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Comments and Addenda

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Effect of the rotation of the central body on the orbit of a satellite

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We discuss the precession of the orbit of a satellite due to the rotation of the central body and express these results in terms of the longitude of the ascending node, Ω', the argument of the perihelion, ω', and the inclination of the orbit, i', measured in the central body's equatorial system. We find that the precession of the perihelion (the Runge-Lenz vector) has a component in the plane of the orbit equal to 2/3 of the precession of the argument of the perihelion, so that the apparent discrepancy in the results of Kalitzin and Bogorodskii is resolved. In addition, we show how the precession of the longitude of the perihelion (which was originally obtained by Lense and Thirring) is related to the results of Kalitzin and Bogorodskii. We also find the precession of the normal to the orbit to be consistent with the original result of Lense and Thirring, which is twice the result recently given by Breen.

I. INTRODUCTION

In a paper[1] which discussed the gravitational interaction of two spinning bodies we presented general expressions for the precession of the orbit of a satellite about a central body. We took into account the quadrupole moment of the central body, in addition to general-relativistic terms (including spin effects). We also showed how these results could be expressed in terms of the angles the astronomers use, viz., the longitude of the ascending node, Ω', the argument of the perihelion, ω', and the inclination of the orbit, i', measured in the central body's equatorial system.[1,2]

In this paper we wish to discuss further the effects of the spin of the central body on the orbit of a satellite, for there exists much confusion and disagreement in the literature on these results. These results were first given by Lense and Thirring[3] and later a discrepancy (a factor of $\cos i'$ instead of a factor of $[1-3\sin^2(i'/2)]$) was found in part of this work by Kalitzin.[4] Bogorodskii[5] found the same discrepancy but expressed his results in a different form from that of Kalitzin,[4] so that there then seemed to be another discrepancy of a factor of $\frac{2}{3}$. Bogorodskii[5] gives the precession of the *argument* of the perihelion while Kalitzin[4] gives the precession of the perihelion in the plane of the orbit. We shall show how these two results are related and thus account for the factor of $\frac{2}{3}$. Furthermore, it turns out that the result obtained by Lense and Thirring is actually the precession of the *longitude* of the perihelion, $d\tilde{\omega}'/dt$, and this accounts for the three different results. Recently Breen[6] calculated the precession of the normal to the orbit and obtained a result one-half of that of Lense and Thirring.[3] We show that the result of Breen[6] is incorrect and that the original result of Lense and Thirring[3] is correct in this respect. Our method of approach[1] is different from all of the above authors[3-6] in that we work with the Runge-Lenz vector and this results in a simplified and transparent treatment of the problem.

II. PRECESSION OF THE ORBIT

Let M and $\vec{S}$ denote the mass and spin angular momentum of the central body, respectively, while m and $\vec{L}$ denote the mass and orbital angular momentum of the satellite. The effect of the rotation of the central body on the orbit of the satellite is that the orbit will precess as a whole with an angular velocity, $\vec{\omega}$, such that[1,7]

$$\left(\frac{d\vec{L}}{dt}\right)_{av}=\vec{\omega}\times\vec{L}, \tag{1}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{av} = \vec{\omega}\times\vec{A}\,, \tag{2}$$

where

$$\vec{\omega} = \frac{2GS}{c^2 a^3 (1-e^2)^{3/2}}\,[\hat{S} - 3(\hat{L}\cdot\hat{S})\,\hat{L}]\,, \tag{3}$$

and $\vec{A}$, a, and e are the Runge-Lenz vector, semimajor axis, and eccentricity, respectively, for the orbit. The quantities G and c are the gravitational constant and the speed of light, respectively, and $(d\vec{L}/dt)_{av}$ and $(d\vec{A}/dt)_{av}$ are the average of $d\vec{L}/dt$ and $d\vec{A}/dt$ over one orbital period. We also note the relation[1]

$$\frac{L/m}{a^2(1-e^2)^{1/2}} = \left(\frac{GM}{a^3}\right)^{1/2} = \frac{2\pi}{T}\,, \tag{4}$$

where T is the orbital period.

We can write $\vec{\omega}$ in the form that astronomers use as

$$\vec{\omega} = \dot{\Omega}'\hat{S} + \dot{\omega}'\hat{L} + \dot{i}'(\hat{S}\times\hat{L})/|\hat{S}\times\hat{L}|\,, \tag{5}$$

where Ω', ω', and i' denote the longitude of the ascending node, the argument of the perihelion, and the inclination of the orbit, respectively, in the central body's equatorial system, and a dot denotes differentiation with respect to time.[1,2] We thus obtain from Eqs. (3) and (5)

$$\dot{\Omega}' = \frac{2GS}{c^2 a^3 (1-e^2)^{3/2}}\,, \tag{6}$$

$$\dot{\omega}' = \frac{-6GS\cos i'}{c^2 a^3(1-e^2)^{3/2}} = -3\,\dot{\Omega}'\cos i'\,, \tag{7}$$

$$\dot{i}' = 0\,, \tag{8}$$

which agree with the results of Bogorodskii.[5] We wish to emphasize the fact that, in general, the unit vectors used in Eq. (5) are *not* orthogonal (orthogonality occurs only in the case of polar orbits) because it seems that confusion on this point has led to errors in the past.

We now proceed by writing $\vec{\omega}$ in terms of three orthogonal unit vectors as

$$\vec{\omega} = \omega_1\hat{A} + \omega_2(\hat{L}\times\hat{A}) + \omega_3\hat{L}\,, \tag{9}$$

where

$$\omega_1 = \dot{\Omega}'\sin i'\sin\omega' + \dot{i}'\cos\omega'\,, \tag{10}$$

$$\omega_2 = \dot{\Omega}'\sin i'\cos\omega' - \dot{i}'\sin\omega'\,, \tag{11}$$

$$\omega_3 = \dot{\Omega}'\cos i' + \dot{\omega}'\,. \tag{12}$$

It should be noted that Ω', i', ω' are just the Eulerian angles ϕ, θ, ψ respectively and ω_1, ω_2, ω_3 are the components of the angular velocity in the orbit or "body" system.[8] Thus from Eqs. (2) and (9) we obtain

$$\left(\frac{d\hat{A}}{dt}\right)_{av} = \vec{\omega}\times\hat{A} = -\omega_2\hat{L} + \omega_3(\hat{L}\times\hat{A})\,, \tag{13}$$

which is the precession of the perihelion—not to be confused with $\dot{\omega}'$, the precession of the argument of the perihelion. It follows that the component of $(d\hat{A}/dt)_{av}$ in the plane of the orbit is ω_3. From Eqs. (7) and (12) we obtain

$$\omega_3 = -2\dot{\Omega}'\cos i' = \tfrac{2}{3}\,\dot{\omega}'\,, \tag{14}$$

which agrees with the result of Kalitzin[4] and Breen.[6] In other words, the component of the precession of the perihelion in the plane of the orbit is $\frac{2}{3}$ of the precession of the argument of the perihelion. Kalitzin[4] noted that the result of Lense and Thirring[3] had a factor $[1-3\sin^2(i'/2)]$ instead of the factor $\cos i'$ appearing in the center of Eq. (14). However, the result presented by Lense and Thirring was actually $d\tilde{\omega}'/dt$, where $\tilde{\omega}'$ is the longitude of the perihelion, i.e.,

$$\tilde{\omega}' = \Omega' + \omega'\,. \tag{15}$$

Explicitly, we have from Eqs. (7) and (15)

$$\begin{aligned}\frac{d\tilde{\omega}'}{dt} &= \dot{\Omega}'(1-3\cos i') \\ &= -2\dot{\Omega}'[1-3\sin^2(i'/2)]\,.\end{aligned} \tag{16}$$

Let us now consider the precession of the normal to the orbit. From Eqs. (1), (3), and (6) we get

$$\left(\frac{d\hat{L}}{dt}\right)_{av} = \vec{\omega}\times\hat{L} = \dot{\Omega}'\sin i'(\hat{S}\times\hat{L})/|\hat{S}\times\hat{L}|\,. \tag{17}$$

Thus the precession of the normal to the orbit is consistent with the original result of Lense and Thirring,[3] which is twice the result recently given by Breen.[6]

III. CONCLUSION

We have analyzed the effect of the spin of the central body on the orbit of a satellite and have clarified the discrepancies (both real and apparent) that exist in the literature on this problem. The use of the Runge-Lenz vector results in a simplified and transparent treatment of the problem.

We have shown that the basic results from which all others are derived are given by Eqs. (1)–(3). The derived quantities which are most used from the point of view of comparison with observations are given by Eqs. (6)–(8).

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Modified electron propagation function in strong magnetic fields*

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The lowest-order radiative correction to the electron propagation function in homogeneous magnetic fields is calculated exactly, using an explicit form of the electron propagation function obtained by Schwinger in 1951.

Recently, the lowest-order radiative correction to the inverse electron propagation function (usually referred to as the mass operator or the renormalized proper-self-energy part) in homogeneous magnetic fields has been calculated exactly by Schwinger,[1-3] using a proper-time method.[4] From this mass operator, we can compute[2,3,5,6] the energy shift, the anomalous magnetic moment, the decay rate, the power spectrum of synchrotron radiation, and the radiative polarization of an electron in the magnetic field. Besides, it is also a building stone in calculating higher-order processes.[7-9] Owing to its great importance and the fact that this is one of the few problems in quantum electrodynamics that can be solved exactly, we offer here another method to compute it.

Our method is particularly simple if we know the Green's function of an electron in homogeneous magnetic fields. Fortunately this task was carried out more than two decades ago by Schwinger.[4] The details of the calculation are illustrated quite clearly in Ref. 4; here we only quote the result (specialized to the pure magnetic field situation, with $F_{12}=-F_{21}=H$)

$$G(x',x'')=\Phi(x',x'')\mathcal{G}(x'-x''), \qquad (1)$$

where

$$\mathcal{G}(x)=\frac{1}{(4\pi)^2}\int_0^\infty \frac{ds_1}{s_1}\,e^{-is_1m^2}\,\frac{z}{\sin z}\,e^{i\zeta z}\times\exp\left[\frac{i}{4s_1}(x_\parallel{}^2+z\cot z\,x_\perp{}^2)\right]\times\left[m-\frac{1}{2s_1}\left(\gamma x_\parallel+\frac{z}{\sin z}e^{-i\zeta z}\gamma x_\perp\right)\right], \qquad (2)$$

$$\Phi(x',x'')=\exp\left[ieq\int_{x''}^{x'}A(\xi)d\xi\right], \qquad (3)$$

$$ab_\parallel=-a^0b^0+a_3b_3,\quad ab_\perp=a_1b_1+a_2b_2,\qquad z=s_1eH,\quad \zeta=q\sigma_3. \qquad (4)$$

We find it is more convenient to cast Eq. (2) in the momentum representation, and to rewrite Eq. (1) in the form

$$G(x',x'')=\Phi(x',x'')\int\frac{(dp)}{(2\pi)^4}\,e^{ip(x'-x'')}\mathcal{G}(p), \qquad (5)$$

where

$$\mathcal{G}(p)=\int\frac{(dx)}{(2\pi)^4}\,e^{-ipx}\,\mathcal{G}(x)=i\int_0^\infty ds_1\exp\left[-is_1\left(m^2+p_\parallel{}^2+\frac{\tan z}{z}\,p_\perp{}^2\right)\right]\times\frac{e^{i\zeta z}}{\cos z}\left(m-\gamma p_\parallel-\frac{e^{-i\zeta z}}{\cos z}\gamma p_\perp\right). \qquad (6)$$

In the absence of external fields ($H=0$), Eq. (6) reduces to

$$\mathcal{G}(p)\to\frac{1}{m+\gamma p}, \qquad (7)$$

as is to be expected.

Before we proceed, we remark that, in obtaining Eq. (1) or Eq. (5), two basic quantities have to be evaluated: $\langle x'|e^{-is\Pi^2}|x''\rangle$ and $\langle x'|e^{-is\Pi^2}\gamma\Pi|x''\rangle$. Schwinger used the proper-time method to evaluate them and obtained (in the momentum representation)