-z_0)^{-1/2}(a+1-z_c e^{i\alpha'})^{-1/2}}{(a+1-z_0)^{1/2}+(a+1-z_c e^{i\alpha'})^{1/2}}\right]. \tag{67}$$

The β integration in the expression for J can be carried out and one obtains

$$J(z_c, z_0) = e^{-\epsilon(z_0)Y}\int_0^{2\pi} d\alpha'\, e^{Y[c(a+1)]^{-1/2} z_c \cos\alpha'} f(z_c, z_0). \tag{68}$$

Here $\epsilon(z_0) = [(a+1-z_0)/c]^{1/2}$ and $f(z_c, z_0)$ is independent of Y. Note that each factor of z_c, including those in f, is accompanied by $\cos\alpha'$ and that $f(0, z_0) = 1$. The $d\alpha'$ integration projects out just even powers of z_c so that only even n_c values can occur.

In order to calculate $\langle n_0(n_c)\rangle$ we expand the integrand in powers of z_c and carry out the α' integration. This gives for J the expression

$$J(z_c, z_0) = 2\pi e^{-\epsilon(z_0)Y}\sum_{n_c\,(\text{even})}\frac{z_c^{n_c}Y^{n_c}}{2^{n_c}[c(a+1)]^{n_c/2}[(n_c/2)!]^2}\left(1+\frac{n_c}{Y[c(a+1)]^{-1/2}}\,\frac{\partial f}{\partial z_c}\bigg|_{z_c=0}\right) \tag{69}$$

Using this in the expression for $\langle n_0(n_c)\rangle$

$$\langle n_0(n_c)\rangle = \frac{\partial}{\partial z_0}\,\frac{\partial^{n_c}}{\partial z_c^{n_c}} J(0,1) \bigg/ \frac{\partial^{n_c}}{\partial z_c^{n_c}} J(0,1), \tag{70}$$

one finds that

$$\langle n_0(n_c)\rangle = \frac{Y}{2(ac)^{1/2}} + O(1/Y). \tag{71}$$

The coefficient of Y is just $-\partial\epsilon(z_0)/\partial z_0$ and the constant term vanishes.

Including now the interaction $b(\vec{\Pi}^2)^2[1+d(\vec{\Pi}^2)^2]^{-1}$, treated by lowest-order perturbation theory, the energy shift is given by

$$\int d^6\Pi\,|\psi_0|^2\,\frac{b(\vec{\Pi}^2)^2}{1+d(\vec{\Pi}^2)^2} \cong \frac{2b}{c}(a+1-z_0)^{-1/2}[(a+1-z_c e^{i\alpha'})^{1/2}+(a+1-z_c e^{-i\alpha'})^{1/2}]^{-1}. \tag{72}$$

Here we have written out the only term containing both z_0 and z_c. This term leads to a Y-independent contribution to $\langle n_0(n_c)\rangle$ which is proprotional to n_c,

$$\frac{\partial\langle n_0(n_c)\rangle}{\partial n_c} = -\frac{b}{c^{1/2}a^{3/2}(a+1)}. \tag{73}$$

Thus if b is negative, the slope of $\langle n_0(n_c)\rangle$ versus n_c is positive. Just as in the discussion of the states in Sec. II, an attractive, nonlinear term in F is necessary in order to give positive values for f_{c0} and the slope $\partial\langle n_0(n_c)\rangle/\partial n_c$.

IV. DISCUSSION

It is well known that the coherent-state representation is useful for studying the statistics of the radiation field in systems in which the average number of photons is large. Here we have seen that this representation can be used to discuss the statistics of boson fields even when non-Abelian symmetry groups are involved. The calculations involved in the models for the production of isovector pions studied in Secs. II and III are not appreciably more difficult than in the corresponding models for the production of isoscalar bosons discussed in I.

It would appear from the models discussed here that the study of correlations between charged and neutral pions will be particularly important for an understanding of the underlying production dynamics. This point has been emphasized by several authors.[3-5] For example, the global state of Sec. II, which is typical of final states obtained in the uncorrelated-jet model, appears to be ruled out by the sign of $\partial\langle n_0(n_c)\rangle/\partial n_c$. The point is that in this type of state the probability of producing n particles is peaked about $\langle n\rangle$ independent of the charges of the produced particles. As a result, an increase in the number of charged particles always leads to a decrease in the number of neutrals, contrary to the existing data.

The short-range correlation models discussed in Secs. II and III have no difficulty in fitting the data for f_{c0} and $\partial\langle n_0(n_c)\rangle/\partial n_c$. The signs of these two quantities are related. What one learns from the data is that the underlying dynamics must have attractive couplings between the charged and neutral components of the pion field. The present data for $\langle n_0(n_c)\rangle$ are consistent with the coefficient B defined in Eq. (13) being zero. Although B can certainly be made small in our models, it appears to be quite difficult to make it identically zero in them or in any model with only short-range correlations.

The short-range correlation models of Secs. II and III seem capable of fitting the major trends in the multiplicity data with a simple parameterization of the effective potential $V(\vec{\Pi})$. However, we have not attempted to make a detailed fit to the data since our main purpose here has been to develop techniques which can be used in a variety of models. Furthermore, the models which we have discussed have the disadvantage that a self-coupled isovector field seems incapable of reproducing the known spectrum of meson Regge trajectories. A more promising approach seems to be to replace the field $\vec{\Pi}$ in the coherent states by an isovector current made up of quark fields. The functional, F, would then depend on these fields. This approach seems flexible enough to reproduce known Regge trajectories without losing any of the attractive features of the models we have presented here. We hope to return to such a model at a later time.

---

*Work supported by the National Science Foundation.

[1]J. C. Botke, D. J. Scalapino, and R. L. Sugar, Phys. Rev. D 9, 813 (1974), hereafter denoted by I.

[2]D. J. Scalapino and R. L. Sugar, Phys. Rev. D 8, 2284 (1973).

[3]L. J. Reinders, Th. W. Ruijgrok, and D. W. Schlitt (unpublished).

[4]D. Campbell and S.-J. Chang, Phys. Rev. D 8, 2996 (1973); L. Caneschi and A. Schwimmer, Phys. Lett. 33B, 577 (1970); E. L. Berger, D. Horn, and G. H. Thomas, Phys. Rev. D 7, 1412 (1973).

[5]There are numerous discussions of models lying between the two extremes we consider. See, for example, E. L. Berger, CERN Report No. TH-1737 (unpublished); E. L. Berger and G. Fox, Phys. Lett. (to be published); C. Chiu and K.-H. Wang, Phys. Rev. D 8, 2929 (1973); A. Białas, K. Fiałkowski, and K. Zalewski, Jagellonian Univ. report (unpublished); K. Fiałkowski, Rutherford Laboratory Report No. RPP/T/34 (unpublished); R. D. Peccei and S. S. Pinsky, Stanford Report No. ITP-428 (unpublished); W. Frazer, R. D. Peccei, S. S. Pinsky, and C.-I Tan, Phys. Rev. D 7, 2647 (1973).

[6]E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963); J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. 6, 734 (1965); K. E. Cahill, Phys. Rev. 180, 1244 (1969).

[7]Compilations of the data are presented in the following works: F. T. Dao *et al.*, NAL-Conf-73/31-Exp 7200.037; M. Bardadin-Otwinowska *et al.*, Warsaw University and Institute of Nuclear Research report (unpublished); A. Krzywicki, Nucl. Phys. B58, 633 (1973).