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Alternative on-shell unitary formalisms in three-particle scattering*

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The on-shell unitary formalism for three-particle scattering found by Cahill is simplified and its connections with the alternative K-matrix formalism is established. The different results obtained in the approximate but fully unitary three-nucleon calculations which have been carried out to date using the two formalisms are found to be attributable to distinct choices of the approximate input. This last circumstance is obscured in the original versions of the two formalisms. It is concluded that most, if not all, differences between alternative on-shell unitary formalisms are illusory and result from input inequivalence in the various approximation schemes that may seem suited to the various formalisms.

I. INTRODUCTION

In the past few years several different formalisms have been developed for generating approximate three-particle scattering amplitudes which satisfy all the requirements of three-particle unitarity.[1-6] These formalisms share the common feature of on-shell integral equations for the physical scattering amplitudes in which the unknown input terms need satisfy only relatively simple conditions in order to generate a unitary theory. The spirit of the approach is analogous to the standard K-matrix formalism in scattering problems involving only two-particle channels.

Several reasons for seriously considering such formalisms have been elaborated upon previously.[1-6] In addition, we point out that the experience gained in the study of multiparticle unitarization techniques in the case of the three-particle problem, where exact numerical solutions are known, may be very useful in applications to situations involving more than three particles. In the latter case the exact solution of the scattering integral equations becomes prohibitive with increasing particle number and the reliance upon approximate techniques appears inevitable.

Two of the extant unitary formalisms[1,2] have been used to carry out approximate calculations of three-nucleon scattering.[7,8] In both cases a few approximate forms of the input functions which are suggested by the structure of the respective formalisms were used in the relevant quasi-Heitler on-shell integral equations. The results in both instances were in qualitative agreement with the exact results[9,10] corresponding to the same two-nucleon interactions. However, the two approximate, albeit unitary, solutions differ even for those choices of input which appear to be of "lowest order" in each formalism.

This circumstance again raises the question of the relationship of the two formalisms and their relative superiority, if any, in generating approximate three-particle amplitudes.[2] Obviously, all possible unitary formalisms are equivalent if no approximations are made. The essential question, then, is the comparison of the various formalisms for given approximate inputs.

We will call two unitary formalisms *input equivalent* if their exact input functions (e.g., the K matrices) are the same and if identical approximate forms of the input functions yield identical physical scattering amplitudes via each formalism. Evidently not all possible on-shell unitary formalisms are input equivalent, but all can be placed in equivalence by redefinitions of their input functions. Such a redefinition is essential if one is to compare calculations carried out with different formalisms such as those of Refs. 7 and 8.

In the present work we establish the input equivalence of Cahill's formalism to a new unitary formalism which is very closely related to the three-body K-matrix formalism.[2] This allows us to compare the approximations employed in the calculations of Refs. 7 and 8, and we find that they are quite distinct, which probably accounts for the different results obtained. A restatement and simplification of Cahill's formalism is necessary for our proof of input equivalence. In particular we show that one member of Cahill's hierarchy of four sets of integral equations is superfluous, and as a consequence the complexity of Cahill's original work is reduced to a level comparable with the formalism of Ref. 2.

The main conclusion to be drawn from our investigation is that there is probably very little to distinguish the various on-shell unitary formalisms in terms of their practical use. Significant apparent differences may arise as a consequence of input inequivalence, but these are, of course, illusory. An open problem with all extant on-shell unitary theories is the lack of compelling reasons for specific choices of approximate input. This is in marked contrast with some *off-shell* unitary formalisms that have been proposed.[11,12]

II. NOTATION AND DEFINITIONS

Most of the notations we will employ for the various two- and three-particle scattering operators are the same as those used by Cahill.[1] Some departures from Ref. 1 do occur, and these are noted in an effort to maintain some notational consistency with our previous publications.[2-5]

Our starting point is the operator $F(z)$ [$\tilde{M}(z)$ in Ref. 1] which satisfies the integral equations

$$F(z) = \bar{\delta}\, G_0(z) + \bar{\delta}\, G_0(z)\, t(z)\, F(z) \tag{2.1a}$$

$$= \bar{\delta}\, G_0(z) + F(z)\, t(z)\, \bar{\delta}\, G_0(z)\,, \tag{2.1b}$$

where we have employed a matrix notation with respect to the channel indices.[13] That is, $F(z)$ represents the 3×3 matrix whose elements are the operators $F_{\beta\alpha}(z)$ ($\beta, \alpha = 1, 2, 3$). $t(z)$ is a diagonal matrix whose elements are the two-particle transition operators $t_\alpha(z)$ on the three-particle Hilbert space; the index α on $t_\alpha(z)$ refers to that channel α in which particle α ($=1, 2, 3$) is asymptotically free. $\bar{\delta}$ is the matrix with elements $1 - \delta_{\beta\alpha}$, and, finally,

$$G_0(z) = (z - H_0)^{-1}\,,$$

where H_0 is the total three-particle kinetic energy operator and z is a complex parametric energy.

The connection between $F(z)$ and the operators corresponding to the physical scattering amplitudes is obtained as follows in the circumstance in which there are no three-body forces, which we suppose is the case. We introduce a projection operator P_α onto the channel α ($=1, 2, 3$):

$$P_\alpha \equiv \sum_{E,\,\eta_\alpha} |\,\phi_\alpha(\eta_\alpha, E)\rangle\langle\phi_\alpha(\eta_\alpha, E)|\,,$$

where the channel states $|\,\phi_\alpha(\eta_\alpha, E)\rangle$ for $\alpha = 1, 2, 3$ refer to noninteracting two-particle states composed of particle α moving freely and a bound state of the other pair. E is the energy of the configuration and η_α refers to any other labels which are needed to specify the asymptotic configuration in channel α including an index covering the possibility of more than a single two-body bound state in that channel. [We will let $|\,\phi_0(\eta_0, E)\rangle$ refer to a three-particle plane-wave state with a similar interpretation for E and η_0.] Then we define

$$M(z) \equiv t(z)F(z)t(z)\,, \tag{2.2a}$$

$$M^R(z) \equiv t(z)F(z)VP\,, \tag{2.2b}$$

$$M^L(z) \equiv PVF(z)t(z)\,, \tag{2.2c}$$

$$M^{LR}(z) \equiv PVF(z)VP\,, \tag{2.2d}$$

where V is the diagonal matrix $(V_\alpha\delta_{\alpha\beta})$ whose elements are the two-particle potentials V_α and P is the diagonal matrix $(P_\alpha\delta_{\alpha\beta})$.

The amplitude for the connected portion of the 3-to-3 process is given by the on-shell values of the matrix element

$$\left\langle \phi_0 \left| \sum_{\alpha,\,\beta} M_{\alpha\beta} \right| \phi_0 \right\rangle\,. \tag{2.3a}$$

The amplitudes for the breakup process ($\alpha \to 0$), the formation process ($0 \to \alpha$), and rearrangement ($\alpha \to \beta$), or elastic for $\alpha = \beta$, scattering are given by the on-shell values of the matrix elements

$$\left\langle \phi_0 \left| \sum_\gamma M^R_{\gamma\alpha} \right| \phi_\alpha \right\rangle\,, \tag{2.3b}$$

$$\left\langle \phi_\alpha \left| \sum_\gamma M^L_{\alpha\gamma} \right| \phi_0 \right\rangle\,, \tag{2.3c}$$

and

$$\langle\, \phi_\beta | M^{LR}_{\beta\alpha} |\, \phi_\alpha\rangle\,, \tag{2.3d}$$

respectively. In what follows we will often refer to the on-shell matrix elements of, for example, the operator $M^{LR}(z)$. We mean by this the matrix with the on-shell elements $\langle\, \phi_\beta | M^{LR}_{\beta\alpha} |\, \phi_\alpha\rangle$. Similar remarks apply to $M(z)$, $M^R(z)$, and $M^L(z)$ as well as to the various auxiliary operators which appear in the course of our development.

III. UNITARY CONSTRAINTS

In this section we find the constraints, in the form of discontinuity equations and Hermitian analyticity conditions, imposed by unitarity on the M operators defined by Eqs. (2.2). This has been done by Cahill.[1] We will repeat this here not only for the sake of completeness and to introduce our particular notations, but primarily because we find that a simple restatement of the discontinuity equations suggests a means for significantly simplifying Cahill's entire formalism.

Using standard techniques[13,14] we find from Eqs. (2.1) and the two-particle off-shell unitarity relation,

$$\Delta t = -2i\, t(\pm)\, D_0\, t(\mp) - 2i\, VDV\,, \tag{3.1}$$

that

$$\Delta F = -2i[F(\pm)\, t(\pm)\, \bar{D}_0\, t(\mp)\, F(\mp) + F(\pm)\, VDVF(\mp)]$$

$$+ [F(\pm)t(\pm)\Delta\zeta + \Delta\zeta t(\mp)\, F(\mp) + \Delta\zeta]\,. \tag{3.2}$$

We have employed the following notational conventions in Eqs. (3.1) and (3.2). For an operator $\mathcal{O}(z)$ which is a function of z

$$\Theta(\pm) \equiv \Theta(E \pm i0)$$

and

$$\Delta\Theta \equiv \Theta(+) - \Theta(-) .$$

Also

$$\zeta(z) \equiv \bar{\delta} G_0(z)$$

and D is a diagonal matrix with elements D_α ($\alpha = 1, 2, 3$) defined by

$$D_\alpha \equiv \sum_{E'_\alpha, \eta_\alpha} |\phi_\alpha(\eta_\alpha, E'_\alpha)\rangle \delta(E - E'_\alpha) \langle \phi_\alpha(\eta_\alpha, E'_\alpha)| . \qquad (3.3)$$

D_0 is simply the unit matrix (in the channel indices) times the operator (3.3) with $\alpha = 0$. Finally,

$$\bar{D}_0 \equiv (1 + \bar{\delta}) D_0 .$$

We find from (3.2) and the definitions (2.2) that *on shell*

$$\Delta M = -2i[M(\pm)\bar{D}_0 M(\mp) + M^R(\pm) D M^L(\mp) + M(\pm)\bar{D}_0 t(\mp) + t(\pm)\bar{D}_0 M(\mp) + t(\pm)\bar{\delta} D_0 t(\mp)], \qquad (3.4a)$$

$$\Delta M^R = -2i[M(\pm)\bar{D}_0 M^R(\mp) + M^R(\pm) D M^{LR}(\mp) + t(\pm)\bar{D}_0 M^R(\mp)], \qquad (3.4b)$$

$$\Delta M^L = -2i[M^L(\pm)\bar{D}_0 M(\mp) + M^{LR}(\pm) D M^L(\mp) + M^L(\pm)\bar{D}_0 t(\mp)], \qquad (3.4c)$$

$$\Delta M^{LR} = -2i[M^L(\pm)\bar{D}_0 M^R(\mp) + M^{LR}(\pm) D M^{LR}(\mp)] . \qquad (3.4d)$$

We stress that Eqs. (3.4) are to be interpreted as on-shell equations in the sense described at the end of the preceding section. In this case Eqs. (3.4) constitute part of the constraints imposed by unitarity on the various amplitudes.

In contrast with the other M operators which are directly related to complete physical scattering amplitudes, M refers only to the connected part of the 3-to-3 amplitude. This separation of the disconnected and connected parts of the 3-to-3 amplitude leads to the complicated forms of Eqs. (3.4a)–(3.4c) and also to an unnecessary complexity in Cahill's entire formalism. Instead, we define

$$\gamma(z) \equiv t(z) + M(z) \qquad (3.5)$$

so that the complete amplitude for the 3-to-3 process is given by the on-shell values of

$$\left\langle \phi_0 \middle| \sum_{\alpha,\beta} \gamma_{\alpha\beta} \middle| \phi_0 \right\rangle .$$

Then instead of Eqs. (3.4a)–(3.4c) we obtain

$$\Delta\gamma = -2i[\gamma(\pm)\bar{D}_0\gamma(\mp) + M^R(\pm) D M^L(\mp)], \qquad (3.4a')$$

$$\Delta M^R = -2i[\gamma(\pm)\bar{D}_0 M^R(\mp) + M^R(\pm) D M^{LR}(\mp)], \qquad (3.4b')$$

$$\Delta M^L = -2i[M^L(\pm)\bar{D}_0\gamma(\mp) + M^{LR}(\pm) D M^L(\mp)], \qquad (3.4c')$$

with (3.4d) unchanged.

Equations (3.4a′)–(3.4c′) and (3.4d) are not yet equivalent to the physical unitarity constraints.[13] The latter are obtained by requiring in addition the Hermitian analyticity conditions

$$\gamma(\pm)^\dagger = \gamma(\mp), \qquad (3.6a)$$

$$[M^R(\pm)]^\dagger = M^L(\mp), \qquad (3.6b)$$

$$[M^{LR}(\pm)]^\dagger = M^{LR}(\mp) . \qquad (3.6c)$$

Note that since we are using a channel-index notation the adjoint operation in Eqs. (3.6) includes a matrix transposition with respect to the channel indices.[13] For example,

$$[M^R_{\beta\alpha}(\pm)]^\dagger = M^L_{\alpha\beta}(\mp) .$$

IV. CAHILL UNITARY FORMALISM

We will proceed in a manner somewhat different from that employed by Cahill.[1] The present treatment is facilitated by the use of the channel matrix notation and $\gamma(z)$ instead of the connected operator $M(z)$. This leads not only to a simpler derivation and proof of consistency but also to a real simplification in the demonstration that one of Cahill's original four sets of equations is superfluous.

From Eqs. (2.1), (2.2a), and (3.5) we find that

$$\gamma(z) = t(z) + t(z)\bar{\delta} G_0(z)\gamma(z)$$
$$= t(z) + \gamma(z)\bar{\delta} G_0(z) t(z) . \qquad (4.1)$$

Upon separating $G_0(\pm)$ into its principal-value and Dirac δ-function parts,

$$G_0(\pm) = G \mp i D_0 ,$$

and letting $\mu(\pm)$ be the solution of

$$\mu(\pm) = t(\pm) + t(\pm)\bar{\delta} G \mu(\pm)$$
$$= t(\pm) + \mu(\pm)\bar{\delta} G t(\pm) , \qquad (4.2)$$

we can convert Eqs. (4.1) into the on-shell integral equations

$$\gamma(\pm) = \mu(\pm) \mp i\mu(\pm)\bar{\delta} D_0\gamma(\pm)$$
$$= \mu(\pm) \mp i\gamma(\pm)\bar{\delta} D_0\mu(\pm) . \qquad (4.3)$$

The operator $\mu(\pm)$ is related to Cahill's $N(\pm)$ by

$$N(\pm) = \mu(\pm) - t(\pm) .$$

Next, we introduce a two-body reaction matrix $R(\pm)$ defined by

$$t(\pm)=R(\pm)\mp iR(\pm)D_0t(\pm)$$
$$=R(\pm)\mp it(\pm)D_0R(\pm). \qquad (4.4)$$

$R(\pm)$ still contains any relevant two-particle bound-state pole singularities; however, $R(+)=R(-)$ except at these poles. Then, if we introduce the operators $\nu(\pm)$ as solutions of the integral equations

$$\nu(\pm)=R(\pm)+R(\pm)\bar{\delta}G\nu(\pm)$$
$$=R(\pm)+\nu(\pm)\bar{\delta}GR(\pm), \qquad (4.5)$$

we see that

$$\mu(\pm)=\nu(\pm)\mp i\nu(\pm)D_0\mu(\pm)$$
$$=\nu(\pm)\mp i\mu(\pm)D_0\nu(\pm). \qquad (4.6)$$

The operator $\nu(\pm)$ is related to Cahill's $Q(\pm)$ by

$$Q(\pm)=\nu(\pm)-R(\pm),$$

and Eqs. (4.6) correspond to Cahill's equations relating $N(\pm)$ and $Q(\pm)$.[15]

We note that Eqs. (4.6) are nonconnected-kernel equations as are the counterparts of these [for $N(\pm)$] derived by Cahill.[1] This circumstance is totally unrelated to the inclusion of disconnected parts in $\mu(\pm)$ and $\nu(\pm)$. Equations (4.6) can be converted into connected-kernel equations using standard techniques,[3] but since they play only a transient role in our development there is no need to do so.

Indeed, from Eqs. (4.3) and (4.6) it is easy to show that

$$\gamma(\pm)=\nu(\pm)\mp i\nu(\pm)\bar{D}_0\gamma(\pm)$$
$$=\nu(\pm)\mp i\gamma(\pm)\bar{D}_0\nu(\pm), \qquad (4.7)$$

and therefore we have eliminated the role of $\mu(\pm)$ in relating $\gamma(\pm)$ to $\nu(\pm)$. This suggests that we should be able to eliminate all of the various N operators, to be defined below, from Cahill's formalism, and this turns out to be the case. The reason for this is that the various N operators have basically the same singularity structure as the original amplitudes. In contrast with this the operators $\nu(\pm)$, for example, have discontinuities across only the two-particle unitary cuts which are generated by the two-particle bound-state poles. Equations (4.7) explicate the singularity structure of $\gamma(\pm)$ associated with the three-particle unitary cut.

We note that Eqs. (4.7) are nonconnected-kernel integral equations. We emphasize that this is unrelated to our elimination of the N-type operators in this instance or in those to follow since Cahill's[1] original equations exhibit similar nonconnectedness properties. The connected-kernel forms of Eqs. (4.7) can be easily obtained using standard techniques,[3] namely,

$$\gamma(\pm)=t(\pm)+Q^l(\pm)\mp it(\pm)\bar{\delta}D_0\gamma(\pm)$$
$$\mp iQ^l(\pm)\bar{D}_0\gamma(\pm)$$
$$=t(\pm)+Q^r(\pm)\mp i\gamma(\pm)\bar{\delta}D_0t(\pm)$$
$$\mp i\gamma(\pm)\bar{D}_0Q^r(\pm), \qquad (4.7')$$

where

$$Q^l(\pm)\equiv[1\mp it(\pm)D_0]Q(\pm),$$
$$Q^r(\pm)\equiv Q(\pm)[1\mp iD_0t(\pm)].$$

We now follow part of Cahill's development rather closely. If we introduce operators $\tilde{N}(\pm)$ as solutions of

$$\tilde{N}(\pm)=\bar{\delta}G+\bar{\delta}Gt(\pm)\tilde{N}(\pm)$$
$$=\bar{\delta}G+\tilde{N}(\pm)t(\pm)\bar{\delta}G, \qquad (4.8)$$

then we find from Eqs. (2.1) that

$$F(\pm)=\tilde{N}(\pm)\mp i[\bar{\delta}D_0+\bar{\delta}D_0t(\pm)F(\pm)+\tilde{N}(\pm)t(\pm)\bar{\delta}D_0+\tilde{N}(\pm)t(\pm)\bar{\delta}D_0t(\pm)F(\pm)]$$
$$=\tilde{N}(\pm)\mp i[\bar{\delta}D_0+F(\pm)t(\pm)\bar{\delta}D_0+\bar{\delta}D_0t(\pm)\tilde{N}(\pm)+F(\pm)t(\pm)\bar{\delta}D_0t(\pm)\tilde{N}(\pm)]. \qquad (4.9)$$

From the definitions (2.2b)–(2.2d) and Eqs. (4.9) it follows that

$$M^R(\pm)=N^R(\pm)\mp i\mu(\pm)\bar{\delta}D_0M^R(\pm)$$
$$=N^R(\pm)\mp i\gamma(\pm)\bar{\delta}D_0N^R(\pm), \qquad (4.10a)$$

$$M^L(\pm)=N^L(\pm)\mp iN^L(\pm)\bar{\delta}D_0\gamma(\pm)$$
$$=N^L(\pm)\mp iM^L(\pm)\bar{\delta}D_0\mu(\pm), \qquad (4.10b)$$

and

$$M^{LR}(\pm)=N^{LR}(\pm)\mp iN^L(\pm)\bar{\delta}D_0M^R(\pm)$$
$$=N^{LR}(\pm)\mp iM^L(\pm)\bar{\delta}D_0N^R(\pm), \qquad (4.10c)$$

where $N^R(\pm)$, $N^L(\pm)$, and $N^{LR}(\pm)$ are defined by Eqs. (2.2) with $F(\pm)$ replaced by $\tilde{N}(\pm)$. Equations (4.3) are of the same class as Eqs. (4.10), and as in the passage from (4.3) to (4.7) we will show that Eqs. (4.10) constitute merely an intermediate step in this unitary formalism rather than an intrinsic part of it.

Next, again following Cahill, we introduce operators $\tilde{Q}(\pm)$ as solutions of the integral equations

$$\tilde{Q}(\pm)=\bar{\delta}G+\bar{\delta}GR(\pm)\tilde{Q}(\pm)$$
$$=\bar{\delta}G+\tilde{Q}(\pm)R(\pm)\bar{\delta}G\,. \tag{4.11}$$

$\tilde{N}(\pm)$ and $\tilde{Q}(\pm)$ are related via

$$\tilde{N}(\pm)=\tilde{Q}(\pm)\mp i\tilde{Q}(\pm)R(\pm)D_0t(\pm)\tilde{N}(\pm)$$
$$=\tilde{Q}(\pm)\mp i\tilde{N}(\pm)t(\pm)D_0R(\pm)\tilde{Q}(\pm)\,. \tag{4.12}$$

It is clear from Eqs. (4.11) that the only contributions to $\Delta\tilde{Q}$ will arise from the two-particle bound-state poles. At this stage all the singularities associated with the three-particle unitary cut have been explicated. However, our objective is to express all quantities in terms of operators with zero discontinuity across the entire unitary cut. To this end we define[16]

$$k\equiv R(\pm)\pm iVDV \tag{4.13}$$

and an operator $\tilde{C}$ as the solution of

$$\tilde{C}=\bar{\delta}G+\bar{\delta}Gk\tilde{C}$$
$$=\bar{\delta}G+\tilde{C}k\bar{\delta}G\,. \tag{4.14}$$

Since $\Delta k=0$ we see that $\Delta\tilde{C}=0$. Combining Eqs. (4.11) and (4.14) we get

$$\tilde{Q}(\pm)=\tilde{C}\mp i\tilde{C}VDV\tilde{Q}(\pm)$$
$$=\tilde{C}\mp i\tilde{Q}(\pm)VDV\tilde{C}\,. \tag{4.15}$$

It is evident that

$$Q(\pm)=R(\pm)\tilde{Q}(\pm)R(\pm)$$

and we define

$$Q^R(\pm)\equiv R(\pm)\tilde{Q}(\pm)VP\,,$$
$$Q^L(\pm)\equiv PV\tilde{Q}(\pm)R(\pm)\,, \tag{4.16}$$
$$Q^{LR}(\pm)\equiv PV\tilde{Q}(\pm)VP\,.$$

Then Eqs. (4.12) imply

$$N^R(\pm)=Q^R(\pm)\mp i\nu(\pm)D_0N^R(\pm)$$
$$=Q^R(\pm)\mp i\mu(\pm)D_0Q^R(\pm)\,, \tag{4.17a}$$

$$N^L(\pm)=Q^L(\pm)\mp iQ^L(\pm)D_0\mu(\pm)$$
$$=Q^L(\pm)\mp iN^L(\pm)D_0\nu(\pm)\,, \tag{4.17b}$$

$$N^{LR}(\pm)=Q^{LR}(\pm)\mp iQ^L(\pm)D_0N^R(\pm)$$
$$=Q^{LR}(\pm)\mp iN^L(\pm)D_0Q^R(\pm)\,. \tag{4.17c}$$

At this point, as with the deviation of Eqs. (4.7), we deviate from Cahill's development and use Eqs. (4.17) to express M^R, M^L, and M^{LR}, as given by Eqs. (4.10), directly in terms of the Q operators. This is easily carried out and results in the set

$$M^R(\pm)=Q^R(\pm)\mp i\nu(\pm)\bar{D}_0M^R(\pm) \tag{4.18a}$$
$$=Q^R(\pm)\mp i\gamma(\pm)\bar{D}_0Q^R(\pm)\,, \tag{4.18b}$$
$$M^L(\pm)=Q^L(\pm)\mp iQ^L(\pm)\bar{D}_0\gamma(\pm) \tag{4.19a}$$
$$=Q^L(\pm)\mp iM^L(\pm)\bar{D}_0\nu(\pm)\,, \tag{4.19b}$$
$$M^{LR}(\pm)=Q^{LR}(\pm)\mp iQ^L(\pm)\bar{D}_0M^R(\pm) \tag{4.20a}$$
$$=Q^{LR}(\pm)\mp iM^L(\pm)\bar{D}_0Q^R(\pm)\,. \tag{4.20b}$$

Equations (4.7) and (4.18)–(4.20) constitute the first set of a hierarchy of equations of our simplified version of the Cahill formalism. We note that Eqs. (4.18a) and (4.19b) are nonconnected-kernel integral equations, while Eqs. (4.18b), (4.19a), and (4.20) are quadrature rules. Once again, as with Eqs. (4.7), we remark that this appearance of nonconnected kernels is not the price paid for the elimination of the N operators since some of Cahill's original equations [cf. Eqs. (4.17)] also have such kernels. The connected-kernel counterparts of Eqs. (4.18a) and (4.19b) are, respectively,

$$M^R(\pm)=[1\mp it(\pm)D_0]Q^R(\pm)$$
$$\mp it(\pm)\bar{\delta}D_0M^R(\pm)\mp iQ^l(\pm)\bar{D}_0M^R(\pm)\,, \tag{4.18a$'$}$$

$$M^L(\pm)=Q^L(\pm)[1\mp iD_0t(\pm)]$$
$$\mp iM^L(\pm)\bar{\delta}D_0t(\pm)\mp iM^L(\pm)\bar{D}_0Q^r(\pm)\,. \tag{4.19b$'$}$$

Our final set of equations of the simplified Cahill unitary formalism connects the Q operators to singularity-free input operators which are related to $\tilde{C}$. We define

$$C\equiv R(\pm)\tilde{C}R(\pm)\,,$$
$$C^R\equiv R(\pm)\tilde{C}VP\,,$$
$$C^L\equiv PV\tilde{C}R(\pm)\,, \tag{4.21}$$
$$C^{LR}\equiv PV\tilde{C}VP\,.$$

We note that on shell

$$C=k\tilde{C}k,$$
$$C^R=k\tilde{C}VP\,, \tag{4.22}$$
$$C^L=PV\tilde{C}k\,,$$

which makes manifest the fact that on the shell

$$\Delta C=\Delta C^R=\Delta C^L=\Delta C^{LR}=0 \tag{4.23a}$$

and

$$C^\dagger=C,\ (C^L)^\dagger=C^R,\ (C^{LR})^\dagger=C^{LR}\,. \tag{4.23b}$$

From Eqs. (4.15) along with definitions (4.21) we obtain our second and final set in our hierarchy of equations of our simplified version of the Cahill formalism[17]:

$$Q(\pm) = C \mp iC^{R}DQ^{L}(\pm) \tag{4.24a}$$

$$= C \mp iQ^{R}(\pm)DC^{L}, \tag{4.24b}$$

$$Q^{R}(\pm) = C^{R} \mp iC^{R}DQ^{LR}(\pm) \tag{4.25a}$$

$$= C^{R} \mp iQ^{R}(\pm)DC^{LR}, \tag{4.25b}$$

$$Q^{L}(\pm) = C^{L} \mp iC^{LR}DQ^{L}(\pm) \tag{4.26a}$$

$$= C^{L} \mp iQ^{LR}DC^{L}, \tag{4.26b}$$

$$Q^{LR}(\pm) = C^{LR} \mp iC^{LR}DQ^{LR}(\pm) \tag{4.27a}$$

$$= C^{LR} \mp iQ^{LR}DC^{LR} \tag{4.27b}$$

We remark that the integral equations among Eqs. (4.24)–(4.27) are of a rather trivial type which become algebraic equations after partial-wave analysis.

The unitary formalism at hand consists in the set of Eqs. (4.7), (4.18)–(4.20), and the set of Eqs. (4.24)–(4.27) with input C, C^R, C^L, and C^{LR} subject to conditions (4.23).[18] To establish this assertion we have to demonstrate that if we are given proper input, then amplitudes $\gamma(\pm)$, $M^R(\pm)$, $M^L(\pm)$, and $M^{LR}(\pm)$ are generated which satisfy Eqs. (3.4′) and Eqs. (3.6). However, in contrast with Cahill's discussion of this point we will not make any use of the original off-shell quantities such as $\tilde{C}$, $\tilde{Q}$, and $\tilde{N}$ which are now irrelevant to the definition of the formalism.

Before doing this it is important to prove that given proper input the two versions of each of Eqs. (4.7), (4.18)–(4.20) and (4.24)–(4.27) are consistent with each other. Evidently the integral equations (4.27a) and (4.27b) define the same $Q^{LR}(\pm)$ so that these equations are mutually consistent. Given this we note using Eqs. (4.27) that

$$[C^{L} \mp iQ^{LR}(\pm)DC^{L}] = C^{L} \mp iC^{LR}D[C^{L} \mp iQ^{LR}(\pm)DC^{L}],$$

which shows that both forms of Eqs. (4.26) generate the same $Q^L(\pm)$. Similarly, given the $Q^{LR}(\pm)$ defined by (4.27) we see that Eqs. (4.25) are consistent. Finally, given Eqs. (4.25) and (4.26) one finds the relation

$$Q^{R}(\pm)DC^{L} = C^{R}DQ^{L}(\pm)$$

so that Eqs. (4.24) are consistent. Note that the only properties of the input we have assumed are (4.23a) which have been implicitly imposed in the writing of Eqs. (4.18)–(4.20).

A similar proof of the consistency of Eqs. (4.7), (4.18)–(4.20) is easily carried out which is entirely independent of the input. One begins from Eqs. (4.7) which evidently generate the same $\gamma(\pm)$. Given the latter it is seen that

$$[Q^{R}(\pm) \mp i\gamma(\pm)\overline{D}_0 Q^{R}(\pm)] = Q^{R}(\pm) \mp i\nu(\pm)\overline{D}_0[Q^{R}(\pm) \mp i\gamma(\pm)\overline{D}_0 Q^{R}(\pm)],$$

which establishes the consistency of Eqs. (4.18). The consistency of Eqs. (4.19) and (4.20) is proved in an analogous manner.

The preceding consistency proof highlights the seminal roles played by $\gamma(\pm)$ and $Q^{LR}(\pm)$ in this formalism. For example, the operators $M^L(\pm)$, $M^R(\pm)$, and $M^{LR}(\pm)$ are, in point of fact, defined by $\gamma(\pm)$, and the integral equations for $M^R(\pm)$ and $M^L(\pm)$ have the same kernels as do Eqs. (4.7). Similar remarks apply to $Q^{LR}(\pm)$.

Next we deduce the implications of the constraints (4.23b) on our proper input. Taking the adjoint of Eq. (4.27a),

$$[Q^{LR}(\pm)]^{\dagger} = C^{LR} \pm i[Q^{LR}(\pm)]^{\dagger}DC^{LR},$$

we infer that

$$[Q^{LR}(\pm)]^{\dagger} = Q^{LR}(\mp). \tag{4.28a}$$

Also, the relation

$$[Q^{R}(\pm)]^{\dagger} = Q^{L}(\mp) \tag{4.28b}$$

follows from (4.28a) in conjunction with Eqs. (4.25) and (4.26). Equation (4.28b), in turn, implies, with Eqs. (4.24),

$$Q(\pm)^{\dagger} = Q(\mp). \tag{4.28c}$$

The Hermitian analyticity satisfied by the (known) two-particle input $[t(\pm)^{\dagger} = t(\mp)]$ implies $R(\pm)^{\dagger} = R(\mp)$ so that

$$\nu(\pm)^{\dagger} = \nu(\mp). \tag{4.28c′}$$

With Eqs. (4.28) one can show that Eqs. (4.7), (4.18)–(4.20) generate operators satisfying the constraints (3.6).

Finally, we will investigate the consequences of conditions (4.23a) on the proper input, which are already embodied in Eqs. (4.24)–(4.27). From the latter we easily deduce the discontinuity relations

$$\Delta Q = -2iQ^{R}(\pm)DQ^{L}(\mp), \tag{4.29a}$$

$$\Delta Q^{R} = -2iQ^{R}(\pm)DQ^{LR}(\mp), \tag{4.29b}$$

$$\Delta Q^{L} = -2iQ^{LR}(\pm)DQ^{L}(\mp), \tag{4.29c}$$

$$\Delta Q^{LR} = -2iQ^{LR}(\pm)DQ^{LR}(\mp). \tag{4.29d}$$

It is now a straightforward matter to verify the discontinuity relations (3.4′) given Eqs. (4.29) in conjunction with Eqs. (4.7) and (4.18)–(4.20) and therefore the unitary nature of the formalism defined by Eqs. (4.7), (4.18)–(4.20), (4.24)–(4.27) along with the proper input.

V. ANOTHER UNITARY FORMALISM

The essential difference between the K matrix[2] and the Cahill (cf. Ref. 1 and Sec. IV) formalisms is in the order of singularity explication. Namely, with reference to the hierarchy of equations associated with each, in the former case the two-particle bound-state pole singularities are explicated first while in the latter they are explicated last. In this section we employ the same starting point as in Sec. IV but we reverse the order of singularity explication.[19] The result is a somewhat more convenient but substantially identical form of the K-matrix theory of Ref. 2. In the next section we show that this rephrasing of the K-matrix formalism is input equivalent to the simplified version of the Cahill formalism developed in Sec. IV.

First, let us decompose $t(\pm)$ into the two parts[13,20]

$$t(\pm)=\bar{t}(\pm)+t_b(\pm), \tag{5.1}$$

where

$$t_b(\pm)=\mp iVDV. \tag{5.2}$$

Using (5.1) Eqs. (2.1), with $z=E\pm i0$, can be rewritten as

$$F(\pm)=\bar{F}(\pm)+\bar{F}(\pm)t_b(\pm)F(\pm)$$
$$=\bar{F}(\pm)+F(\pm)t_b(\pm)\bar{F}(\pm), \tag{5.3}$$

where

$$\bar{F}(\pm)=\bar{\delta}G_0(\pm)+\bar{\delta}G_0(\pm)\bar{t}(\pm)\bar{F}(\pm)$$
$$=\bar{\delta}G_0(\pm)+\bar{F}(\pm)\bar{t}(\pm)\bar{\delta}G_0(\pm). \tag{5.4}$$

Let us again define $M(\pm)$, $M^{L,R}(\pm)$, and $M^{LR}(\pm)$ by Eqs. (2.2) but with $t(\pm)$ replaced by $\bar{t}(\pm)$. Obviously these two definitions are identical on shell, which is the only case of interest to us. Also we define operators $\bar{M}(\pm)$, $\bar{M}^{L,R}(\pm)$, and $\bar{M}^{LR}(\pm)$ by expressions similar to Eqs. (2.2) but with $t(\pm)$ replaced by $\bar{t}(\pm)$ and $F(\pm)$ replaced by $\bar{F}(\pm)$. Then from Eqs. (5.2) and (5.3) we find the first set of our hierarchy of equations:

$$M(\pm)=\bar{M}(\pm)\mp i\bar{M}^R(\pm)DM^L(\pm) \tag{5.5a}$$
$$=\bar{M}(\pm)\mp iM^R(\pm)D\bar{M}^L(\pm), \tag{5.5b}$$

$$M^R(\pm)=\bar{M}^R(\pm)\mp i\bar{M}^R(\pm)DM^{LR}(\pm) \tag{5.6a}$$
$$=\bar{M}^R(\pm)\mp iM^R(\pm)D\bar{M}^{LR}(\pm), \tag{5.6b}$$

$$M^L(\pm)=\bar{M}^L(\pm)\mp i\bar{M}^{LR}(\pm)DM^L(\pm) \tag{5.7a}$$
$$=\bar{M}^L(\pm)\mp iM^{LR}(\pm)D\bar{M}^L(\pm), \tag{5.7b}$$

and

$$M^{LR}(\pm)=\bar{M}^{LR}(\pm)\mp i\bar{M}^{LR}(\pm)DM^{LR}(\pm) \tag{5.8a}$$
$$=\bar{M}^{LR}(\pm)\mp iM^{LR}(\pm)D\bar{M}^{LR}(\pm). \tag{5.8b}$$

The net content of Eqs. (5.5)–(5.8) is the explication of the singularities arising from the two-particle bound-state pole singularities.

The subsequent development is simplified if we introduce

$$\gamma(\pm)\equiv\bar{t}(\pm)+M(\pm) \tag{5.9}$$

which is equivalent to the $\gamma(\pm)$ defined by Eq. (3.5) on shell. Also we let[21]

$$\Gamma(\pm)\equiv\bar{t}(\pm)+\bar{M}(\pm). \tag{5.10a}$$

Then Eqs. (5.5) become

$$\gamma(\pm)=\Gamma(\pm)\mp i\bar{M}^R(\pm)DM^L(\pm) \tag{5.5a$'$}$$
$$=\Gamma(\pm)\mp iM^R(\pm)D\bar{M}^L(\pm). \tag{5.5b$'$}$$

One can deal with the three-particle singularities in a manner formally identical to that followed in Sec. IV, except that Q-type operators never appear since these were associated with the explication of the two-body bound-state pole singularities. The result is the second, and last, in the hierarchy of equations of this particular unitary formalism:

$$\Gamma(\pm)=\kappa\mp i\kappa\bar{D}_0\Gamma(\pm) \tag{5.11a}$$
$$=\kappa\mp i\Gamma(\pm)\bar{D}_0\kappa, \tag{5.11b}$$

$$\bar{M}^R(\pm)=C^R\mp i\Gamma(\pm)\bar{D}_0C^R \tag{5.12a}$$
$$=C^R\mp i\kappa\bar{D}_0\bar{M}^R(\pm), \tag{5.12b}$$

$$\bar{M}^L(\pm)=C^L\mp iC^L\bar{D}_0\Gamma(\pm) \tag{5.13a}$$
$$=C^L\mp iM^L(\pm)\bar{D}_0\kappa, \tag{5.13b}$$

$$\bar{M}^{LR}(\pm)=C^{LR}\mp iC^L\bar{D}_0\bar{M}^R(\pm) \tag{5.14a}$$
$$=C^{LR}\mp i\bar{M}^L(\pm)\bar{D}_0C^R. \tag{5.14b}$$

Here the C operators are defined by Eqs. (4.22),

$$\kappa=k+C, \tag{5.15a}$$

and we note that the off-shell κ operator satisfies

$$\kappa=k+k\bar{\delta}G$$
$$=k+\kappa\bar{\delta}Gk. \tag{5.16}$$

The connected-kernel forms of Eqs. (5.11), (5.12b), and (5.13b) can be obtained using standard techniques[3] in the same manner as in the derivation of Eqs. (4.7$'$), (4.18a$'$), and (4.19b$'$).

The present unitary formalism consists in the set of Eqs. (5.5)–(5.8) and the set of Eqs. (5.11)–(5.14) with proper input[18] given by C, C^R, C^L, and C^{LR}. The proof of this assertion follows along the same lines as in Sec. IV and need not be repeated here; we note [cf. Eq. (5.15a)] that we require $k^\dagger=k$ as part of our explicit proper two-particle input. In connection with the consistency of our various equations we observe that in the present instance $M^{LR}(\pm)$ and $\Gamma(\pm)$ now possess the seminal roles played by $\gamma(\pm)$ and $Q^{LR}(\pm)$ in the Cahill formalism.

We conclude this section by establishing the connection between the present and the K-matrix[2] formalism malisms. The latter has its input defined entirely in terms of the operator κ. If we note that

$$\bar{C}=\bar{\delta}G+\bar{\delta}G\kappa\bar{\delta}G \tag{5.17}$$

then with the aid of Eqs. (5.16) we find the alternative expressions for the C operators [as defined by Eqs. (4.22)] in addition to Eq. (5.15a):

$$C^R=\kappa\bar{\delta}P\,, \tag{5.15b}$$

$$C^L=P\bar{\delta}\kappa\,, \tag{5.15c}$$

$$C^{LR}=P[\bar{\delta}G_0(\pm)^{-1}+\bar{\delta}\kappa\bar{\delta}]P. \tag{5.15d}$$

Equations (5.15) express the C-operator input in terms of the κ operator. However, the present formalism which is phrased in terms of the C operators makes manifest the fact that the various C's need not be interrelated except for the Hermitian analytic property $(C^R)^\dagger=C^L$.

It is also easy to show in addition to Eq. (5.10a) that

$$\bar{M}^R(\pm)=\Gamma(\pm)\bar{\delta}P, \tag{5.10b}$$

$$\bar{M}^L(\pm)=P\bar{\delta}\Gamma(\pm)\,, \tag{5.10c}$$

$$\bar{M}^{LR}(\pm)=P[\bar{\delta}G_0(\pm)^{-1}+\bar{\delta}\Gamma(\pm)\bar{\delta}]P\,. \tag{5.10d}$$

Equations (5.10) establish the connection between the $\bar{M}$ operators and the $\Gamma(\pm)$ operator of Ref. 2. Note that $\Gamma(\pm)$ is employed in two different ways in the present and in the formalism of Ref. 2. In the present unitary formalism only on-shell matrix elements of $\Gamma(\pm)$ of the form $\langle\phi_0|\Gamma_{\beta\alpha}(\pm)|\phi_0\rangle$ appear in accordance with our previous connection on the usage of the terminology *on shell*. With respect to this convention $\Gamma(\pm)$ is used in an off-shell sense both in Ref. 2 and in Eqs. (5.10). Similar remarks apply to Eqs. (5.10). Again the present formulation is preferable in that its full generality is manifest (cf. the remarks at the end of the preceding paragraph).

Finally, if we introduce the Alt *et al.*[22] operators

$$U(z)=G_0(z)^{-1}F(z)G_0(z)^{-1}\,,$$

$$\bar{U}(\pm)=\bar{\delta}G_0(\pm)^{-1}+\bar{\delta}\Gamma(\pm)\bar{\delta}\,,$$

then

$$\bar{U}_{00}(\pm)=\sum_{\lambda,\gamma=1}^{3}\Gamma_{\lambda\gamma}(\pm)\,,$$

$$\bar{U}_{0\alpha}(\pm)P_\alpha=\sum_{\gamma=1}^{3}\bar{M}^R_{\gamma\alpha}(\pm)\,,$$

$$P_\alpha\bar{U}_{\alpha 0}(\pm)=\sum_{\gamma=1}^{3}\bar{M}^L_{\alpha\gamma}(\pm)\,,$$

$$P_\beta\bar{U}_{\beta\alpha}(\pm)P_\alpha=\bar{M}^{LR}_{\alpha\beta}(\pm)\,,$$

and we see that Eqs. (5.5)–(5.8) are just the so-called reduced K-matrix equations of Ref. 2. Thus the present unitary formalism is a rephrasing of the formalism of Ref. 2 in a manner in which the complete flexibility available in the choice of input is made manifest.

VI. INPUT EQUIVALENCE

In this section we establish the input equivalence of the two unitary formalisms of Secs. IV and V. The method of proof is as follows. We assume the unitary formalism of Sec. IV as emobdied in Eqs. (4.7), (4.18)–(4.20), and (4.24)–(4.27) with a given set of proper input. We define κ by Eq. (5.15a) and $\Gamma(\pm)$ as the solution of Eqs. (5.11). Equations (5.12a), (5.13a), and (5.14a) are regarded as the definitions of $\bar{M}^R(\pm)$, $\bar{M}^L(\pm)$, and $\bar{M}^{LR}(\pm)$, respectively; Eqs. (5.12b), (5.13b), and (5.14b) then follow with the use of Eqs. (5.11). What is to be shown, then, is that the operators $\gamma(\pm)$, $M^R(\pm)$, $M^L(\pm)$, and $M^{LR}(\pm)$, which are directly related to the physical scattering amplitudes, as defined by the Cahill formalism are also determined by Eqs. (5.5)–(5.8). The proof is reversible, thus establishing the input equivalence of the two formalisms.

If we write $\gamma(\pm)=\Gamma(\pm)\mp iA(\pm)$ in the first of Eqs. (4.7) it follows with the aid of Eqs. (4.19a) and (4.24a) that $A(\pm)$ satisfies

$$A(\pm)=C^R DM^L(\pm)\mp i\kappa\bar{D}_0A(\pm)$$

and so from (5.12b) we conclude that

$$A(\pm)=\bar{M}^R(\pm)DM^L(\pm)$$

and therefore we obtain Eq. (5.5a′). Proceeding in a similar manner from the second of Eqs. (4.7a) we deduce Eq. (5.5b′).

Next writing $M^R(\pm)=\bar{M}^R(\pm)\mp iB(\pm)$ in Eq. (4.18a) and making use of Eqs. (4.20a), (4.24a), (4.26a), and (4.27) we find the integral equation

$$B(\pm)=C^R DM^{LR}(\pm)\mp i\kappa\bar{D}_0B(\pm)$$

and therefore from (5.12b) we find

$$B(\pm)=\bar{M}^R(\pm)DM^{LR}(\pm)$$

and thereby obtain Eq. (5.6a). From Eqs. (4.18b), (4.27a), (5.5b′), (5.12a), and (5.14b) one determines the relation

$$[M^R(\pm)-\bar{M}^R(\pm)\pm iM^R(\pm)D\bar{M}^{LR}(\pm)][1\mp iDQ^{LR}(\pm)]=0\,,$$

which, provided $[1\mp iDQ^{LR}(\pm)]^{-1}$ exists,[23] implies Eq. (5.6b). Equations (5.7) follow from Eqs. (4.19) in an analogous fashion.

Finally, from Eqs. (4.20a), (4.26b), (4.27a), (5.6b), and (5.14a) one establishes that

$$[1\mp iQ^{LR}(\pm)D][M^{LR}(\pm)-\bar{M}^{LR}(\pm)\pm i\bar{M}^{LR}(\pm)DM^{LR}(\pm)]=0\,,$$

which, provided $[1 \mp iQ^{LR}(\pm)D]^{-1}$ exists,[23] implies Eq. (5.8a). Equation (5.8b) follows in a similar manner.

Under the same conditions[23] it is evident that the preceding proof is reversible. This completes the proof of the input equivalence of the two formalisms.

It is now meaningful to compare the approximate input employed in the calculations[24] of Refs. 7 and 8. Tandy *et al.*[7] consider two sets of approximate proper input. The first (the exact unitary zero-order model of Ref. 7) corresponds to the choice[1,7]

$$C = k\bar{\delta}Gk\,, \tag{6.1a}$$

$$C^R = k\bar{\delta}GVP = k\bar{\delta}P\,, \tag{6.1b}$$

$$C^L = PV\bar{\delta}Gk = P\bar{\delta}k\,, \tag{6.1c}$$

$$C^{LR} = PV\bar{\delta}GVP = P\bar{\delta}G_0^{-1}P\,. \tag{6.1d}$$

The second set is the same as given by Eqs. (6.1b)–(6.1d) with (6.1a) replaced by a separable form for C.

Two different sets of approximate proper input are also considered in Ref. 8. The first of these (which is also called the exact unitary zero-order model) is the completely unitary impulse approximation introduced in Ref. 2:

$$C = 0\,, \tag{6.2a}$$

$$C^R = k\bar{\delta}P\,, \tag{6.2b}$$

$$C^L = P\bar{\delta}k\,, \tag{6.2c}$$

$$C^{LR} = P(\bar{\delta}G_0^{-1} + \bar{\delta}k\bar{\delta})P\,. \tag{6.2d}$$

The second approximation in Ref. 8 is called the exact unitary first-order model and corresponds to the choice

$$C = k\bar{\delta}Gk\,, \tag{6.3a}$$

$$C^R = (k + k\bar{\delta}Gk)\bar{\delta}P\,, \tag{6.3b}$$

$$C^L = P\bar{\delta}(k + k\bar{\delta}Gk)\,, \tag{6.3c}$$

$$C^{LR} = P[\bar{\delta}G_0^{-1} + \bar{\delta}(k + k\bar{\delta}Gk)\bar{\delta}]P. \tag{6.3d}$$

Obviously the sets (6.1)–(6.3) of proper input differ substantially and therefore the different results of Refs. 7 and 8 are not surprising. However, it is quite difficult to conjecture under what conditions one set can be preferred over another. Indeed, all three sets yield cross sections in the case of N-d scattering in qualitative agreement with the corresponding exact results,[9,10] although the results of Ref. 7 using the set (6.1) at 14 MeV are perhaps in closest agreement. This may be fortuitous since only the spin-averaged cross sections are reported in Ref. 7. In Ref. 8 it is found that the latter cross sections are generally better reproduced than those for the quartet and doublet states individually.

The results of Refs. 7 and 8 are interesting to the extent that they show how even with rather crude input the requirement of exact three-particle unitarity yields reasonable scattering amplitudes with a relatively small amount of numerical effort. However, these techniques are likely to be of little quantitative use in the three-particle problem unless the choices of approximate input can be justified and systematically improved under certain physical circumstances (such as at very low or at relatively high energies) in a more convincing manner. Thus far no on-shell unitary formalism has been able to do this in contrast with some off-shell unitary theories.[11,12] Therefore, at the present time there is little reason to prefer one unitary formalism over another, and among those which are input equivalent, no substantive reasons at all.

As a final remark we note that we can generate the so-called minimal three-particle scattering model[5] from either of the formalisms considered in this paper via the choice

$$C = C^R = C^L = C^{LR} = 0\,. \tag{6.4}$$

Equations (6.4) imply that

$$Q(\pm) = Q^R(\pm) = Q^L(\pm) = Q^{LR}(\pm) = 0\,,$$

$$\bar{M}^R(\pm) = \bar{M}^L(\pm) = \bar{M}^{LR}(\pm) = 0\,,$$

and consequently

$$\gamma(\pm) = \Gamma(\pm)\,,$$

$$\kappa = k\,,$$

and

$$M^R(\pm) = M^L(\pm) = M^{LR}(\pm) = 0\,.$$

Obviously this approximation makes physical sense at best only under those circumstances in which the 3-to-3 scattering is the only possible three-particle process. Examples of the latter are discussed in Ref. 5.

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[1]R. T. Cahill, Nucl. Phys. A194, 599 (1972).

[2]K. L. Kowalski, Phys. Rev. D 5, 395 (1972).

[3]K. L. Kowalski, Phys. Rev. D 6, 3705 (1972).

[4]K. L. Kowalski, Phys. Rev. D 7, 1806 (1973).

[5]K. L. Kowalski, Phys. Rev. D 7, 2957 (1973).

[6]T. Sasakawa, Nucl. Phys. A203, 496 (1973). This unitary formalism is of the K-matrix type introduced in Ref. 2.

[7]P. C. Tandy, R. T. Cahill, and I. E. McCarthy, Phys. Lett. 41B, 241 (1972).

[8]D. K. Sharma, thesis, Case Western Reserve Univ., 1973 (unpublished); D. K. Sharma and K. L. Kowalski, (unpublished).

[9]R. Aaron, R. D. Amado, and Y. Yam, Phys. Rev. 140, B1291 (1965).

[10]I. H. Sloan, Phys. Rev. 185, 1361 (1969).

[11]J. Carew and L. Rosenberg, Phys. Rev. D 7, 1113 (1973).

[12]K. L. Kowalski, Phys. Rev. C 3, 1973 (1973).

[13]K. L. Kowalski, Phys. Rev. 188, 2235 (1969); Phys. Rev. D 2, 812(E) (1970).

[14]D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento 43A, 258 (1966).

[15]Reference 1, Eq. (4.22a).

[16]$\tilde{k}$ and $\tilde{C}$ correspond to $\bar{R}$ and $\bar{F}$, respectively, of Ref. 1.

[17]The set of equations for the C operators defined by Eqs. (4.21) which follow with the use of Eqs. (4.14) (cf. Ref. 1) could be considered as a third member of the hierarchy. However, the exploitation of the unitary formalism does not depend upon the specific structures implied by Eqs. (4.14) and (4.21).

[18]A set of operators C, C^R, C^L, and C^{LR} satisfying conditions (4.23) will be called *proper input*.

[19]Some of the operators contained in Sec. V have different off-shell extensions than their counterparts in Sec. IV but have identical on-shell matrix elements. This is clearly not an essential difference between the two developments.

[20]I. H. Sloan, Phys. Rev. 165, 1587 (1968).

[21]$\Gamma(\pm)$ is the same as the similarly denoted operator in Ref. 2.

[22]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl Phys. B2, 167 (1967).

[23]By virtue of Eq. (4.27) this is equivalent to assuming a unique correspondence between $Q^{LR}(\pm)$ and C^{LR}.

[24]It should be remarked that the calculations of Ref. 7 employ the original (Ref. 1) Cahill formalism in its nonconnected kernel form. On the other hand, connected-kernel equations were used in all cases in Ref. 8. This does not, however, affect our comparisons of the various sets of approximate input.

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Current quarks and constituent quarks: Symmetry breaking and interaction*

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A study is made of the question of how different the correct transformation V from the current-quark to the constituent-quark basis is expected to be from the free mass-degenerate quark model discussed by Melosh. The effect of SU(3) mass breaking and of mutual interaction of the quarks is discussed in the context of a simple model. The algebraic properties of V are more complicated than that of V_{free}, the transformation constructed by Melosh; nevertheless, it still is tractable enough so an attack may be made on the problem of mass splitting in SU(6) multiplets.

There has been much discussion recently[1] of the relation between current quarks and constituent quarks. On the one hand, there is the SU(6) algebra of integrated weak and electromagnetic current densities[2] and related operators, which is denoted by $\mathrm{SU}(6)_{W,\,\text{currents}}$; on the other hand, there is the SU(6) algebra of operators which form an approximate symmetry of the strong-interaction Hamiltonian,[3] which is denoted by $\mathrm{SU}(6)_{W,\,\text{strong}}$. Assuming these two algebras to be connected by a unitary transformation V, there are several requirements that this V must satisfy; Melosh has described these requirements and furthermore, for the free-quark model with degenerate quark masses, he has constructed an operator V_{free} which satisfies the constraints. Although V_{free} most certainly does not have all the correct properties that V must have, by abstracting some of the algebraic structure of V_{free} which might reasonably be expected to carry over in a more realistic situation one may make predictions for pionic decays of meson and baryon resonances, recover many of the good results of the old $\mathrm{SU}(6)_W$ scheme for the matrix elements of weak charges, and correct some of the poor results.

The basic problem which remains, however, is the determination of how different one should expect the correct transformation V to be from the explicitly constructed model transformation V_{free}. For example, the extremely simple property of