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$\pi\pi$ parameters from K_{e4} data and a rigorous representation for $\pi\pi$ amplitudes

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Previous attempts to extract the $\pi\pi$ scattering length a_0 from K_{e4} data have been handicapped by the lack of any one-parameter model for δ_0^0 which is valid over the entire energy range of the data. In this paper, we present a simple but reliable one-parameter model for the $\pi\pi$ S waves and P wave. The model is based on rigorous representations for $\pi\pi$ amplitudes, and is valid from threshold to $M_{\pi\pi} = 400$ MeV or more. We apply the model to all available K_{e4} data, and extract the value of a_0 together with the corresponding values for several other low-energy $\pi\pi$ parameters. We find that the world average of the data yields $a_0 = (0.26 \pm 0.08)\mu^{-1}$, in agreement with the current-algebra prediction of Weinberg. As an appendix, we present in detail the rigorous one-parameter equations for the S waves and the P wave upon which our simple model is based. We make a precise prediction for the $I = 2$ S wave between threshold and 900 MeV, and we predict the P wave for given values of the ρ mass and width.

I. INTRODUCTION AND SUMMARY

The primary purpose of present K_{e4} experiments is to measure the $I=0$ $\pi\pi$ S-wave phase shift δ_0^0, and thereby determine the scattering length a_0. In order to determine a_0 in this way, it is of course necessary to extrapolate δ_0^0 between threshold and the upper limit of the data, i.e., up to about 400 MeV. Previous efforts to determine a_0 have been handicapped by the lack of any one-parameter model for δ_0^0 which is valid over the entire energy range of the data.

In this paper, we present a simple but reliable one-parameter model for the $\pi\pi$ S waves and P wave. The model is based on rigorous representations for $\pi\pi$ amplitudes, and is valid from threshold up to 400 MeV or more. We apply the model to all available K_{e4} data, and extract the value of a_0 together with the corresponding values of several other low-energy $\pi\pi$ parameters. We find that the world average of the data yields $a_0 = (0.26 \pm 0.08)\mu^{-1}$, in agreement with the current-algebra prediction of Weinberg.[1]

In the Appendix, we present in detail the rigorous one-parameter equations for the S waves and the P wave upon which our simple model is based. We make a precise prediction for the $I=2$ S wave between threshold and 900 MeV, and we predict the P wave for given values of the ρ mass and width.

II. NOTATION AND CONVENTIONS

We shall denote the $\pi\pi$ amplitude with isospin I in the s (direct) channel by $A^I(s,t)$, where

$$s = (M_{\pi\pi})^2 ,$$

$$t = \tfrac{1}{2}(s - 4\mu^2)(z - 1) ,$$

with $z \equiv \cos\theta$. We normalize the A^I such that

$$A^I(s,t) = \sum (2l+1) A^{(l)I}(s) P_l(z) ,$$

$$A^{(l)I}(s) = Q^{-1} \exp(i\delta_l^I) \sin\delta_l^I ,$$

$$Q = \left(\frac{s - 4\mu^2}{s}\right)^{1/2} ,$$

where $\mu = 138$ MeV denotes the mean pion mass. As a final remark on conventions, we use units wherein $\hbar = c = 1$.

III. LOW-ENERGY MODELS FOR S WAVES

The $A^{(l)I}$ are analytic in s, except for left and right cuts for $s<0$ and $s>4\mu^2$. Since

$$Q \cot\delta_l^I = [A^{(l)I}]^{-1} + iQ$$

is *real* for $s > 4\mu^2$, it has no right cut and is analytic for $s>0$, *except for poles corresponding to whatever zeros may occur in* $A^{(l)I}$.

Of the four groups[2-5] which have extracted $\pi\pi$ phase shifts from K_{e4} data, only two[2,3] have extrapolated δ_0^0 to threshold to obtain a_0. Beier *et al.*[2] used the scattering-length approximation

$$Q \cot\delta_0^0 \simeq (\mu a_0)^{-1} . \tag{1}$$

Equation (1) has the virtue of simplicity, but the right-hand side contains no pole term, so the approximation must break down near threshold if $A^{(0)0}$ has a zero near threshold. Since $\pi\pi$ S waves are expected to have zeros somewhere near threshold, Eq. (1) may (and indeed does) fail over part of the K_{e4} region.

Zylbersztejn *et al.*[3] used the approximation

$$Q \cot\delta_0^0 \simeq [\mu a_0 + 2b(s-4\mu^2)]^{-1}, \quad b \equiv (32\pi f_\pi^2)^{-1} \simeq 0.022\mu^{-2}, \tag{2}$$

where $f_\pi \approx 95$ MeV denotes the pion decay constant.[6] Equation (2) is a simple unitarization of Weinberg's current-algebra amplitude,[1] wherein no assumption is made about the σ commutator, so that a_0 is a free parameter. Equation (2) has the virtue of accommodating a zero in $A^{(0)0}$, but it is limited by the fact that b is regarded as independent of a_0 (also see Ref. 6). In reality, the optimal value for b depends on the strength of the cuts in $A^{(0)0}$, and both cuts depend on a_0. Furthermore, the right-hand side of Eq. (2) is a pure pole term, whereas one should expect a slowly varying additive term as well.

In this paper, we deal with the aforementioned problems by using an S-wave approximation of the form

$$Q \cot\delta_0^I \simeq \frac{\xi_I}{s-s_I} + \eta_I, \tag{3}$$

where ξ_I, s_I, and η_I are independent of s. The right-hand side of Eq. (3) has a superficial dependence on three unknown parameters, but we shall see that all three can be expressed in terms of a single parameter λ. Toward this end, we next consider rigorous representations for $\pi\pi$ amplitudes.

IV. RIGOROUS REPRESENTATIONS FOR $\pi\pi$ AMPLITUDES

Roskies[7] and Roy[8] have shown that analyticity proven in field theory is sufficient to derive twice-subtracted dispersive representations for the A^I. These representations are valid over a substantial portion of the physical region, and constitute a powerful tool for studying the low-energy $\pi\pi$ interaction.

The two subtraction parameters in the representations of Roskies and Roy are usually chosen as the S-wave scattering lengths a_0 and a_2. However, it seems firmly established that the $\pi\pi$ charge-exchange cross section tends asymptotically to zero [in Regge language, $\alpha_\rho(0)<1$], in which case $(2a_0-5a_2)$ satisfies a well-known sum rule.[9] The resulting relation between a_0 and a_2 is summarized by the "universal curve" of Morgan and Shaw.[10]

Recently the present author[11] has combined the analyticity used by Roskies and Roy with the fact that $\alpha_\rho(0)<1$, and has derived a representation for the A^I in which only one subtraction parameter appears, namely the symmetry-point parameter λ of Chew and Mandelstam.[12] In the same paper,[11] the $\rho\pi\pi$ Regge residue function was determined within about 15% from recent $\pi\pi$ data. Since the asymptotic contributions to the A^I are dominated by Reggeized ρ exchange, we now have theoretical tools and empirical information which are sufficient for constructing reliable low-energy amplitudes in terms of λ.

V. SOLUTIONS FOR LOW-ENERGY AMPLITUDES

We have constructed the S waves and P wave for the cases $\lambda = 0.03$, -0.01, -0.05, and -0.09 (see the Appendix). Upon comparing the solutions with Eq. (3), we find that δ_0^0 can be expressed within[13] $\pm 0.5°$ between threshold and 400 MeV by Eq. (3) with

$$\xi_0 = (17.0 + 42.6\lambda)\mu^2, \tag{4a}$$

$$s_0 = (1.42 + 105.5\lambda)\mu^2, \tag{4b}$$

$$\eta_0 = -0.401 - 2.94\lambda. \tag{4c}$$

Therefore, Eqs. (3) and (4) comprise a one-parameter representation for δ_0^0 which is valid over the K_{e4} region for $0.03 \geq \lambda \geq -0.09$, which corresponds to $0.0 \leq a_0 \leq 1.1\mu^{-1}$. After λ has been determined by fitting the δ_0^0 of Eqs. (3) and (4) to the data, a_0 can be obtained from

$$a_I \simeq \mu^{-1}\left(\frac{\xi_I}{4\mu^2 - s_I} + \eta_I\right)^{-1}, \tag{5}$$

or more simply (within[14] $\pm 0.001\mu^{-1}$) from

$$a_0 \simeq (0.163 - 7.28\lambda + 28.6\lambda^2)\mu^{-1}. \tag{6}$$

To illustrate the limitations of the approximations (1) and (2), we present in Fig. 1 the δ_0^0 generated by Eqs. (1) and (2) for $a_0 = 0.0$, $0.25\mu^{-1}$, and $0.50\mu^{-1}$, together with the δ_0^0 generated by Eqs. (3) and (4). The scattering-length approximation (1) is rather poor above 325 MeV, and the approximation (2) is only fair above 325 MeV.

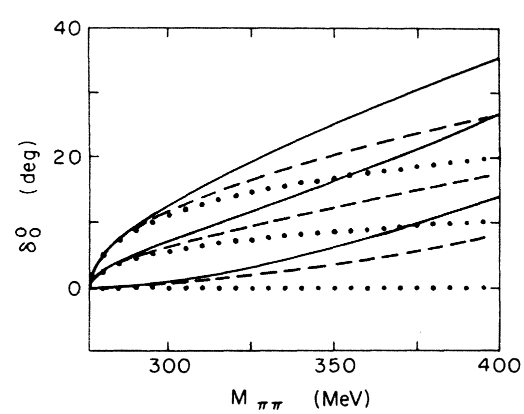

FIG. 1. Different models for δ_0^0 for $a_0 = 0.0$, $0.25\mu^{-1}$, and $0.50\mu^{-1}$. The dotted curves are based on Eq. (1), the dashed ones on Eq. (2), and the solid ones on Eqs. (3) and (4).

Since about half the K_{e4} data lie above 325 MeV, a significant improvement can be made in determinations of a_0 by using Eqs. (3) and (4).

The phase shift actually measured in K_{e4} decay is the difference $(\delta_0^0 - \delta_1^1)$. We find that the P-wave scattering length a_1 is given within[15] 0.5% by

$$a_1 \simeq (0.0378 - 0.148\lambda + 0.26\lambda^2)\mu^{-2}, \tag{7}$$

and that δ_1^1 is given within $\pm 0.3°$ between threshold and 400 MeV by

$$Q \cot\delta_1^1 \simeq \frac{4}{a_1(s - 4\mu^2)}. \tag{8}$$

(A formula for δ_1^1 between threshold and 900 MeV is given in the Appendix.) Equations (7) and (8) imply that δ_1^1 is less that 1° below 350 MeV and less than 2° below 400 MeV, so that its neglect in previous fits to K_{e4} data has been a good approximation. However, we shall include δ_1^1 in the fits of this paper.

Although δ_0^2 is not involved in K_{e4} decay, it does depend on λ, and can therefore be inferred from K_{e4} data. We find that δ_0^2 can be expressed within[16] $\pm 0.5°$ between threshold and 900 MeV by Eq. (3) with

$$\xi_2 = (-45.2 + 43.4\lambda)\mu^2, \tag{9a}$$

$$s_2 = (1.08 - 74.2\lambda)\mu^2, \tag{9b}$$

$$\eta_2 = -0.947 + 2.06\lambda. \tag{9c}$$

The scattering length a_2 is given by Eq. (5) or, within[17] $\pm 0.001\mu^{-1}$, by

$$a_2 \simeq (-0.061 - 1.55\lambda + 0.3\lambda^2)\mu^{-1}. \tag{10}$$

It is worth noting that δ_0^2 depends only weakly on λ.[18] For example, Eqs. (3) and (9) imply that when $M_{\pi\pi} = m_\rho$, $-23° \leq \delta_0^2 \leq -18°$ if $-0.05 \leq \lambda \leq 0.03$, i.e., if a_0 lies in the rather broad range $0.0 \leq a_0 \leq 0.6\mu^{-1}$. Hence the model has strong predictive power for δ_0^2.[18] An unfortunate corollary is that a very precise measurement of δ_0^2 would be required in order for us to gain useful information about λ or a_0.

The symmetry-point derivative parameter[12] λ_1 is given within[19] 1% by

$$\lambda_1 \simeq (0.105 - 0.60\lambda + 1.7\lambda^2)\mu^{-2}. \tag{11}$$

The $\pi\pi$ matrix elements of the σ commutator are measured by the values of A^0 and A^2 at the Dashen-Weinstein point,[20] where $s = 2\mu^2$, $t = \mu^2$. We find within ± 0.001 that

$$A^0(2\mu^2, \mu^2) \simeq 0.036 - 5.23\lambda + 0.9\lambda^2, \tag{12a}$$

$$A^2(2\mu^2, \mu^2) \simeq -0.017 - 1.90\lambda - 0.2\lambda^2. \tag{12b}$$

Having summarized our one-parameter model for the low-energy $\pi\pi$ amplitudes, we proceed now to a discussion of the data.

VI. ANALYSIS OF DATA

Beier *et al.*[2] presented values for $(\delta_0^0 - \delta_1^1)$ at three distinct energies. They recognized that the scattering-length approximation (1) is good only near threshold, and chose to use only the data below 353 MeV in their determination of a_0. Their result (neglecting δ_1^1) was $a_0 = (0.17 \pm 0.13)\mu^{-1}$, and their best fit to the data below 353 MeV is shown in Fig. 2(a).

We have fitted all three data points of Beier *et al.* to the $(\delta_0^0 - \delta_1^1)$ of Eqs. (3), (4), (7), and (8). Our result[21] for a_0 is $a_0 = (0.21 \pm 0.08)\mu^{-1}$, and our best fit is shown in Fig. 2(a). We remark that our curve differs from that of Beier *et al.* by 50% or more above 310 MeV, so it is coincidental that our result for a_0 is so close to theirs. The explanation lies in the fact that the second data point deviates appreciably (though acceptably[22])

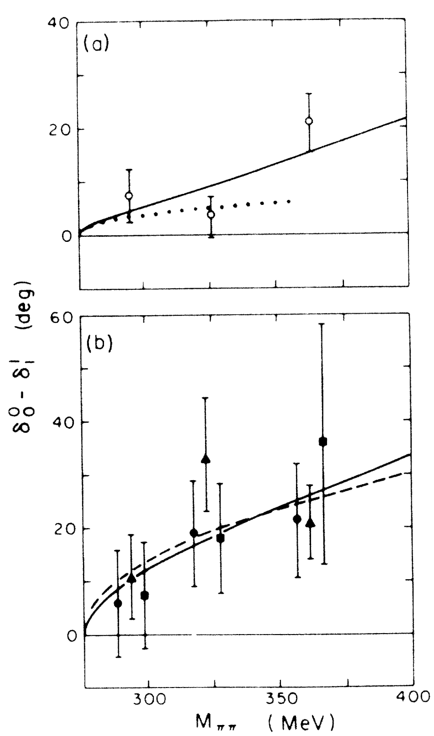

FIG. 2. (a) Data of Beier *et al*. The dotted curve is their best fit of Eq. (1) to the data below 353 MeV. The solid curve is our best fit of Eqs. (3), (4), (7), and (8) to all three data points. (b) Data of Zylbersztejn *et al*. [Triangles: maximum-likelihood method; circles: χ^2 points; squares: Pais-Treiman method. Each datum within each cluster of three points is taken at the energy of the center (triangle) point.] The dashed curve is their best fit of Eq. (2) to the data; the solid curve is our best fit of Eqs. (3), (4), (7), and (8).

from our best fit to all three points.

Zylbersztejn *et al.*[3] determined $(\delta_0^0 - \delta_1^1)$ at three distinct energies, and did so by three different methods. The data are displayed in Fig. 2(b). Their result for a_0 [using Eq. (2) and neglecting δ_1^1] was $a_0 = (0.60 \pm 0.25)\mu^{-1}$, and their best fit is shown in Fig. 2(b). Using the same data, we obtain[23] $a_0 = (0.50^{+0.29}_{-0.26})\mu^{-1}$, and our best fit is also shown in Fig. 2(b).

Schweinberger *et al.*[4] reported an average value for $(\delta_0^0 - \delta_1^1)$ of $(11 \pm 13)^\circ$, but did not state the corresponding energy. Since phase space peaks near 330 MeV, we assume this value for the energy, and obtain $a_0 = (0.24^{+0.42}_{-0.37})\mu^{-1}$.

Ely *et al.*[5] reported an average for $(\delta_0^0 - \delta_1^1)$ of $(25 \pm 9)^\circ$, but did not report the energy. We again use 330 MeV, and obtain $a_0 = (0.70^{+0.39}_{-0.31})\mu^{-1}$.

Within the stated uncertainties, the four values we have deduced for a_0 are mutually consistent, except for the values of Beier *et al.* and Ely *et al.* With regard to this discrepancy, we note that a standard deviation is defined in such a way that one out of every three measured values is expected to lie more than one standard deviation away from the "true value." Since the Ely value is only $1\frac{1}{2}$ (Ely) standard deviations away from the Beier value, this discrepancy is fully consistent with the extent of agreement to be expected among four measurements.[24]

VII. RESULTS FOR $\pi\pi$ PARAMETERS

The weighted average of the four measured values for a_0 is[25]

$$a_0 = (0.26 \pm 0.08)\mu^{-1}. \tag{13a}$$

The weighted average is clearly dominated by the Beier value, because of its relatively small uncertainty.

The weighted average for a_0 corresponds to

$$\lambda = -0.013 \mp 0.010, \tag{13b}$$

which we combine with Eqs. (7), (10), (11), and (12) to conclude that[25]

$$a_1 = (0.040 \pm 0.003)\mu^{-2}, \tag{13c}$$

$$a_2 = (-0.041 \pm 0.016)\mu^{-1}, \tag{13d}$$

$$\lambda_1 = (0.113 \pm 0.007)\mu^{-2}, \tag{13e}$$

$$A^0(2\mu^2, \mu^2) = 0.104 \pm 0.053, \tag{13f}$$

$$A^2(2\mu^2, \mu^2) = 0.008 \pm 0.019. \tag{13g}$$

The uncertainties stated in Eqs. (13a)–(13g) are all correlated, since the value of λ determines all the other parameters. Note that Eqs. (13a)–(13g) are all in good agreement with Weinberg's predictions,[1] and that Eqs. (13f) and (13g) give direct support to his assumption that the σ commutator is predominantly isoscalar.[20]

We wish to emphasize that Eqs. (3), (9), and (13b) imply a very precise curve for δ_0^2 between threshold and 900 MeV. For example, taking all uncertainties into account (see the Appendix), we predict that $\delta_0^2 = (-9.3 \pm 0.8)^\circ$, $(-17.6 \pm 1.6)^\circ$, and $(-24.4 \pm 3.1)^\circ$ at $M_{\pi\pi} = 500$, 700, and 900 MeV, respectively. These values for δ_0^2 are consistent with the values inferred from pion-production data by Walker *et al.*,[26] Colton *et al.*,[27] Cohen *et al.*,[28] Baubillier *et al.*,[29] and Hoogland *et al.*,[30] and lend support to them. Our results are inconsistent for some energies with the δ_0^2 of Baton *et al.*,[31] Katz *et al.*,[32] and Baker,[33] and cast doubt on them. Because of the precision of our results, and because our results are free from the systematic uncertainties characteristic of Chew-Low extrapolations, we regard the combination of theory and experiment described herein as the best determination of δ_0^2 available at present.[34]

ACKNOWLEDGMENTS

It is a pleasure to acknowledge discussions with E. W. Beier and A. K. Mann.

APPENDIX

In this appendix, we discuss the S waves and the P wave implied by Eqs. (3b) and (24) of Ref. 11. Our notation and conventions are identical to those of Ref. 11, except that we now adopt mass units wherein $\mu = 1$.

The $A^{(l)I}(s)$ could of course be obtained from the $A^I(s, t)$ by the standard projection procedure. However, it is advantageous to exploit Bose symmetry and thereby write

$$A^{(l)I}(s) = \int_0^1 dz\, P_l(z) A^I(s, t), \tag{A1}$$

where the relation between s, t, and z has been given in Sec. II.

The advantages of Eq. (A1) derive from the fact that t remains smaller for z on the interval $0 \leq z \leq 1$ than would be the case for the standard interval $-1 \leq z \leq 1$. At high energies the $\operatorname{Im} A^I$ are best known near the forward direction, so it is desirable to keep t as small as possible. Furthermore, the $A^I(s, t)$ of Ref. 11 which we utilize here are only valid for $-32 \leq t \leq 4$. Hence Eq. (A1) results in $A^{(l)I}$ valid for $0 \leq s \leq 68$, whereas use of the interval $-1 \leq z \leq 1$ would limit the validity of $A^{(l)I}$ to $0 \leq s \leq 36$. Therefore, we shall use Eq. (A1) to obtain the $A^{(l)I}$.

We note that for energies less than about 1.5

GeV, only the S, P, and D waves have non-negligible absorptive parts. Furthermore, Regge theory provides good approximations for absorptive parts above roughly 1.5 GeV. Therefore, it is convenient to introduce a parameter $\Lambda \sim (1.5\text{ GeV})^2$, and to treat the regions $s<\Lambda$ and $s>\Lambda$ in different ways which exploit these facts. In this spirit, we write the S waves for $I=\binom{0}{2}$ as

$$A^{(0)I}(s)\simeq \binom{-5}{-2}\lambda+\frac{1}{\pi}\int_4^{\Lambda}ds'\left\{\frac{s-c_0}{s'-c_0}\left[\frac{\operatorname{Im}A^{(0)I}(s')}{s'-s}+5\operatorname{Im}A^{(2)I}(s')\frac{(s'-2-\frac{1}{2}s)}{(s'-4)^2}\right]+\sum_{I'=0}^{2}C_{II'}\sum_{l=0}^{2}\operatorname{Im}A^{(l)I'}(s')K_l(s',s)\right\}$$
$$+A_{\rm HE}^{(0)I}(s,\Lambda)\,, \tag{A2}$$

where $c_0\equiv\frac{4}{3}$,

$$C\equiv\begin{pmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix},$$

$$K_0(s',s)\equiv\left(F^+-F^--\frac{2}{s'-c_0}\right),$$

$$K_1(s',s)\equiv\frac{3}{s'-4}\left[(s'+2s-4)F^++(c_0-s')F^--2\right],$$

$$K_2(s',s)\equiv\frac{5}{(s'-4)^2}\left[(s'^2-8s'+16+6s's-24s+6s^2)F\;+(\tfrac{16}{3}-s'^2)F^-+2\left(4-3s+\frac{\frac{16}{3}-s'^2}{s'-c_0}\right)\right],$$

$$F^{\pm}\equiv\frac{2}{s-4}\ln\left(1\pm\frac{s-4}{2s'+s-4}\right),$$

$$A_{\rm HE}^{(0)I}(s,\Lambda)\equiv\int_0^1 dz\,\frac{1}{\pi}\int_{\Lambda}^{\infty}ds'\left\{\frac{2(t-c_0)^2\operatorname{Im}T^I(s',c_0)}{(s'-c_0)(s'-t)(s'+t-2c_0)}\right.$$
$$\left.+(s-c_0)\sum_{I'=0}^{2}C_{II'}\left[\frac{1}{(s'-c_0)(s'-s)}-\frac{(-1)^{I'}}{(s'+t-2c_0)(s'-u)}\right]\operatorname{Im}T^{I'}(s',t)\right\},$$

$$u\equiv 4-s-t\,,$$

$$T^I(s,t)\equiv\sum_{I'=0}^{2}C_{II'}A^{I'}(s,t)\,.$$

We have expressed $A_{\rm HE}^{(0)I}$ in terms of the T^I because these are the amplitudes for which Regge theory prescribes simple asymptotic behavior.

The only approximation made in Eq. (A2) lies in the fact that $\operatorname{Im}A^{(l)I}(s')$ has been set equal to zero for $l\geq 3$, $4\leq s'\leq\Lambda$. Aside from this approximation, Eq. (A2) is rigorously valid for $0\leq s\leq 68$, as will be the case for our equation for $A^{(1)1}$.

For the P wave, we obtain

$$A^{(1)1}(s)\simeq\frac{1}{\pi}\int_4^{\Lambda}ds'\left[\frac{(s-4)\operatorname{Im}A^{(1)1}(s')}{(s'-4)(s'-s)}+\sum_{I=0}^{2}C_{1I}\sum_{l=0}^{2}\operatorname{Im}A^{(l)I}(s')M_l(s',s)\right]+A_{\rm HE}^{(1)1}(s,\Lambda)\,, \tag{A3}$$

where

$$M_0(s',s)\equiv\frac{1}{s-4}(G^+-G^--4)\,,$$

$$M_1(s',s)\equiv\frac{3}{(s-4)(s'-4)}\left[(s'+2s-4)G^++(3s'-4)G^-+2(2s'-s-4)\right],$$

$$M_2(s',s)\equiv\frac{5}{(s-4)(s'-4)^2}\left[(s'^2-8s'+6s's+16-24s+6s^2)G^++(32s'-13s'^2-16)G^-\right.$$
$$\left.+2(52s'+20s-14s'^2-9s's-48-4s^2)\right],$$

$$G^{\pm} \equiv 2\left(1+\frac{2s'}{s-4}\right)\ln\left(1\pm\frac{s-4}{2s'+s-4}\right),$$

$$A_{\rm HE}^{(1)1}(s,\Lambda) \equiv \int_0^1 dz\, P_1(z)\frac{t-u}{\pi}\int_\Lambda^\infty ds' \sum_{I=0}^{2} C_{1I}\left[\frac{1}{(s'-s)(s'+2t-4)} - \frac{(-1)^I}{(s'-t)(s'-u)}\right]\mathrm{Im}\,T^I(s',t).$$

In this paper, we use $\Lambda = (1.5\ \mathrm{GeV})^2$.

To construct solutions to Eq. (A2) which satisfy unitarity in the low-energy region, we use the trial-function method[35] developed earlier by the present author. Between threshold and 900 MeV, we represent the $\mathrm{Im}\,A^{(0)I}$ by functions of the form

$$\mathrm{Im}\,A^{(0)I}(s) \simeq \nu^{1/2}\left(b_1^I + \sum_{n=2}^{8} b_n^I \nu^{n/2}\right), \qquad \text{(A4)}$$

where $\nu \equiv \frac{1}{4}(s-4)$. The b_n^I are to be determined by imposing Eq. (A2) at a set of closely spaced mesh points, while simultaneously imposing (in a least-squares sense) the unitarity relation

$$\mathrm{Re}A^{(l)I} = [\mathrm{Im}\,A^{(l)I}(Q^{-1} - \mathrm{Im}\,A^{(l)I})]^{1/2}. \qquad \text{(A5)}$$

Above 900 MeV, we consider two cases for $\mathrm{Im}\,A^{(0)0}$, corresponding to the data of Protopopescu *et al.*[36] and, alternatively, the data of Martin and Estabrooks.[37] We feed the data for $\mathrm{Im}\,A^{(0)0}$ into our integrals, and constrain the b_n^0 by imposing continuity on the zeroth and first derivatives of $\mathrm{Im}\,A^{(0)0}$ at 900 MeV.

Our treatment of $\mathrm{Im}\,A^{(0)2}$ above 900 MeV is iterative. In zeroth order, we assume that $\mathrm{Im}\,A^{(0)2}$ is an unknown constant above 900 MeV. We constrain the b_n^2 by imposing continuity on the zeroth and first derivatives of $\mathrm{Im}\,A^{(0)2}$ at 900 MeV, and feed the resulting absorptive parts (which involve seven free parameters) into Eq. (A2). Upon imposing the unitarity relation (A5) in a least-squares sense below 900 MeV, we obtain a zeroth-order solution for δ_0^2 in this region.

In the next stage of our iteration, we approximate the zeroth-order δ_0^2 below 900 MeV by an equation of the form (3). We use the result to extrapolate the zeroth-order $\mathrm{Im}\,A^{(0)2}$ above 900 MeV, and assume that the first-order $\mathrm{Im}\,A^{(0)2}$ is given above 900 MeV by the extrapolated zeroth-order result plus an unknown constant. The additive constant is determined by imposing continuity on the zeroth and first derivatives of $\mathrm{Im}\,A^{(0)2}$ at 900 MeV, while imposing unitarity (in a least-squares sense) below 900 MeV. In this way we obtain our first-order result for δ_0^2, which is then approximated by Eq. (3) to begin the next cycle of the iteration. This procedure converges within five cycles to a stable result for δ_0^2. [The unitarity relation (A5) is satisfied within 1% below 900 MeV.] We remark that for each given value of λ it is essential to solve *simultaneously* for δ_0^0 and δ_0^2, since each depends on the other through their respective left cuts.

When constructing the S waves, we approximate $\mathrm{Im}\,A^{(1)1}$ and $\mathrm{Im}\,A^{(2)0}$ by ρ and f_0 energy-dependent Breit-Wigner formulas,[38] and we neglect $\mathrm{Im}\,A^{(2)2}$. We assume $m_\rho = 770$ MeV, $\Gamma_\rho = 146$ MeV, $m_f = 1270$ MeV, and $\Gamma(f\to\pi\pi) = 130$ MeV.[39] (Our assumptions for the region $s \geq \Lambda$ will be stated later.)

After constructing the S waves for $\lambda = 0.03$, -0.01, -0.05, and -0.09, we construct the corresponding P waves by an iterative procedure, assuming at each stage that δ_1^1 can be approximated between threshold and Λ by

$$Q\cot\delta_1^1 \simeq \frac{\xi_1}{s-4} + \eta_1 + \tau_1 s, \qquad \text{(A6)}$$

where ξ_1, η_1, and τ_1 are assumed to be linear functions of λ. Thus the right-hand side of Eq. (A6) contains six initially unknown parameters.

To begin our iteration, we utilize the fact that $\xi_1 = 4/a_1$, and we set $a_1 = 0.035$ for all four values of λ. At this and every stage of the iteration, we regard η_1 and τ_1 as functions of ξ_1, m_ρ, and Γ_ρ; specifically, we impose on Eq. (A6) the constraints

$$\delta_1^1(770\ \mathrm{MeV}) = 90^\circ, \qquad \text{(A7)}$$

$$\delta_1^1(853\ \mathrm{MeV}) = 135^\circ. \qquad \text{(A8)}$$

We shall find that Eqs. (A7) and (A8) correspond to a ρ width of about 146 MeV, which is our justification for using them. We wish these two equations to be satisfied for arbitrary values of λ, and that is why we use linear forms in λ for ξ_1, η_1, and τ_1: The right-hand side of Eq. (A6) is then a linear function of λ, so it is possible to satisfy Eqs. (A7) and (A8) for arbitrary values of λ.

The first-order ξ_1 is obtained by fitting Eq. (A6) below 600 MeV to the δ_1^1 implied by Eq. (A3), when the zeroth-order δ_1^1 is used in the latter. The λ dependence of δ_1^1 arises from the fact that both S waves depend on λ, and the S waves contribute to the right-hand side of Eq. (A3). The first-order η_1 and τ_1 are determined by ξ_1, together with the fact that Eqs. (A7) and (A8) are required to hold for all λ. Successive stages of the iteration are obvious, and the procedure converges within five cycles. The result is

$$\xi_1 = 101.25 + 266.18\lambda ,$$
$$\eta_1 = -3.032 - 18.71\lambda , \qquad \text{(A9)}$$
$$\tau_1 = -0.02254 + 0.2860\lambda .$$

Equations (A6) and (A9) imply that $\delta_1^1 = 45°$ at $M_{\pi\pi} = 709$, 707, 705, and 703 MeV for $\lambda = 0.03$, -0.01, -0.05, and -0.09, respectively. Measuring Γ_ρ from the 45° point to the 135° point, we conclude that Eqs. (A6) and (A9) describe a ρ resonance with $144 \le \Gamma_\rho \le 150$ MeV, in good agreement with the current experimental value[39] $\Gamma_\rho = (146 \pm 10)$ MeV.

For $s \ge \Lambda = (1.5\text{ GeV})^2$, we assume that

$$\operatorname{Im} T^0(s,t) = \gamma_P(t)(s/\bar{s})^{\alpha_P(t)} + \gamma_f(t)(s/\bar{s})^{\alpha_f(t)} ,$$
$$\operatorname{Im} T^1(s,t) = \gamma_\rho(t)(s/\bar{s})^{\alpha_\rho(t)} ,$$
$$\operatorname{Im} T^2(s,t) = 0 .$$

We use $\bar{s} = 1\text{ GeV}^2$, which defines the scale of the γ's.

With standard assumptions for the α's, simple power counting reveals that the $A_{\rm HE}^{(l)I}$ are all dominated by the contributions of $\operatorname{Im} T^1$. In our calculations, we assume that

$$\alpha_\rho(t) = 0.50 + 0.90(t/\bar{s}) ,$$

and we use the result of Ref. 11 that

$$\gamma_\rho(t) \cong 0.82 + 2.04(t/\bar{s}) + 0.88(t/\bar{s})^2 .$$

We exploit duality in assuming that

$$\alpha_f(t) = \alpha_\rho(t) ,$$
$$\gamma_f(t) = \tfrac{3}{2}\gamma_\rho(t) ,$$

and we incorporate an asymptotic total cross section of 15 mb (Ref. 40) in our assumptions

$$\alpha_P(t) = 1 ,$$
$$\gamma_P(t) = 1.18 .$$

The lack of t dependence in our α_P and γ_P is a crude but harmless assumption: The resulting Pomeranchon contributes less than 0.01 to the $\operatorname{Re} A^{(l)I}$ below 600 MeV, and less than 0.04 below 800 MeV, so that even a 50% correction would be almost negligible.

Having described the input used in our equations, we shall now discuss briefly the uncertainties in our solutions.

Below 400 MeV, the primary uncertainty in δ_0^0 arises from the experimental uncertainty of ± 10 MeV in Γ_ρ, and from an estimated uncertainty of $\pm 15\%$ in $\gamma_\rho(t)$. If we regard these uncertainties as statistically independent, the resulting uncertainty in δ_0^0 is less than $\pm 1.0°$ below 350 MeV, and less than $\pm 1.4°$ below 400 MeV.[41] The few remaining uncertainties are much smaller and also statistically independent, so that their net effect is negligible. (For example, alternate usage of the data of Refs. 36 and 37 above 900 MeV affects δ_0^0 by less than $\pm 0.3°$ below 400 MeV.[41]) Of course, the preceding remarks are valid only when λ has been given; in practice, further uncertainties arise from the imprecision in our knowledge of λ. The latter uncertainties are readily estimated from Eqs. (3) and (4).

For $\lambda = -0.01$ (near our favored value of -0.013 ± 0.010), we find that $\delta_0^0 = 40°$, 54°, 64°, and 75° at $M_{\pi\pi} = 500$, 600, 700, and 800 MeV, respectively. However, the results for δ_0^0 become increasingly sensitive to input above 900 MeV as one approaches 900 MeV. We have not made careful estimates of the uncertainties in δ_0^0 at these higher energies, but we believe the uncertainties to be appreciable. Hence the δ_0^0 stated above should not be taken too seriously. Fortunately, it is $\operatorname{Im} A^{(0)0}$ which enters into our dispersive integrals, and $\operatorname{Im} A^{(0)0}$ is insensitive to moderate uncertainties in δ_0^0 when $60° \lesssim \delta_0^0 \lesssim 120°$. Hence our low-energy results for δ_0^0 should be reliable despite the uncertainties in the ρ region.

The primary uncertainties in δ_0^2 are those resulting from the aforementioned uncertainties in Γ_ρ and $\gamma_\rho(t)$. Above 700 MeV, an estimated uncertainty of 30% in the Pomeranchon contribution also becomes significant. The resulting net uncertainty in δ_0^2 is $\pm 0.5°$, $\pm 1.5°$, and $\pm 3.0°$ at $M_{\pi\pi} = 500$, 700, and 900 MeV, respectively.

The uncertainties in δ_0^2 resulting from imprecision in our knowledge of λ can easily be estimated from Eqs. (3) and (9). For $\lambda = -0.013 \pm 0.010$, the corresponding uncertainty in δ_0^2 is less than $\pm 0.6°$ between threshold and 900 MeV. Combining this with the preceding uncertainties, we estimate the total uncertainty in δ_0^2 to be $\pm 0.8°$, $\pm 1.6°$, and $\pm 3.1°$ at $M_{\pi\pi} = 500$, 700, and 900 MeV, respectively.

Finally we consider the P wave. When the $\operatorname{Im} A^{(1)1}$ implied by Eqs. (A6) and (A9) is substituted into the right-hand side of Eq. (A3), the resulting $\operatorname{Re} A^{(1)1}$ is not precisely consistent with Eqs. (A7) and (A8). Consistency can be obtained by adding to the right-hand side of Eq. (A3) a phenomenological term

$$\Delta A^{(1)1} = -10^{-5}(3.15 + 15.0\lambda)(s-4)^2 .$$

Since $\Delta A^{(1)1}$ is quadratic in $(s-4)$, it can be interpreted as a contribution from distant singularities, in which case its nonzero value simply rectifies small errors in our treatment of high-energy contributions to the right-hand side of Eq. (A3). When $\Delta A^{(1)1}$ is added to the right-hand side of Eq. (A3), the result agrees with the Re$A^{(1)1}$ implied by Eqs. (A6) and (A9) within ± 0.002 between threshold and 900 MeV. This agreement justifies *a fortiori* our use of the approximation (A6), and indicates that if Eqs. (A7) and (A8) are presumed to be exact, then Eqs. (A6) and (A9) yield a δ_1^1 valid within ± 0.2° between threshold and 900 MeV. In practice, the aforementioned uncertainties in Γ_ρ and $\gamma_\rho(t)$ make a_1 uncertain by about ± 7%, and make δ_1^1 uncertain by a similar amount below 400 MeV. In the ρ region, δ_1^1 is of course sensitive to the precise values of m_ρ and Γ_ρ. The uncertainties in a_1 and in δ_1^1 below 400 MeV resulting from imprecision in our knowledge of λ may be estimated from Eqs. (7) and (8), and are quite small—only ± 3% for $\lambda = -0.013 \pm 0.010$, yielding a net uncertainty of ± 8% in a_1 (and also in λ_1).

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[5]R. P. Ely, Jr. *et al.*, Phys. Rev. 180, 1319 (1969).

[6]Although Zylbersztejn *et al.* (Ref. 3) used the experimental value of 95 MeV for f_π, better agreement with Regge sum rules is obtained if one uses the Goldberger-Treiman value $f_\pi \approx 83$ MeV.

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[13]This uncertainty refers only to the goodness of fit of Eqs. (3) and (4) to our preferred solutions to the rigorous equations. Our solutions for δ_0^0 contain additional small uncertainties due to uncertainties in the input (see Appendix). For $-0.05 \le \lambda \le 0.03$, the additional uncertainty in δ_0^0 is less than ± 1.4° below 400 MeV; for $\lambda = -0.09$, it is less than ± 2.0° below 400 MeV.

[14]This uncertainty refers only to the goodness of fit of Eq. (6) to our preferred solutions of the rigorous equations. For $-0.05 \le \lambda \le 0.03$, an additional uncertainty in a_0 of $\pm 0.01\mu^{-1}$ results from uncertainties in the input (see Appendix). For $\lambda = -0.09$, the additional uncertainty in a_0 is $\pm 0.04\mu^{-1}$.

[15]This uncertainty refers only to the goodness of fit of Eq. (7) to our preferred solutions of the rigorous equations. An additional uncertainty in a_1 of ± 7% results from uncertainties in the input (see Appendix).

[16]This uncertainty refers only to the goodness of fit of Eqs. (3) and (9) to our preferred solutions of the rigorous equations. An additional uncertainty in δ_0^2 of ± 0.5°, ± 1.5°, and ± 3.0° at $M_{\pi\pi} = 500$, 700, and 900 MeV, respectively, results from uncertainties in the input (see Appendix).

[17]This uncertainty refers only to the goodness of fit of Eq. (10) to our preferred solutions of the rigorous equations. An additional uncertainty in a_2 of $\pm 0.003\mu^{-1}$ results from uncertainties in the input (see Appendix).

[18]For any given value of λ, it is essential to solve *simultaneously* for δ_0^0 and δ_0^2, since each depends on the other through their respective left cuts. We find that when λ is varied, δ_0^0 changes rapidly and counteracts the effect on δ_0^2 of the change in λ.

[19]This uncertainty refers only to the goodness of fit of Eq. (11) to our preferred solutions of the rigorous equations. An additional uncertainty in λ_1 of ± 7% results from uncertainties in the input (see Appendix).

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[21]We determine λ (hence a_0) by minimizing χ^2 for the fit to the data. The uncertainty in λ is that variation which increases χ^2 by unity.

[22]$\chi^2 = 3.42$ with two degrees of freedom, which is exceeded 18% of the time for a normal distribution.

[23]Zylbersztejn *et al.* present three sets of values for $(\delta_0^0 - \delta_1^1)$, based on three different methods of analysis (see Fig. 2). For the maximum-likelihood set, we obtain $a_0 = (0.54^{+0.18}_{-0.16})\mu^{-1}$, with $\chi^2 = 2.71$ for two degrees of freedom. For the "χ^2 set," we obtain $a_0 = (0.43^{+0.21}_{-0.19})\mu^{-1}$, with $\chi^2 = 0.22$ for two degrees of freedom. For the Pais-Treiman set, we obtain $a_0 = (0.51^{+0.28}_{-0.24})\mu^{-1}$, with $\chi^2 = 0.34$ for two degrees of freedom. The weighted average for a_0 is $0.50\,\mu^{-1}$. Since all three results for a_0 are based on the same data, we assume the uncertainty in a_0 to be determined by the maximum and minimum values consistent with *any* of the three results.

[24]$\chi^2 = 2.76$ with three degrees of freedom, which is exceeded 44% of the time for a normal distribution.

[25]The uncertainties stated in Eqs. (13a)–(13g) include all the known sources of uncertainty (see Appendix).

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[34]The K_{e4} value for λ may suffer from systematic uncertainties not represented in Eq. (13b), but our result for δ_0^2 is quite insensitive to moderate variations in

λ. The primary uncertainty in our result for δ_0^2 arises from uncertainties in various absorptive parts (see Appendix).

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[41]These uncertainties apply to the range $-0.05 \leq \lambda \leq 0.03$. For $\lambda = -0.09$, the uncertainties in δ_0^0 are about 50% larger.