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Production of the R, S, T, and U resonances in the dual resonance model*

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The dual resonance model (DRM) is used to estimate the cross sections and the background for the production of narrow high-spin mesons in an idealized missing-mass (MM) experiment $a+b\to c+\text{MM}$, in which the particles a, b, and c are spinless and do not carry any internal quantum numbers. The theoretical results are compared with the recent Northeastern–Stony Brook (NU/SUNY) experiment. It is found that, while the calculated backgrounds agree reasonably well, the S, T, and U cross sections are substantially higher than their experimental upper limits obtained from this experiment. In a mathematical appendix we also present a general method for calculating N-point dual amplitudes with two external particles of arbitrary spin.

I. INTRODUCTION

The first indication of the presence of narrow resonances in the high-mass region ($\gtrsim$1.5 GeV) of the meson spectrum came from the CERN missing-mass spectrometer (CMMS) group, who reported three narrow resonances—the $R_1(1630)$, $R_2(1700)$, and $R_3(1748)$—in the region of the R enhancement (mass ~1700 MeV).[1,2] The CMMS group also observed the $S(1929)$, $T(2195)$, and $U(2382)$ mesons.[3] The widths of all these resonances were compatible with zero, and had upper limits of 21, 30, 38, 35, 13, and 30 MeV for the R_1, R_2, R_3, S, T, and U, respectively. A subsequent experiment performed by the NU/SUNY group, however, showed no evidence of the aforementioned narrow-resonance structure.[4,5]

In a DRM with indefinitely rising linear Regge trajectories, the existence of narrow mesons of high mass (and, hence, of high spin) has been explicitly demonstrated by Chan and Tsou.[6] The purpose of this paper is to use the DRM to estimate (i) the production cross section for these mesons,[7] and (ii) the background in a simplified MM reaction,

$$a+b\to c+\text{MM}, \tag{1}$$

involving only scalar particles a, b, and c, and then to compare the predictions with the experimental data of the NU/SUNY group. In addition, a method is developed for calculating general N-point dual amplitudes with two external spinning particles. This is presented in the Appendix.

II. THE MODEL

Consider, instead of (1), the slightly more general process

$$a+b\to x(j)+\text{anything}, \tag{2}$$

in which $x(j)$ represents a resonance of spin j. Figure 1 illustrates the kinematics of the process; the invariants of interest are

$$s=(P_a+P_b)^2,\quad t=(P_a-P_x)^2,\quad M^2=(P_a+P_b-P_x)^2. \tag{3}$$

Restriction of M^2 in (3) to its lowest allowed value m^2,[8] so that only one other particle c is produced besides x in the final state, yields the production cross section of the spin-j resonance. For $j=0$ reaction (2) is identical to (1), and most of the contribution to the cross section at high M in the DRM comes from the daughter resonances whose degeneracy increases exponentially with M. Since the daughters are also much broader than the parents at the same M,[6] this cross section provides the background in the missing-mass reaction (1).

The generalized optical theorem[9] relates the inclusive cross section for the process (2) to an "appropriate" discontinuity[10] in the missing-mass variable of the "forward" elastic six-line amplitude $T(ab\bar{x}(j)\to ab\bar{x}(j))$ (Ref. 11):

$$\frac{d^2\sigma_{ab}^{x}}{dt\,dM^2}=-\frac{1}{32\pi^2\lambda(s,m_a{}^2,m_b{}^2)}\times\text{Disc}_{M^2}T_f(ab\bar{x}(j)\to ab\bar{x}(j)) \tag{4}$$

(see Ref. 12) where

$$\lambda(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx. \tag{5}$$

Strictly speaking, dual amplitudes do not possess any discontinuity[13]; however, on using the identification

$$\text{Disc}_{M^2}T_f=2\pi i\delta(\alpha(M^2)-\{\alpha(M^2)\})\times\underset{\alpha(M^2)=\{\alpha(M^2)\}}{\text{Residue}}T_f, \tag{6}$$

where

$$\alpha(M^2)\equiv\alpha=\alpha_0+\alpha' M^2 \tag{7}$$

(see Ref. 14) and $\{\alpha(M^2)\}$ denotes the integer part

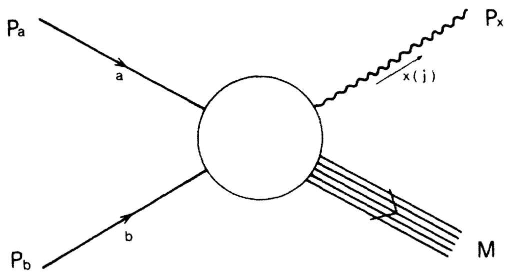

FIG. 1. Kinematics for the inclusive production of a resonance x of spin j in ab collisions.

of the trajectory function $\alpha(M^2)$, the average differential cross section over $(\alpha')^{-1}$ units of M^2, from $\hat{M}^2-1/2\alpha'$ to $\hat{M}^2+1/2\alpha'$, becomes

$$\frac{d^2\sigma^x_{ab}}{dtdM^2}=\frac{1}{16\pi\lambda(s,m_a^2,m_b^2)}\times\underset{\alpha=N}{\text{Residue}}\,T_f(ab\bar{x}(j)\to ab\bar{x}(j)), \tag{8}$$

$N\equiv\alpha(\hat{M}^2)$ being a non-negative integer.

The problem is now reduced to calculating the six-point amplitude T in the DRM. As is well known, this has contributions from 60 distinct diagrams, corresponding to the different permutations of the external lines. Of these only the 18 diagrams which have a, b, and $\bar{x}$ adjacent can make a nonvanishing contribution to the discontinuity in the $ab\bar{x}$ channel. For the purposes of a crude estimate we have calculated the contribution

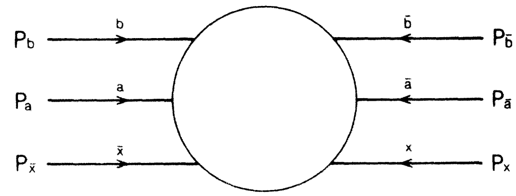

FIG. 2. One of the 18 dual diagrams which makes a contribution to $\text{Disc}_{M^2}T_f$.

to T from only one of these dual diagrams, and then suitably adjusted the over-all normalization constant as explained in the next section. This diagram is shown in Fig. 2; all the particles are considered incoming and the invariants of interest are defined as follows:

$$s_{ab}=(P_a+P_b)^2,\quad s_{a\bar{x}}=(P_a+P_{\bar{x}})^2,\quad s_{ab\bar{x}}=(P_a+P_b+P_{\bar{x}})^2,\ \text{etc.} \tag{9}$$

A general method for the calculation of $\text{Res}_{\alpha=N}T_f$ is given in the Appendix [Eqs. (A47) and (A52)]. Below we will simply quote two special cases of the general formula which are of immediate interest. The first case consists of taking x in reaction (2) to be a spinless particle. Then we obtain the following expression for the background in the production of a spin-N resonance in a missing-mass experiment:

$$\frac{d^2\sigma^c_{ab}}{dtdM^2}=\frac{\kappa}{16\pi\lambda(s,m_a^2,m_b^2)}\sum_{\substack{p,q,r,s\\(p+q+r+s=N)}}R(e,p)R(f,q)R(g,r)R(h,s)\times B_4(-\alpha_{a\bar{x}}+q+s;\,-\alpha_{ab})B_4(-\alpha_{\bar{a}x}+r+s;\,-\alpha_{\bar{a}\bar{b}}), \tag{10}$$

where κ is a normalization constant, $R(n,m)$ is an mth degree polynomial in n,

$$R(n,m)=\frac{1}{m!}n(n+1)\cdots(n+m-1), \tag{11}$$

$B_4(u;v)$ is the usual Euler beta function,[15] and

$$\begin{aligned}e&=\alpha_{b\bar{b}}+1,\\ f&=\alpha_{ab\bar{b}}-\alpha_{ab}-\alpha_{b\bar{b}},\\ g&=\alpha_{b\bar{b}\bar{a}}-\alpha_{\bar{a}\bar{b}}-\alpha_{b\bar{b}},\\ h&=\alpha_{x\bar{x}}+\alpha_{b\bar{b}}-\alpha_{ab\bar{b}}-\alpha_{b\bar{b}\bar{a}}.\end{aligned} \tag{12}$$

On the other hand, if we put $N=0$ in $\text{Res}_{\alpha=N}T_f$ we obtain the production cross section of a spin-j resonance:

$$\frac{d\sigma}{dt}=\frac{\kappa}{16\pi\lambda(s,m_a^2,m_b^2)}\frac{1}{(2j-1)!!}\times\sum_{q=-j}^{j}\sum_{l_1=0}^{j}\sum_{l_2=0}^{j}(j+q)!(j-q)!W^*(j,q,l_1;\vec{p}_a,\vec{p}_b)W(j,q,l_2;\vec{p}_a,\vec{p}_b)\times B_4(-\alpha_{a\bar{x}}+l_1;\,-\alpha_{ab})B_4(-\alpha_{\bar{a}x}+l_2;\,-\alpha_{\bar{a}\bar{b}}), \tag{13}$$

in which $\vec{p}_a$ and $\vec{p}_b$ denote the three-vector parts of P_a and P_b, respectively,

$$W(j,q,l;\vec{p}_a,\vec{p}_b)=\sum_{n=\max(-l,q-j+l)}^{\min(l,q+j-l)}\mathcal{Y}_{j-l,q-n}(\vec{p}_a)\mathcal{Y}_{ln}(\vec{p}_b), \tag{14}$$

and

$$\mathcal{Y}_{kq}(\vec{r})=\left[\frac{4\pi}{(2k+1)(k+q)!(k-q)!}\right]^{1/2}Y_{kq}(\vec{r}), \tag{15}$$

with $Y_{kq}(\vec{r})$ being a solid spherical harmonic.

In the numerical computation of Eqs. (10) and (13) two problems arise: (a) The lack of a vacuum trajectory with intercept 1 in the DRM implies that, in the phase-space region of fixed missing mass in reaction (2), the cross section does not have the correct Regge behavior as $s\to\infty$. In order to simulate this behavior crudely we have multiplied all cross sections obtained from Eq. (8) by the factor $s^{2-2\alpha_0}$.[7] (b) The B_4 factors have poles whenever the trajectories α_{ab} and $\alpha_{\bar{a}\bar{b}}$ assume non-negative integral values. This problem is avoided by giving these trajectories nonzero imaginary parts. Since the precise discontinuity in M^2 which is to be considered in Eq. (4) is obtained by keeping s_{ab} fixed above and $s_{\bar{a}\bar{b}}$ fixed below their respective cuts,[16-18] the trajectories α_{ab} and $\alpha_{\bar{a}\bar{b}}$ must have phases of opposite sign. We therefore give α_{ab} a positive imaginary part and $\alpha_{\bar{a}\bar{b}}$ a negative imaginary part. $\mathrm{Im}\,\alpha_{ab}$ may be roughly calculated by assuming that it vanishes below the 3π threshold and then rises linearly with s_{ab} through the A_2 resonance. Then since

$$\mathrm{Im}\,\alpha_{ab}(s_{ab}=M_{A_2}{}^2)=\alpha' M_{A_2}\Gamma_{A_2}, \tag{16}$$

where M_{A_2} is the mass and Γ_{A_2} is the width of the A_2 meson, we obtain

$$\mathrm{Im}\,\alpha_{ab}\simeq 0.07\alpha' s_{ab},\quad s_{ab}\gg 9m_\pi{}^2. \tag{17}$$

$\mathrm{Im}\,\alpha_{\bar{a}\bar{b}}$ is assumed to be negative and of the same magnitude as $\mathrm{Im}\,\alpha_{ab}$:

$$\mathrm{Im}\,\alpha_{\bar{a}\bar{b}}\simeq -0.07\alpha' s_{\bar{a}\bar{b}},\quad s_{\bar{a}\bar{b}}\gg 9m_\pi{}^2. \tag{18}$$

We find that the cross sections evaluated from Eqs. (10) and (13) are quite insensitive to the exact value of the imaginary parts used for the trajectories α_{ab} and $\alpha_{\bar{a}\bar{b}}$. The numerical results of the next section are therefore only for the values given in Eqs. (17) and (18).

III. RESULTS AND DISCUSSION

(a) The trajectory slope α' is taken to be 1 GeV^{-2}. Following Ref. 7 we have chosen the value −0.2 for the intercept α_0.[19] The only other parameter in the calculation is the over-all normalization constant κ. This is fixed by equating the total cross section for the process $a+b\to c+x(j=2)$, as obtained from Eq. (13), to that of A_2 production in the reaction $\pi^-p\to pA_2^-$ at a center-of-mass energy squared of $s=30.92\ \mathrm{GeV}^2$.[20] From CERN compilations of π^- induced reactions[21]

$$\sigma(\pi^-p\to pA_2^-)=0.18\ \mathrm{mb},\quad s=30.92\ \mathrm{GeV}^2. \tag{19}$$

(b) The NU/SUNY group has recently measured the missing-mass spectrum of the reaction $\pi^-+p\to(\mathrm{MM})^-+p$ at various incident energies in the interval $0.2\leq|t|<0.3\ \mathrm{GeV}^2$. The results of their fits to the R region at a beam momentum of 8 GeV/c, and to the S, T, and U regions at 11, 13.4, and 16 GeV/c, are summarized in Table I. The S, T, and U data were generally compatible with smooth backgrounds quadratic in M^2; the cross sections for these mesons listed in Table I are upper limits determined from fits with quadratic backgrounds plus resonances whose masses were fixed at the CMMS values and whose widths were given by the CMMS upper-limit widths ($\Gamma_S=35$, $\Gamma_T=13$, $\Gamma_U=30$ MeV).

The theoretical results are reported in Table II, averaged over the t interval $0.2\leq|t|\leq 0.3\ \mathrm{GeV}^2$ to facilitate easier comparison with experiment. The calculated cross section for the production of the R meson is about twice as large as the experimental cross section at $s\simeq 15\ \mathrm{GeV}^2$. This discrepancy increases to approximately a factor of 10 for the S meson at $s\simeq 20\ \mathrm{GeV}^2$. The situation is even worse for the T and U mesons. Experimentally, the T and U cross sections are compatible with zero, while theoretically, even at $s=30\ \mathrm{GeV}^2$, these cross sections are predicted to be 68 and 51 $\mu\mathrm{b/GeV}^2$, respectively. However, considering the crude nature of the calculation, the agreement between the experimental and the calculated backgrounds is quite reasonable.

In spite of the several inherent difficulties of the DRM the above analysis strongly suggests a failure of the DRM to reproduce the experimentally observed cross sections for the production of the S, T, and U mesons. In seeking an explanation for this failure two speculations might be suggested. The first is that, since a proper vacuum trajectory is absent from the DRM, the Pomeron couplings to high-spin states decrease appreciably faster than meson couplings with increasing spin. On the other hand, the discrepancy might be avoided by revising the concept of indefinitely rising linear trajectories.

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TABLE I. R, S, T, and U cross sections and background (experimental).[a]

Center-of-mass energy squared (GeV^2)	R $\langle d\sigma/dt\rangle$[c] ($\mu$b/GeV^2)	R Backgound[d] (mb/GeV^4)	S $\langle d\sigma/dt\rangle$[c] ($\mu$b/GeV^2)	S Backgound[d] (mb/GeV^4)	T $\langle d\sigma/dt\rangle$[c] ($\mu$b/GeV^2)	T Backgound[d] (mb/GeV^4)	U $\langle d\sigma/dt\rangle$[c] ($\mu$b/GeV^2)	U Backgound[d] (mb/GeV^4)	Widths (MeV)
15.9 (8)[b]	95 ± 23	1.2							139 ± 31[e]
21.5 (11)[b]			11.6 ± 4.5	1.0	5.2 ± 5.9	1.1			f
26.0 (13.4)[b]			3.6 ± 5.4	0.7	1.5 ± 3.0	0.7(5)	−6.4 ± 4.5	0.8	f
30.9 (16)[b]			2.9 ± 3.8	0.6	−6.3 ± 2.4	0.6	7.0 ± 3.9	0.6	f

[a] Table compiled from Refs. 4 and 5.

[b] Number in parentheses denotes equivalent beam momentum in GeV/c.

[c] For t interval $0.2 \le |t| \le 0.3$ GeV^2. Errors shown are statistical only. Additional normalization error for the R less than ±25%, and for the S, T, and U less than ±15%.

[d] For t interval $0.2 \le |t| \le 0.3$ GeV^2. Numbers shown are very approximate, and have been gleaned visually from the graphs of Refs. 4 and 5.

[e] Fitted width of the R.

[f] Assumed widths $\Gamma_S = 35$, $\Gamma_T = 13$, and $\Gamma_U = 30$ MeV.

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APPENDIX: DUAL AMPLITUDE FOR AN N-POINT MULTIPERIPHERAL GRAPH WITH TWO EXTERNAL SPINNING PARTICLES

We present below a general method for calculating N-point dual amplitudes in the tree configuration with two external spinning particles. The amplitude, A, of the diagram of Fig. 2 will appear as a special case. The calculation is performed using the operator formalism of the Veneziano model,[22] according to which B_N, the N-point function with only external scalar particles, is expressible in terms of an infinite set of creation and annihilation operators which satisfy the commutation relations

$$[a_\mu^{(m)}, a_\nu^{(n)\dagger}] = -g_{\mu\nu}\delta_{mn} \tag{A1}$$

(see Ref. 23) where μ and ν are Lorentz indices, and $m, n = 1, 2, \ldots, \infty$.

Consider the multiperipheral configuration of Fig. 3. This diagram does not involve any "twisted propagators"; p_0, $p_1, \ldots, p_{N+1}$ represent the four-momenta of the external particles, and $x_1, \ldots, x_{N-1}$ are the internal Chan variables. The wiggly lines denote spinning particles on the leading trajectory. For applications to inclusive reactions only the case where the spins j and j' of the excited particles are equal is of interest. In the rest frame in which both the spinning particles are simultaneously at rest the amplitude $A(jm; N; j'm')$ for this diagram is given by

$$A(jm;N;j'm') = (\psi(j'm'), V(p_1)D(\alpha(s_1))V(p_2)\cdots \times D(\alpha(s_{N-1}))V(p_N)\psi(jm)), \tag{A2}$$

where $\psi(jm)$ denotes the eigenvector of a particle with angular momentum j and definite magnetic quantum number m, and the "vertex" operator $V(p_i)$ and the "propagator" $D(\alpha(s_i))$ are defined as follows:

$$V(p_i) = \exp\left[-\sum_{n=1}^{\infty} p_i \cdot \frac{a^{(n)\dagger}}{\sqrt{n}}\right] \exp\left[\sum_{n=1}^{\infty} p_i \cdot \frac{a^{(n)}}{\sqrt{n}}\right], \tag{A3}$$

$$D(\alpha(s_i)) = \int_0^1 dx_i x_i^{-\alpha(s_i)+H-1}(1-x_i)^{\alpha_0 - 1}, \tag{A4}$$

TABLE II. R, S, T, U, and X cross sections and background (theoretical).[a]

Center-of-mass energy squared (GeV^2)[b]	R $\langle d\sigma/dt\rangle$[c]	R Background[d]	S $\langle d\sigma/dt\rangle$[c]	S Background[d]	T $\langle d\sigma/dt\rangle$[c]	T Background[d]	U $\langle d\sigma/dt\rangle$[c]	U Background[d]	X $\langle d\sigma/dt\rangle$[c]	X Background[d]
15	177	0.69	139	0.82	105	0.94	82	1.06	61	1.17
20	148	0.59	115	0.69	87	0.80	66	0.89	49	0.99
25	132	0.52	104	0.61	77	0.70	57	0.79	42	0.87
30	121	0.48	91	0.56	68	0.64	51	0.72	37	0.78

[a] For the t interval $0.2 \le |t| \le 0.3$ GeV^2.
[b] Numbers in the column do not correspond exactly to experiment, but are sufficiently close to permit a useful comparison.
[c] In units of μb/GeV^2.
[d] In units of mb/GeV^4.

$$H = -\sum_{n=1}^{\infty} n a^{(n)\dagger}\cdot a^{(n)}, \tag{A5}$$

$$s_i = \left(\sum_{j=0}^{i} p_j\right)^2, \tag{A6}$$

$$\alpha(s_i) = \alpha_0 + \tfrac{1}{2}s_i \tag{A7}$$

(see Ref. 24). The lemma[22]

$$V(p_N)x_{N-1}^H \cdots V(p_2)x_1^H V(p_1) = \exp\left[-\sum_{n=1}^{\infty} P_{\leftarrow}^{(n)}\cdot\frac{a^{(n)\dagger}}{\sqrt{n}}\right]x_{1N}^H \exp\left[\sum_{n=1}^{\infty} P_{\rightarrow}^{(n)}\cdot\frac{a^{(n)}}{\sqrt{n}}\right] \times \prod_{1\le i<j\le N}(1-x_{ij})^{-p_ip_j}, \tag{A8}$$

where

$$x_{ij} = x_i x_{i+1}\cdots x_{j-1}, \tag{A9}$$

$$P_{\leftarrow}^{(n)} = p_N + (x_{N-1N})^n p_{N-1} + \cdots + (x_{1N})^n p_1, \tag{A10}$$

and

$$P_{\rightarrow}^{(n)} = p_1 + (x_{12})^n p_2 + \cdots + (x_{1N})^n p_N \tag{A11}$$

enables us to write

$$V(p_1)D(\alpha(s_1))\cdots D(\alpha(s_{N-1}))V(p_N) = \int d\phi_{N+2}\exp\left[-\sum_{n=1}^{\infty} P_{\rightarrow}^{(n)}\cdot\frac{a^{(n)\dagger}}{\sqrt{n}}\right]x_{1N}^H\exp\left[\sum_{n=1}^{\infty} P_{\leftarrow}^{(n)}\cdot\frac{a^{(n)}}{\sqrt{n}}\right] \equiv \int d\phi_{N+2}V_N \tag{A12}$$

(see Ref. 25), where we have used the abbreviation $d\phi_{N+2}$ to denote the integrand of the Bardakci-Ruegg expression for the $(N+2)$-point function:

$$d\phi_{N+2} = \left[\prod_{i=1}^{N-1} dx_i\, x_i^{-\alpha(s_i)-1}(1-x_i)^{\alpha_0-1}\right] \times \prod_{1\le i<j\le N}(1-x_{ij})^{-p_ip_j}. \tag{A13}$$

Further, since a generating function for the angular momentum eigenvectors, $\psi(jm)$, is provided by[26]

$$\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\phi_{jm}(v)\chi(jm) = \exp(\vec{d}\cdot\vec{a}^{(1)})\psi_0 \tag{A14}$$

(see Ref. 27), where $v=(v_+, v_-)$ is an arbitrary spinor,

$$\phi_{jm}(v) = \frac{v_+^{\,j+m}v_-^{\,j-m}}{[(j+m)!(j-m)!]^{1/2}}, \tag{A15}$$

$$\chi(jm) = [(2j-1)!!]^{1/2}2^j\psi(jm), \tag{A16}$$

and the components of the vector $\vec{d}$ are

$$d_1 = \sqrt{2}\,v_+^2,\quad d_2 = \sqrt{2}\,v_-^2,\quad d_3 = 2v_+v_-, \tag{A17}$$

it appears profitable, in evaluating the matrix element (A2), to consider the scalar product

$$N = (\psi_0, \exp(\vec{d}^{(1)*}\cdot\vec{a}^{(1)})V_N\exp(\vec{d}^{(2)}\cdot\vec{a}^{(1)\dagger})\psi_0). \tag{A18}$$

Calling

$$\vec{a}^{(1)}\equiv\vec{a},\quad P_{\rightarrow}^{(1)}\equiv P_{\rightarrow},\ \text{and}\ P_{\leftarrow}^{(1)}\equiv P_{\leftarrow}, \tag{A19}$$

and denoting the three-dimensional parts of the four-vectors $P_{\rightarrow}$ and $P_{\leftarrow}$ by $\vec{P}_{\rightarrow}$ and $\vec{P}_{\leftarrow}$, respectively, we may write the expression for N as

$$N = \exp(\vec{d}^{(1)}\cdot\vec{P}_{\rightarrow})^*\exp(x_{1N}\vec{d}^{(1)*}\cdot\vec{d}^{(2)}) \times\exp(-\vec{d}^{(2)}\cdot\vec{P}_{\leftarrow}). \tag{A20}$$

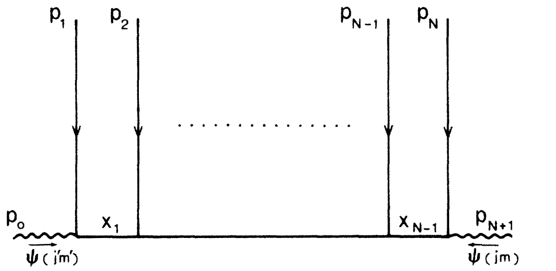

FIG. 3. A particular $(N+2)$-point multiperipheral graph with two external spinning particles and no "twisted propagators."

The last step involves the use of the following two identities: (1) For any two operators A and B whose commutator $[A, B]$ is a c number,

$$e^A e^B = e^B e^A e^{[A,B]}; \tag{A21}$$

and (2) for any four-vector $f^{(n)}$

$$\exp\left[\sum_{n=1}^{\infty} f^{(n)}\cdot a^{(n)}\right] x^H = x^H \exp\left[\sum_{n=1}^{\infty} x^n f^{(n)}\cdot a^{(n)}\right]. \tag{A22}$$

Now, we make the following two remarks:

(1) Since the vectors $\vec{d}^{(1)}$ and $\vec{d}^{(2)}$ are defined in terms of the arbitrary spinors $v^{(1)}=(v_+^{(1)}, v_-^{(1)})$ and $v^{(2)}=(v_+^{(2)}, v_-^{(2)})$, respectively, by

$$d_1^{(i)}=\sqrt{2}\,(v_+^{(i)})^2,$$
$$d_2^{(i)}=\sqrt{2}\,(v_-^{(i)})^2, \qquad i=1,2 \tag{A23}$$
$$d_3^{(i)}=2v_+^{(i)}v_-^{(i)},$$

we have

$$\vec{d}^{(1)*}\cdot\vec{d}^{(2)}=2(v^{(1)*}v^{(2)})^2, \tag{A24}$$

where

$$(v^{(1)*}v^{(2)}) \equiv v_+^{(1)*}v_+^{(2)}+v_-^{(1)*}v_-^{(2)} \tag{A25}$$

and

$$\sum_{m=-j}^{j}\phi_{jm}^*(v^{(1)})\phi_{jm}(v^{(2)})=\frac{(\vec{d}^{(1)*}\cdot\vec{d}^{(2)})^j}{2^j(2j)!} \tag{A26}$$

so that, from Eqs. (A24) and (A26),

$$\exp(x_{1N}\vec{d}^{(1)*}\cdot\vec{d}^{(2)})=\sum_{k=0}^{\infty}\sum_{q=-k}^{k}2^{2k}(2k-1)!!\,x_{1N}{}^k \times\phi_{kq}^*(v^{(1)})\phi_{kq}(v^{(2)}). \tag{A27}$$

(2) If $Y_{kq}(\vec{r})$ represents a solid spherical harmonic including the factor r^k, then

$$\exp(\vec{d}\cdot\vec{r})=\sum_{k=0}^{\infty}\sum_{q=-k}^{k}2^k\left[\frac{4\pi}{2k+1}\right]^{1/2}\phi_{kq}(v)Y_{kq}(\vec{r}). \tag{A28}$$

Then we obtain, upon substituting Eqs. (A27) and (A28) into (A20),

$$N=\sum_{k_1=0}^{\infty}\sum_{q_1=-k_1}^{k_1}\sum_{k_2=0}^{\infty}\sum_{q_2=-k_2}^{k_2}\sum_{k_3=0}^{\infty}\sum_{q_3=-k_3}^{k_3}2^{k_1+2k_2+k_3}(2k_2-1)!!\,x_{1N}{}^{k_2}$$
$$\times\frac{[(k_1+k_2+q_1+q_2)!(k_1+k_2-q_1-q_2)!(k_2+k_3+q_2+q_3)!(k_2+k_3-q_2-q_3)!]^{1/2}}{(k_2+q_2)!(k_2-q_2)!}$$
$$\times\phi_{k_1+k_2,q_1+q_2}^*(v^{(1)})\phi_{k_2+k_3,q_2+q_3}(v^{(2)})\mathcal{Y}_{k_1q_1}^*(\vec{P}_\rightarrow)\mathcal{Y}_{k_3q_3}(-\vec{P}_\leftarrow). \tag{A29}$$

The definition of the $\mathcal{Y}$'s is

$$\mathcal{Y}_{kq}(\vec{r})=\left[\frac{4\pi}{(2k+1)(k+q)!(k-q)!}\right]^{1/2}Y_{kq}(\vec{r}). \tag{A30}$$

A trivial change in the summation indices in Eq. (A29) yields

$$N=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{j'=0}^{\infty}\sum_{m'=-j'}^{j'}\sum_{k=\max(0,j-j')}^{j}\sum_{q=q_{\min}}^{q_{\max}}\phi_{j'm'}^*(v^{(1)})\phi_{jm}(v^{(2)})$$
$$\times 2^{j+j'}(2j-2k-1)!!\frac{[(j+m)!(j-m)!(j'+m')!(j'-m')!]^{1/2}}{(j+m-k-q)!(j-m-k+q)!}x_{1N}{}^{j-k}$$
$$\times\mathcal{Y}_{j'-j+k,m'-m+q}^*(\vec{P}_\rightarrow)\mathcal{Y}_{kq}(-\vec{P}_\leftarrow), \tag{A31}$$

where

$$q_{\min}=\max[m-m'-(j'-j+k), m-(j-k), -k] \tag{A32}$$

and

$$q_{\max}=\min[m-m'+j'-j+k, m+j-k, k]. \tag{A33}$$

Furthermore, N may be alternatively expressed as

$$N=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{j'=0}^{\infty}\sum_{m'=-j'}^{j'} 2^{j+j'}[(2j-1)!!(2j'-1)!!]^{1/2}\phi^{*}_{j'm'}(v^{(1)})\phi_{jm}(v^{(2)})\,(\psi(j'm'),\ V_N\psi(jm)). \tag{A34}$$

On comparing Eqs. (A31) and (A34) we obtain the amplitude $A(jm;N;j'm')$, which for $j=j'$ and $m=m'$ becomes

$$A(jm;N;jm)=\sum_{k=0}^{j}\sum_{q=\max(-k,m-j+k)}^{\min(k,m+j-k)}\frac{(2j-2k-1)!!}{(2j-1)!!}\,\frac{(j+m)!(j-m)!}{(j+m-k-q)!(j-m-k+q)!}$$
$$\times\int d\phi_{N+2}x_{1N}{}^{j-k}\mathcal{Y}^{*}_{kq}(\vec{P}_{\rightarrow})\mathcal{Y}_{kq}(-\vec{P}_{\leftarrow}). \tag{A35}$$

Note that the limits on q in Eq. (A35) may be changed to simply $-k\leq q\leq k$. Then the sum over m of the amplitude $A(jm;N;jm)$ may be explicitly performed, since

$$\sum_{m=-j}^{j}\frac{(j+m)!(j-m)!}{(j+m-k-q)!(j-m-k+q)!}=(k+q)!(k-q)!\,{}^{2j+1}C_{2k+1}. \tag{A36}$$

Doing this sum and using the spherical harmonic addition theorem give the result

$$\sum_{m=-j}^{j}A(jm;N;jm)=(2j+1)j!\sum_{k=0}^{j}\frac{(-1)^k2^k}{(2k+1)!(j-k)!}\times\int d\phi_{N+2}x_{1N}{}^{j-k}P_k(\vec{P}_{\rightarrow}\cdot\vec{P}_{\leftarrow}), \tag{A37}$$

where P_k is the usual Legendre polynomial. It is clear that the above method can be readily generalized to include other multiperipheral graphs like Fig. 3, in which some of the propagators are "twisted."

The amplitude of Fig. 2 (summed over m) corresponds to the special case $N=4$ of Eq. (A37). Identifying the momenta p_1, p_2, p_3, and p_4 with P_a, P_b, $P_{\bar{b}}$, and $P_{\bar{a}}$, respectively, we have

$$\vec{P}_{\rightarrow}=\vec{p}_a+x_{12}\vec{p}_b+x_{13}\vec{p}_{\bar{b}}+x_{14}\vec{p}_{\bar{a}} \tag{A38}$$

and

$$\vec{P}_{\leftarrow}=\vec{p}_{\bar{a}}+x_{34}\vec{p}_{\bar{b}}+x_{24}\vec{p}_b+x_{14}\vec{p}_a, \tag{A39}$$

where $\vec{p}_a$ is the three-vector part of P_a, etc. The conditions of "forward" scattering for the process $a+b+\bar{x}\rightarrow a+b+\bar{x}$,

$$P_{\bar{a}}=-P_a,\quad P_{\bar{b}}=-P_b, \tag{A40}$$

simplify Eqs. (A38) and (A39) to

$$\vec{P}_{\rightarrow}=(1-x_1x_2x_3)\vec{p}_a+x_1(1-x_2)\vec{p}_b, \tag{A41}$$

$$\vec{P}_{\leftarrow}=-(1-x_1x_2x_3)\vec{p}_a-x_3(1-x_2)\vec{p}_b, \tag{A42}$$

so that

$$\mathcal{Y}^{*}_{kq}(\vec{P}_{\rightarrow})\mathcal{Y}_{kq}(-\vec{P}_{\leftarrow})=\sum_{l_1=0}^{k}\sum_{l_2=0}^{k}(1-x_1x_2x_3)^{2k-l_1-l_2}x_1{}^{l_1}x_3{}^{l_2}(1-x_2)^{l_1+l_2}\,W^{*}(k,q,l_1;\vec{p}_a,\vec{p}_b)W(k,q,l_2;\vec{p}_a,\vec{p}_b), \tag{A43}$$

where we have adopted the abbreviation

$$W(k,q,l;\vec{p}_a,\vec{p}_b)=\sum_{n=\max(-l,q-k+l)}^{\min(l,q+k-l)}\mathcal{Y}_{k-l,q-n}(\vec{p}_a)\mathcal{Y}_{ln}(\vec{p}_b) \tag{A44}$$

and employed the identity

$$\mathcal{Y}_{kq}(\vec{r}+\vec{s})=\sum_{l=0}^{k}W(k,q,l;\vec{r},\vec{s}). \tag{A45}$$

Then, from Eq. (A37)

$$A_f\equiv\sum_{m=-j}^{j}A_f(jm;4;jm)$$
$$=(2j+1)j!\sum_{k=0}^{j}\sum_{q=-k}^{k}\frac{2^k}{(2k+1)!(j-k)!}(k+q)!(k-q)!\sum_{l_1=0}^{k}\sum_{l_2=0}^{k}\tilde{B}_6(j,k,l_1,l_2)\,W^{*}(k,q,l_1;\vec{p}_a,\vec{p}_b)W(k,q,l_2;\vec{p}_a,\vec{p}_b), \tag{A46}$$

where

$$\tilde{B}_6(j,k,l_1,l_2)=\int_0^1\int_0^1\int_0^1 dx_1dx_2dx_3x_1^{-\alpha_{a\bar{x}}+j-k+l_1-1}(1-x_1)^{-\alpha_{ab}-1}$$
$$\times x_2^{-\alpha+j-k-1}(1-x_2)^{-\alpha_{b\bar{b}}+l_1+l_2-1}x_3^{-\alpha_{\bar{a}\bar{x}}+j-k+l_2-1}(1-x_3)^{-\alpha_{\bar{a}\bar{b}}-1}$$
$$\times(1-x_1x_2)^{-\alpha_{ab\bar{b}}+\alpha_{ab}+\alpha_{b\bar{b}}}(1-x_2x_3)^{-\alpha_{b\bar{b}\bar{a}}+\alpha_{\bar{a}\bar{b}}+\alpha_{b\bar{b}}}$$
$$\times(1-x_1x_2x_3)^{-\alpha_{b\bar{b}}-\alpha_{x\bar{x}}+\alpha_{ab\bar{b}}+\alpha_{b\bar{b}\bar{a}}+2k-l_1-l_2}, \qquad \text{(A47)}$$

with $\alpha=\alpha_0+\alpha' M^2$, $\alpha_{a\bar{x}}=\alpha_0+\alpha' s_{a\bar{x}}$, etc. The quantity $\tilde{B}_6$ has poles in α whenever $\alpha \geqslant j-k$. Defining

$$e=\alpha_{b\bar{b}}-l_1-l_2+1, \qquad \text{(A48)}$$

$$f=\alpha_{ab\bar{b}}-\alpha_{ab}-\alpha_{b\bar{b}}, \qquad \text{(A49)}$$

$$g=\alpha_{b\bar{b}\bar{a}}-\alpha_{\bar{a}\bar{b}}-\alpha_{b\bar{b}}, \qquad \text{(A50)}$$

and

$$h=\alpha_{x\bar{x}}+\alpha_{b\bar{b}}-\alpha_{ab\bar{b}}-\alpha_{b\bar{b}\bar{a}}-2k+l_1+l_2 \qquad \text{(A51)}$$

we see that the residue of $\tilde{B}_6$ at the pole $\alpha=N$ ($N\geqslant j-k$) is given by

$$\operatorname*{Res}_{\alpha=N}\tilde{B}_6(j,k,l_1,l_2)=\sum_{\substack{p,q,r,s=0\\ [p+q+r+s=N-j+k]}}^{\infty} R(e,p)R(f,q)R(g,r)R(h,s)$$
$$\times B_4(-\alpha_{a\bar{x}}+j-k+l_1+q+s;\,-\alpha_{ab})B_4(-\alpha_{\bar{a}x}+j-k+l_2+r+s;\,-\alpha_{\bar{a}\bar{b}}), \qquad \text{(A52)}$$

where $B_4(x,y)$ is the Euler beta function, and

$$R(n,m)=\frac{1}{m!}n(n+1)\cdots(n+m-1). \qquad \text{(A53)}$$

Combining Eqs. (A47) and (A52) we obtain the final expression for the quantity of interest $\mathrm{Res}_{\alpha=N}A_f$.

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[8]m is the mass of the first particle on the universal Regge trajectory in the DRM.

[9]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[10]We shall presently return to the question of the precise discontinuity.

[11]$\bar{x}$ denotes the antiparticle of x.

[12]A summation over the $(2j+1)$ spin states of the particle x is implied in this equation. m_a and m_b are the masses of the particles a and b, respectively, while T_f denotes the forward elastic amplitude.

[13]For this and other related problems occurring in the DRM see, for example, C. E. DeTar, K. Kang, Chung-I Tan, and J. H. Weis, Phys. Rev. D 4, 425 (1971).

[14]α_0 and α' denote the intercept and slope, respectively, of the trajectory.

[15]$\alpha_{a\bar{x}}=\alpha_0+\alpha' s_{a\bar{x}}$, etc.

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[17]J. C. Polkinghorne, Nuovo Cimento 7A, 555 (1972).

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[19]Only negative values of the intercept are considered in order to avoid the presence of tachyons on the leading trajectory.

[20]Here, and subsequently, we shall need to compare the results of our calculations with processes involving fermions. Although this procedure is questionable we hope that our results are qualitatively correct.

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[23]Our convention for the metric is $g_{\mu\nu}=(1,-1,-1,-1)$.

[24]Note that the units chosen are such that $2\alpha'=1\ \mathrm{GeV}^{-2}$.

[25]Strictly speaking, the momentum $P^{(n)}_{\rightarrow}$ must be complex conjugated (*). We shall proceed without the asterisk for the moment but will introduce it at a later stage.

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[27]ψ_0 denotes the vacuum state.