Pseudo-Goldstone pion masses in a gauge model*

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A $SU_L(2)\otimes U(1)\otimes U(1)$ gauge model of electromagnetic and weak interactions is presented, which has three pseudo-Goldstone bosons identified as pions. We can include strong interactions in the pion masses without generating mass counterterms, if the strong gauge symmetry is U(1) or "color" SU(3). At the one-loop level (without strong interactions) the π^0 remains massless, while the $\pi^\pm$ pick up masses. From our analysis, the one-loop calculation should be interpreted as the electromagnetic and weak contribution to the pion mass difference. Our result is not too large in magnitude as compared with the results found in other models. The pion mass difference including strong interactions has the same form as in the Weinberg model where pions are not pseudo-Goldstone bosons. We also exhibit formulas for the pion masses including strong interactions, which however we can not evaluate.

I. INTRODUCTION

The idea of pions as pseudo-Goldstone bosons,[1,2] perhaps a unique feature of gauge theory, has received much attention. Most notably, the pion masses are calculable.[1-4] Models with pseudo-Goldstone pions have many distinctive features, some of which we shall discuss below. But in the models[3] studied so far, the contribution of weak and electromagnetic interactions to the pseudo-Goldstone boson masses come out to be too large (of the order $e^2 m_W{}^2$) to be the pion masses (mass difference). In this note we present a model which contains three pseudo-Goldstone bosons whose masses can be of the order of the pion masses (mass difference). Our model differs from previous ones in that the pseudo-Goldstone boson masses are roughly related to the mass differences of the vector mesons. We shall henceforth refer to the pseudo-Goldstone bosons as pions.

It is well known that gauge theories in which the symmetry is spontaneously broken are renormalizable. Because of this property, some physical quantities are in principle calculable, among which are the masses of the pseudo-Goldstone bosons. If the scalar potential of a Lagrangian has a larger symmetry, called pseudosymmetry, than the gauge symmetry, then certain physical fields remain massless after spontaneous symmetry breaking. These are the pseudo-Goldstone bosons, which pick up finite and calculable masses in higher order. Such a theory offers the following unique and desirable features.

(a) The role of the pions. The Higgs mechanism in gauge theories renders the ordinary explanation of pions as Goldstone bosons (for example, in the σ model) invalid. Ordinary Goldstone bosons in gauge theory cannot be pions. But if pions are pseudo-Goldstone bosons, they pick up masses in higher-order corrections. The origin of the pion mass is due to the correction to the pseudosymmetry, in contrast to *explicit* symmetry-breaking terms in nongauge theories (as in the σ model).

(b) Hadronic mass scale. Although many hadronic mass relations have been derived from strong-interaction symmetries, none of the existing theories of hadrons relates the scale of the hadronic masses to that of other masses. In renormalizable theories, if the mass is a calculable quantity, then the square of the ratio of the scalar-meson mass to the vector-meson mass can be calculated in perturbation series. An example is the model of Coleman and Weinberg.[5] The mass of the pseudo-Goldstone boson is calculable because of the absence of a mass counterterm; the mass ratio m_ϕ/m_V can be expressed as a perturbation series.

(c) Zeroth-order symmetry. An important feature of some gauge theories is the zeroth-order symmetry. Corrections to such a symmetry are of higher order, and are finite and in general small, and this might account for approximate symmetry relations observed in nature. In order to have pseudo-Goldstone bosons, the scalar potential must have a symmetry larger than the gauge symmetry. Such a symmetry (the pseudosymmetry) is a zeroth-order symmetry. The pseudo-Goldstone pions are the consequence of the pseudosymmetry.

(d) Partial conservation of axial-vector current (PCAC) in gauge theory. The pseudo-Goldstone pions are massless in zeroth order. Thus, naturally, to first order the relevant axial-vector current has a massless pion pole. Picking up the pion pole we obtain the soft-pion result. It should be emphasized that the soft-pion result is obtained not because the chiral $SU(2)\otimes SU(2)$ symmetry is a good symmetry as ordinarily interpreted for the soft-pion calculations using PCAC, but simply due

to the fact that the pion mass vanishes to first order in perturbation theory. The success of the soft-pion result could be taken to mean that the first-order approximation works so well that higher-order corrections are in fact small. Note that if the pion is not a pseudo-Goldstone boson, it need not be massless in zeroth order, and in this case something other than the perturbation argument is needed to justify the use of PCAC or the smallness of the pion mass. Thus, we note that the success of the soft-pion mass differences of Das *et al.*[6] can be easily understood if pions are pseudo-Goldstone bosons. Furthermore, higher-order contributions to pion masses (on mass shell) are finite, thus resolving the old problem of the divergence of on-mass-shell pion mass differences. This last point in (d) has not generally been noticed.

The above four points are our motivation for investigating the pseudo-Goldstone boson problem. The outline of the paper is as follows: In Sec. II, we give the formulas for the pion masses and mass differences. In Sec. III, the model is presented and the pion masses calculated. In Sec. IV, we give a concluding discussion. Some mathematical details are given in the Appendix.

II. THE PION MASS FORMULA

While previous attention has been directed to pseudo-Goldstone pions in gauge theories of weak and electromagnetic interactions or of strong interactions separately, it is essential to take all three basic interactions into account. In other words, if one has a gauge model of weak and electromagnetic interactions which contains pseudo-Goldstone pions, one must ensure that incorporating strong interaction does not spoil the calculability of the pion mass. The problem arises from the fact that pions are strongly interacting particles. Thus the strong interaction, if not properly incorporated, may require mass counterterms in order to preserve renormalizability, which would make the pion mass uncalculable. To avoid such counterterms the potential with pseudosymmetry should be the most general one, with respect not only to the weak and electromagnetic interactions but the strong interaction as well. From parity considerations, it was previously noted[7] that the strong gauge symmetry should be neutral to the weak and electromagnetic gauge symmetries, which suggests that it should be U(1) or "color" SU(3). If the (most general) potential has a pseudosymmetry with respect to the weak and electromagnetic gauge group, and is also color-invariant, then we see that the pseudo-Goldstone boson mass is finite and calculable.

Let us recall the origin of the pseudo-Goldstone pion masses. We have a Lagrangian in which the scalar potential has a larger symmetry $\overline{G}$ than the gauge symmetry G. When G is spontaneously broken to a subgroup S, $\overline{G}$ is correspondingly broken to $\overline{S}$. With each generator in G not in S, there is associated the unphysical Goldstone boson which is transformed away. Corresponding to each generator in $\overline{G}$, not in $\overline{S}$, and also not in $G-S$, there is associated the pseudo-Goldstone boson. Since $\overline{G}$ is a zeroth-order symmetry, the pseudo-Goldstone boson remains massless in zeroth order. Higher-order corrections due to pure scalar-meson exchanges do not contribute to the pion mass since the scalar potential is $\overline{G}$-invariant to all orders. With respect to the $\overline{G}$-invariant potential the pions are like the ordinary Goldstone boson, which remains massless to all orders. To phrase it differently, if $\overline{G}$ were the symmetry of the total Lagrangian, then after the spontaneous symmetry breakdown the pions remain massless to all orders. Thus the pion masses come from interactions which do not respect $\overline{G}$ symmetry (higher-order corrections). The strong interaction alone being neutral to $\overline{G}$, is $\overline{G}$-invariant to all orders. But the interactions of the weak and electromagnetic gauge mesons and the Yukawa interaction need not be $\overline{G}$-invariant, and in general they will contribute to the pseudo-Goldstone pion mass. For most interesting cases, the Yukawa interaction is also $\overline{G}$-invariant, it then does not contribute to the pion mass (at the one-loop level). We shall not discuss it in this note.

At the one-loop level, the calculation has been carried out by Weinberg[2] for any group and representations and for arbitrary gauge. The result can be most easily illustrated by using the Landau gauge. The propagators of the vector mesons A_μ^α and the scalar mesons ϕ_i in this gauge are given by

$$\Delta_{\mu\nu}^{A,\alpha}=\frac{\eta_{\mu\nu}-k_\mu k_\nu/k^2}{k^2+\mu_\alpha{}^2},$$

$$\Delta^{\phi,i}=\frac{1}{k^2+M_i{}^2},$$

where μ_α and M_i are the masses of A_μ^α and ϕ_i. There are four diagrams, shown in Fig. 1, which contribute to the pseudo-Goldstone pion masses. Their contributions are as follows:

$$m_B{}^2=m_B{}^{2(A,A)}+m_B{}^{2(A,\phi)}+m_B{}^{2(A)}+m_B{}^{2(T)},$$

where B denotes the quantum numbers of the pseudo-Goldstone pions, and

$$m_B{}^{2(AA)} = \frac{i}{(2\pi)^4} \sum_{\alpha,\beta} g_{B\alpha\beta}{}^2 \int d^4k \, \Delta^{A,\alpha}_{\mu\nu} \Delta^{A,\beta}_{\lambda\rho} \eta^{\mu\lambda} \eta^{\nu\rho} ,$$

$$m_B{}^{2(A,\phi)} = \frac{i}{(2\pi)^4} \sum_{\alpha,j} g_{Bj\alpha}{}^2 \int d^4k \, k^\mu k^\nu \Delta^{A,\alpha}_{\mu\nu} \Delta^{\phi,j} = 0 ,$$

$$m_B{}^{2(A)} = \frac{i}{(2\pi)^4} \sum_{\alpha} g_{BB\alpha\alpha} \int d^4k \, \Delta^{A,\alpha}_{\mu\nu} \eta^{\mu\nu} ,$$

$$m_B{}^{2(T)} = \frac{i}{(2\pi)^4} \sum_{\alpha,j} f_{BBj} M_j{}^{-2} g_{j\alpha\alpha} \int d^4k \, \Delta^{A,\alpha}_{\mu\nu} \eta^{\mu\nu} ,$$

where $g_{B\alpha\beta}$, $g_{Bj\alpha}$, $g_{BB\alpha\alpha}$, and f_{BBj} are the coupling constants of the interaction terms $\phi_B A^\mu_\alpha A_{\beta\mu}$, $\partial_\mu \phi_B \phi_j A^\mu_\alpha$, $\phi_B \phi_B A^\mu_\alpha A_{\alpha\mu}$, and $\phi_B \phi_B \phi_j$, respectively. We then find

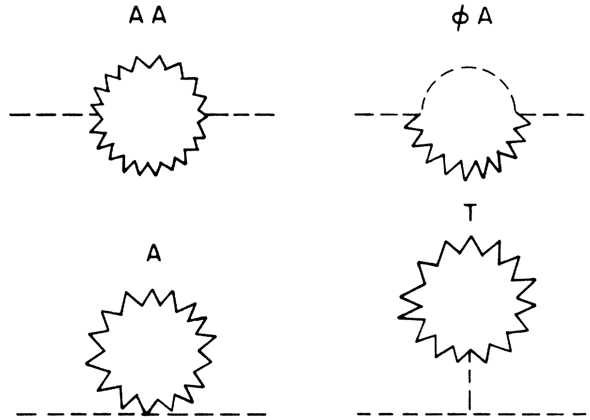

FIG. 1. Feynman diagrams for the scalar self-energy. (Here dashed lines refer to scalar fields, wavy lines refer to gauge fields.)

$$m_B{}^2 = \frac{3i}{(2\pi)^4} \left[\sum_{\alpha,\beta} 2 g_{B\alpha\beta}{}^2 \int d^4k (k^2 + \mu_\alpha{}^2)^{-1} (k^2 + \mu_\beta{}^2)^{-1} + \sum_\alpha g_{BB\alpha\alpha} \int d^4k (k^2 + \mu_\alpha{}^2)^{-1} + \sum_{\alpha,j} \frac{f_{BBj} g_{j\alpha\alpha}}{M_j^2} \int d^4k (k^2 + \mu_\alpha{}^2)^{-1} \right] . \tag{1}$$

We can easily check the above formula with Weinberg's result which can be written as

$$m^2{}_{AB} = -(\bar{\mu}^{-1} \Pi \bar{\mu}^{-1})_{AB} , \tag{2}$$

where

$$\bar{\mu}^2{}_{AB} \equiv -(\theta_B \lambda)_i (\theta_A \lambda)_i , \tag{3}$$

$$\Pi_{AB} = \frac{3i}{(2\pi)^4} \left[-\tfrac{1}{2} (\theta_A \lambda)_i (\{ \theta_\alpha, \theta_\beta \} \lambda)_i (\theta_B \lambda)_j (\{ \theta_r, \theta_\delta \} \lambda)_j \int d^4k (k^2 + \mu^2)^{-1}{}_{\alpha r} (k^2 + \mu^2)^{-1}{}_{\beta\delta} + \tfrac{1}{2} (\theta_A \lambda)_i (\{ \theta_\beta, \theta_\alpha \} \theta_B \lambda)_i \int d^4k (k^2 + \mu^2)^{-1}{}_{\alpha\beta} + \tfrac{1}{4} (\{ \theta_A, \theta_B \} \lambda)_i (\{ \theta_\alpha, \theta_\beta \} \lambda)_i \int d^4k (k^2 + \mu^2)^{-1}{}_{\alpha\beta} \right] , \tag{4}$$

where λ_i are the vacuum expectation value of the real scalar fields, θ_α are antisymmetric and Hermitian matrices of the generators in the (real) scalar space. Capital A, B denotes the axes which project out the pseudo-Goldstone boson subspace and repeated indices are summed over. In the representation of physical particles (with diagonalized mass matrix), one finds that Eqs. (1) and (2) are the same, with the aid of the following relations:

$$g_{B\alpha\beta} \equiv \tfrac{1}{2} (\theta_B \lambda)_i (\{ \theta_\alpha, \theta_\beta \} \lambda)_i / |\bar{\mu}_B| , \tag{5a}$$

$$g_{BB\alpha\alpha} \equiv -(\theta_B \lambda)_i (\theta_\alpha{}^2 \theta_B \lambda)_i / |\bar{\mu}_B| , \tag{5b}$$

$$\sum_{\alpha,j} f_{BBj} g_{j\alpha\alpha} / M_j{}^2 = -(\theta_B{}^2 \lambda)_i (\theta^2 \lambda)_i / |\bar{\mu}_B|^2 , \tag{5c}$$

where $|\bar{\mu}_B| = [-(\theta_B \lambda)_i (\theta_B \lambda)_i]^{1/2}$, and α, β correspond to indices of the diagonalized vector mesons. Eq. (5) can be easily checked from Ref. 2.

As we discussed before, if strong gauge symmetry is U(1) or color SU(3), whereas the weak and electromagnetic gauge symmetry is color-singlet, then the scalar potential with the pseudosymmetry is the most general gauge-invariant one within the pseudo-Goldstone pion sector. The pion mass to first-order perturbation in the weak and electromagnetic coupling constants is again given in Fig. 1. Therefore to all orders of strong interaction, we can use current algebra and write

$$m_\pi{}^2 = \frac{i}{(2\pi)^4} \sum_\alpha g_\alpha{}^2 \int d^4k \, \Delta^{A,\alpha}_{\mu\nu}(k) \times \int d^4x \, e^{ikx} \langle \pi | T^*(j^\alpha_\mu(x) j^\alpha_\nu(0)) | \pi \rangle , \tag{6}$$

where g_α is the weak and electromagnetic coupling constant defined by

$$\mathcal{L}_{\text{int}} = \sum_\alpha g_\alpha A^\alpha_\mu j^\alpha_\mu ,$$

and j_μ is the hadronic current coupled to A_μ. The essential element in obtaining Eq. (6) is that the strong interaction does not generate infinite corresponding to the mass counterterm of the pseudo-Goldstone pions, thus the pion mass including all orders of strong interaction should be finite.

III. THE MODEL

We shall consider a $SU_L(2)\otimes U(1)\otimes U(1)$ gauge model for weak and electromagnetic interactions. We have an extra U(1) gauge symmetry as compared with the Weinberg $SU_L(2)\otimes U(1)$ model.[8] As expected, many properties will be the same as in the Weinberg model, but there are two differences: (1) Because of the extra U(1) gauge gluon, the neutral currents are different in this model than in Weinberg's case, since the fermions can couple to the two neutral vector gluons arbitrarily. The neutral current induced process will have a different amplitude, which is more flexible than in the Weinberg model. (2) In the $SU_L(2)\otimes U(1)$ model, one cannot incorporate the pseudo-Goldstone pions.[9] But in the $SU_L(2)\otimes U(1)\otimes U(1)$ model, the (doublet) representations can be unlocked, by requiring a discrete symmetry, namely, charge conjugation invariance. As a result, the scalar potential has a pseudosymmetry, which gives rise to three pseudo-Goldstone bosons.

We shall not go into the fermion sector, since the couplings of the fermions with the physical gluons follow easily from the spontaneous symmetry breakdown discussed below. For the purpose of calculating the pseudo-Goldstone boson masses, we shall only discuss the scalar Lagrangian. Let g be the $SU_L(2)$ gauge coupling constant, and for simplicity, we have the same coupling constant g' for both of the U(1) groups.[10] We have three doublets of spinless mesons,

$$\phi=\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}=\begin{pmatrix}i\phi^1+\phi^2\\ \phi^4-i\phi^3\end{pmatrix},$$

$$\psi=\begin{pmatrix}i\pi^1+\pi^2\\ \sigma-i\pi^3\end{pmatrix},\quad \xi=\begin{pmatrix}i\Pi^1+\Pi^2\\ \Sigma-i\Pi^3\end{pmatrix},$$

with the following gauge-invariant Lagrangian:

$$\begin{aligned}\mathcal{L}=&\tfrac{1}{2}\left|(\partial_\mu-ig\tau_\alpha A_\mu^\alpha-ig'B_\mu)\phi\right|^2\\ &+\tfrac{1}{2}\left|(\partial_\mu-ig\tau_\alpha A_\mu^\alpha-ig'C_\mu)\psi\right|^2\\ &+\tfrac{1}{2}\left|(\partial_\mu-ig\tau_\alpha A_\mu^\alpha-ig'C_\mu)\xi\right|^2+P(\phi,\psi,\xi),\end{aligned}\tag{7}$$

where A_μ^α, $\alpha=1,2,3$, and B_μ, C_μ are the $SU_L(2)$ and $U(1)\otimes U(1)$ gauge vector mesons, and $P(\phi,\psi,\xi)$ is the scalar potential. The Lagrangian is like the σ model, with the field $\sigma+i\pi^\alpha\tau_\alpha$, $\Sigma+i\Pi^\alpha\tau_\alpha$, and $\phi_4+i\phi^\alpha\tau_\alpha$ transforming like $(\frac{1}{2},\frac{1}{2})$ under chiral $SU(2)\otimes SU(2)$ symmetry. We have only gauged the left-handed SU(2) symmetry. If we turn off the weak and electromagnetic interaction, the above $\mathcal{L}$ is chiral $SU(2)\otimes SU(2)$-invariant with the discrete symmetry imposed below.[11] The isospin and parity of the (pion) fields are defined as in the σ model. This point is relevant, if pions are going to have strong interactions also. The color SU(3) strong-interaction symmetry will be dealt with in Appendix B. But as far as the masses are concerned, we can forget it at present.

Let us first of all see the pseudo-Goldstone bosons in the model. We impose a discrete symmetry, namely, a special charge-conjugation invariance on the Lagrangian, which plays a crucial role. The Lagrangian invariant under the discrete symmetry is also renormalizable, if it has the most general (quartic) polynomial which is gauge-invariant but also discrete-symmetry-invariant. In Eq. (7) one notes that the field ϕ couples to B_μ, whereas ψ and ξ couple to C_μ, in such a way that the covariant derivatives are invariant under the following charge-conjugation transformation: $\phi\xrightarrow{C}(\tau_2)\phi^*$, $B_\mu\to -B_\mu$. In the scalar polynomial, terms like $(\overline{\phi}\psi)$, $(\overline{\phi}\xi)$, and so on are not charge-conjugation-invariant. Therefore by imposing the charge-conjugation invariance, the most general gauge-invariant polynomial is a function of ϕ^2, ψ^2, ξ^2, and $\psi\cdot\xi$ only.[11] As a result, the polynomial has a pseudosymmetry, namely, it is also invariant under separate SU(2) transformation on ϕ, and/or on ψ and ξ together, i.e., $\overline{G}=SU_L^\phi(2)\otimes SU_L^{\psi,\xi}(2)\otimes U(1)\otimes U(1)$. The gauge symmetry $G=SU_L(2)\otimes U(1)\otimes U(1)$ is spontaneously broken to a subgroup $S=U(1)$, i.e., the electromagnetic gauge; correspondingly $\overline{G}$ is also broken to $\overline{S}=U(1)$. Thus we have $\overline{G}-\overline{S}-(G-S)=3$ pseudo-Goldstone bosons, where in this equation $\overline{G}$ is defined to be the number of generators of the group $\overline{G}$, etc.[1]

Let the vacuum expectation values of the fields be $\langle\phi_0\rangle=\lambda$, $\langle\sigma\rangle=\sigma$, and $\langle\Sigma\rangle=\Sigma$. The symmetry is spontaneously broken to the electromagnetic gauge, with the photon

$$A_\mu=\frac{1}{(2g^2+g'^2)^{1/2}}(g'A_\mu^3+gB_\mu+gC_\mu).\tag{8}$$

The charged vector meson has mass

$$m_{W^\pm}{}^2=g^2(\lambda^2+\sigma^2+\Sigma^2),\quad W_\mu^\pm\equiv\frac{1}{\sqrt{2}}(A^1\pm iA^2).$$

The other two neutral vector mesons are also massive. They can be expressed analytically, if we set $\lambda^2=\sigma^2+\Sigma^2$. In general, the vacuum expectation values are unrelated, but in order to carry out the calculation analytically, we shall assume this relation holds without changing the qualitative features of our model. We then get $m_Z^2 = (2g^2 + g'^2)\lambda^2$, where

$$Z_\mu = \frac{1}{(2g^2 + g'^2)^{1/2}} \left[\sqrt{2}\, g A_\mu^3 + \frac{g'}{\sqrt{2}} (B_\mu + C_\mu) \right], \tag{9}$$

and $m_X^2 = g'^2 \lambda^2$, where

$$X_\mu \equiv \frac{1}{\sqrt{2}} (B_\mu - C_\mu). \tag{10}$$

Note that our definition of g and g' [in Eq. (7)] is a factor of 2 smaller than the ordinary gauge coupling constants. The electric charge is given by $e = 2gg'/(2g^2 + g'^2)^{1/2}$ and $Q = \frac{1}{2} I_{3L} + Y$, where Y is the sum of hypercharges corresponding to the two U(1) groups.

It is straightforward to find that the four Goldstone bosons are

$$\frac{1}{\sqrt{2}\,\lambda} (\lambda \phi^i + \langle \sigma \rangle \pi^i + \langle \Sigma \rangle \Pi^i), \quad i = 1 \pm i2, 3 \tag{11}$$

and

$$\frac{1}{\sqrt{2}} (-\lambda \phi^4 + \langle \sigma \rangle \sigma + \langle \Sigma \rangle \Sigma),$$

and the three pseudo-Goldstone pions are

$$\pi^i_{\text{phys}} = \frac{1}{\sqrt{2}\,\lambda} (-\lambda \phi^i + \langle \sigma \rangle \pi^i + \langle \Sigma \rangle \Pi^i), \qquad i = 1 \pm i2, 3 \tag{12}$$

where $\lambda^2 = \langle \sigma \rangle^2 + \langle \Sigma \rangle^2$. One recalls that pseudo-Goldstone bosons correspond to generators of the pseudosymmetry group (but not the gauge group). This is evidenced by the different sign in Eq. (12), which corresponds to $\mathrm{SU}_L^\phi(2) \otimes \mathrm{SU}_L^{\psi,\xi}(2)$ transformation as compared with gauge transformation, which acts on all fields simultaneously [as in Eq. (11)]. This point will be clearly seen in Appendix A, which is patterned after Ref. 2.

To calculate the pseudo-Goldstone pion masses at the one-loop level, we use Eq. (1) [or (2)]. (Here the strong interaction is turned off, setting the strong coupling constant, which we call f, equal to zero.) The coupling constants in Eq. (1) can either be calculated directly by expressing the interaction Lagrangian [Eq. (7)] in terms of the physical gluons, $W_\mu^\pm$, Z_μ, X_μ, and the photon, and in terms of the pseudo-Goldstone pion fields, or be evaluated by using Eq. (5). The latter method is simple, once the representation of the generators in Eq. (5) is specified. The details will be presented in Appendix A.

(i) Mass of the neutral pion. We find by the use of (5a) that the coupling of π^0 with any two vector mesons is zero; thus the two-vector-meson exchange diagram (Fig. 1) does not contribute. We have checked this by direct calculation from the Lagrangian (7). The four-point vertex (5b) is nonvanishing for $(\pi^0)^2 W_\mu^+ W_\mu^-$, $(\pi^0)^2 Z_\mu Z_\mu$, and $(\pi^0)^2 X_\mu X_\mu$ with coupling constants $g_{\pi^0\pi^0 W^+W^-} = g^2$, $g_{\pi^0\pi^0 ZZ} = g^2 + \frac{1}{2} g'^2$, and $g_{\pi^0\pi^0 XX} = \frac{1}{2} g'^2$, which one can also easily check from the Lagrangian (7). But its contribution (i.e., $m^{2(A)}$) is exactly canceled by the tadpole term ($m^{2(T)}$). One can see this point from Eqs. (5b) and (5c), where the two equations are equal but opposite in sign if θ_α^2 commutes with θ_B (here $B = \pi^0$, note that θ is antisymmetric). Thus, we find $m_{\pi^0}^2 = 0$ at the one-loop level (and $f = 0$).

(ii) Mass of the charged pion. The only nonvanishing contribution of $m_{\pi^\pm}^{2(AA)}$ comes from W- and X-exchange diagram with

$$g_{\pi^+ W X} = i2gg'\lambda, \tag{13}$$

thus

$$m_{\pi^\pm}^{2(AA)} = -\frac{3i}{(2\pi)^4} 4 g^2 g'^2 \lambda^2 \times \int d^4k \, \frac{1}{(k^2 + m_W^2)(k^2 + m_X^2)}.$$

$m_{\pi^\pm}^{2(A)}$ has a contribution from Z exchange, X exchange, and photon exchange, with coupling of the four-point vertices as follows:

$$\begin{aligned} g_{\pi\pi W^+W^-} &= g^2, \\ g_{\pi\pi ZZ} &= \frac{(2g^2 - g^2)^2}{2(2g^2 + g^2)}, \\ g_{\pi\pi XX} &= g'^2, \\ g_{\pi\pi AA} &= \frac{4g^2 g'^2}{2g^2 + g'^2} = e^2. \end{aligned} \tag{14}$$

We have

$$m_{\pi^\pm}^{2(A)} = \frac{3i}{(2\pi)^4} \left[g^2 \int d^4k \, \frac{1}{k^2 + m_W^2} + \frac{(2g^2 - g'^2)^2}{2(2g^2 + g'^2)} \int d^4k \, \frac{1}{k^2 + m_Z^2} + \tfrac{1}{2} g'^2 \int d^4k \, \frac{1}{k^2 + m_X^2} + e^2 \int d^4k \, \frac{1}{k^2} \right].$$

Equations (13) and (14) can be checked directly from Lagrangian (7). Similarly, the tadpole term has nonvanishing contribution from Z and X. We find [Eq. (5c)]

$$m_{\pi^\pm}^{2(T)} = -\frac{3i}{(2\pi)^4} \left[\tfrac{1}{2}(2g^2 + g'^2) \int d^4k \, \frac{1}{k^2 + m_Z^2} + \tfrac{1}{2} g'^2 \int d^4k \, \frac{1}{k^2 + m_X^2} + g^2 \int d^4k \, \frac{1}{k^2 + m_W^2} \right]. \tag{15}$$

Summing up the above contributions, we have

$$m_{\pi^\pm}{}^2(f=0)=\frac{3i}{(2\pi)^4}\left[-4g^2g'^2\int d^4k\frac{1}{(k^2+m_W{}^2)(k^2+m_X{}^2)}\right.$$

$$-e^2\int d^4k\,\frac{1}{k^2+m_Z{}^2}$$

$$\left.+e^2\int d^4k\,\frac{1}{k^2}\right]. \qquad (16)$$

We see that the quadratic divergence and logarithmic divergence cancel; we have a finite integral,

$$m_{\pi^\pm}{}^2(f=0)=\frac{e^2}{4\pi}\,\frac{3}{2\pi}\,\frac{m_Z{}^2}{4}$$

$$\times\left(\frac{m_W{}^2\ln m_W{}^2-m_X{}^2\ln m_X{}^2}{m_W{}^2-m_X{}^2}-\ln m_Z{}^2\right). \qquad (16')$$

The last term in the parentheses makes this model different from previous ones. Without it, the mass will be too big to be interpreted as the pion mass or mass difference.

Higher-order corrections to the pion masses, including the π^0, will presumably be nonzero. Unless π^0 is associated with an additional global symmetry which requires it to be massless to all orders (which we do not find here), π^0 will pick up mass by the very nature of pseudo-Goldstone bosons. It should be noted that a next-order correction of form $g^4m_W{}^2$ will be of the right order of magnitude for the pion mass. Next, we shall include the effects of strong interactions in the pion mass formulas, and from the analysis below, it will be seen that the isospin breaking in this model is of the order e^2. In view of this, (16′) should be interpreted as the lowest-order contribution of electromagnetic and weak interactions to the pion mass differences. Our result is satisfactory in that this contribution is not too large as is the case in other models.[3] Since $\Delta m_\pi{}^2\approx\frac{1}{4}e^2(3/2\pi)m_\rho{}^2$, we have

$$\frac{m_W{}^2\ln m_W{}^2-m_X{}^2\ln m_X{}^2}{m_W{}^2-m_X{}^2}-\ln m_Z{}^2\approx\frac{4m_\rho{}^2}{m_Z{}^2}$$

$$\simeq\frac{10^{-5}\cos^2\theta\sin^2\theta}{e^2}$$

where the mixing angle is defined by $g'/2g=\tan\theta$. Expressing the masses in terms of the mixing angle, one has

$$\cos^2\theta\ln\cos^2\theta-\sin^2\theta\ln\sin^2\theta\simeq\frac{10^{-5}}{e^2}\,\sin^2\theta\cos^2\theta$$

$$\times(\cos^2\theta-\sin^2\theta).$$

We note that if the mixing angle is (near) 0 or $\frac{1}{2}\pi$ the pion mass difference is (near) zero. It should be noted that the mixing angle is different from the Weinberg angle and the neutral current in this model (which measures the angle) is different from that in the Weinberg model.

A remark: Suppose we treat the second U(1) gauge symmetry as the strong-interaction gauge group, where C_μ is the strong vector gluon. Then we have the Weinberg model for weak and electromagnetic interaction and an Abelian gluon for the strong interaction. The strong U(1) group should presumably be characterized by a different coupling constant than g, but we expect that the qualitative features will not be changed. From the previous section, we should expect no pure strong interaction contribution to the pion mass, and this is checked from Eq. (16) where diagrams with X-meson exchanges alone do not contribute. If the strong-interaction gauge symmetry is non-Abelian, [e.g., color SU(3)], then gauge invariance (color conservation) prohibits diagrams with one weak vector-meson and one strong vector-meson exchange. On the other hand, for Abelian strong gauge symmetry, this is allowed [i.e., W-meson and X-meson exchanges in Eq. (16)].

How to incorporate color SU(3) gauge group in the present model will be dealt with in Appendix B. The crucial points are: (1) The pions are pseudo-Goldstone bosons even in the presence of strong interaction; (2) the pions interact strongly also.

Let us now turn to the pion masses including all orders of strong interaction. Using Eq. (6) we find

$$m_{\pi^0}{}^2=\frac{i}{(2\pi)^4}\left[g^2\int d^4k\Delta^W_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^0|T^*(j^{W^+}_\mu(x)j^{W^-}_\nu(0))+T^*(j^{W^-}_\mu(x)j^{W^+}_\nu(0))|\pi^0\rangle\right.$$

$$+\tfrac{1}{2}(2g^2+g'^2)\int d^4k\,\Delta^Z_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^0|T^*(j^Z_\mu(x)j^Z_\mu(0))|\pi^0\rangle$$

$$\left.+\tfrac{1}{2}g'^2\int d^4k\,\Delta^X_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^0|T^*(j^X_\mu(x)j^X_\nu(0))|\pi^0\rangle\right], \qquad (17)$$

$$m_{\pi^\pm}{}^2=\frac{i}{(2\pi)^4}\Big[\,g^2\int d^4k\,\Delta^W_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^{W^+}_\mu(x)j^{W^-}_\nu(0))+T^*(j^{W^-}_\mu(x)j^{W^+}_\nu(0))|\pi^\pm\rangle$$
$$+\tfrac{1}{2}\frac{(2g^2-g'^2)^2}{(2g^2+g'^2)}\int d^4k\,\Delta^Z_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^Z_\mu(x)j^Z_\nu(0))|\pi^\pm\rangle$$
$$+\tfrac{1}{2}g'^2\int d^4k\,\Delta^X_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^X_\mu(x)j^X_\nu(0))|\pi^\pm\rangle$$
$$+e^2\int d^4k\,\Delta^A_{\mu\nu}(k)\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^{em}_\mu(x)j^{em}_\nu(0))|\pi^\pm\rangle\Big]\,,\qquad(18)$$

where $\Delta^A_{\mu\nu}$ ($A=W, Z, X,$ and photon) is the propagator of the vector meson A, j^A is the associated current and so on. The strong-interaction effect is represented by T^* product of the currents, with the lowest-order weak and electromagnetic coupling constants given by the Lagrangian (7). The coupling constants [in (17), (18)] are the three-point vertices $g_{\pi\phi A}{}^2$ (ϕ the exchanged meson) or the coefficient of the seagull term, i.e., $g_{\pi\pi AA}$, which were given before.

Since $\frac{1}{2}(2g^2-g'^2)^2/(2g^2+g^2)=\frac{1}{2}(2g^2+g'^2)-e^2$, we see that the pion mass difference is given by

$$\Delta m_\pi{}^2=\frac{i}{(2\pi)^4}\,e^2\Big[\int d^4k\,\Delta^A_{\mu\nu}\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^{em}_\mu(x)j^{em}_\nu(x))|\pi^\pm\rangle$$
$$-\int d^4k\,\Delta^Z_{\mu\nu}\int d^4x\,e^{ikx}\langle\pi^\pm\,|T^*(j^Z_\mu(x)j^Z_\nu(x))|\pi^\pm\rangle\Big]\,,\qquad(19)$$

which has the same form[12] as in the Weinberg model where pions are not pseudo-Goldstone bosons. In the soft-pion limit, the first term seems to dominate,[12] which is the result of Das *et al.*[6] One could evaluate the pion mass itself as well in the soft-pion limit. The evaluation of the spectral function becomes a problem, since low-lying pole dominance seems unwarranted.

From (19), we see that the pion mass difference is characterized by the square of the electromagnetic coupling constant e^2. We have neutral pion mass $m_{\pi^0}{}^2=m_{\pi^\pm}{}^2-\Delta m_\pi{}^2$ as given by (17). The neutral pion mass need not be too small; the value depends on the evaluation of the spectral functions as we have commented above. In this model, as in the Weinberg model, the isospin breaking is of the order e^2, which can also be seen directly from the mixing of the SU(2) and U(1) gauge gluons.[12] In comparison, the isospin breaking in some other models[3] need not be of order e^2, which would presumably spoil the result of Das *et al.*[6]

IV. DISCUSSION AND CONCLUSION

We have generalized the Weinberg $SU_L(2)\otimes U(1)$ model by introducing an additional U(1) group. By so doing, one notes the following: (1) The model has more flexibility, since two neutral vector mesons can be exchanged. If experiments should turn out not to agree consistently with the Weinberg model, an additional gluon exchange will be the most simple and natural explanation. (2) The model has three pseudo-Goldstone bosons if one imposes a charge conjugation invariance as specified before. (3) The masses of the pseudo-Goldstone bosons are given by functions of the masses of the gauge vector mesons which can be made small by properly restricting the mixing angle of the coupling constants. The pseudo-Goldstone boson masses in models studied before turn out to be related to the masses of the gauge vector mesons, which will be an order of magnitude too large to be pion masses. The present model does not have such a drawback. We identify the pseudo-Goldstone bosons as the pions.

One notes that the pseudo-Goldstone boson masses have no contribution from the scalar-meson exchange diagrams. Thus at the one-loop level we are able to write the masses in terms of time-ordered products of vector or axial-vector currents (corresponding to vector-meson exchange). On the other hand, if pions are bound states of quarks, then there is no *a priori* reason why the scalar current contribution (corresponding to scalar-meson exchange) is small or not present. Since one lacks knowledge about commutators involving scalar currents, this presents an obstacle to the evaluation of the pion mass by the use of current algebra.

In the study of PCAC and symmetry breaking of strong interactions, one recalls that the σ model with Goldstone pions has proven to be very useful. We hope that the study of the idea of pseudo-Goldstone pions will shed some light on the origin of the pion mass, and may be helpful in the bound-state model of pions.

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APPENDIX A

In this appendix, we evaluate the coupling constants by the use of Eq. (5). We note that Eq. (5) is defined (see Ref. 2) by grouping all the real scalar fields as a big column matrix, and θ_α are the real representations of the generators times the coupling constants acting on the big column matrix; λ is the vacuum expectation value of the column matrix. One of the (equivalent) real representations of the τ matrices can be written as

$$\theta_1=\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix},\quad \theta_2=\begin{bmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{bmatrix},$$

$$\theta_3=\begin{bmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&i&0\end{bmatrix},\quad \theta_0=\begin{bmatrix}0&i&0&0\\-i&0&0&0\\0&0&0&i\\0&0&-i&0\end{bmatrix}, \tag{A1}$$

which satisfy the algebra $\theta_i\theta_j=i2\epsilon_{ijk}\theta_k$, $i,j,k=1,2,3$, and $[\theta_i,\theta_0]=0$. The corresponding scalar fields can be represented by $\phi^T=(-\phi^1,\phi^2,\phi^3,\phi^4)$, $\psi^T=(-\pi^1,\pi^2,\pi^3,\sigma)$, and $\xi^T=(-\Pi^1,\Pi^2,\Pi^3,\Sigma)$.

Writing all scalar fields as a big column matrix Ψ, where $\Psi^T\equiv(\phi^T,\psi^T,\xi^T)$, the Lagrangian (7) can be written as

$$\mathcal{L}=\tfrac{1}{2}\,|(\partial_\mu-i\theta_\alpha A_\mu^\alpha-i\theta_B B_\mu-i\theta_C C_\mu)\Psi|^2 + \text{potential},$$

where

$$\theta_\alpha=g\begin{pmatrix}\theta_\alpha&&\\&\theta_\alpha&\\&&\theta_\alpha\end{pmatrix},\quad \alpha=1,2,3$$

$$\theta_B=g'\begin{pmatrix}\theta_0&&\\&0&\\&&0\end{pmatrix},\quad \theta_C=g'\begin{pmatrix}0&&\\&\theta_0&\\&&\theta_0\end{pmatrix}.$$

In order to use Eq. (5), we need to find out the representation θ_A corresponding to the physical vector mesons A, where $A=W^1$, W^2, Z, X, and the photon A. The θ_A's are defined by

$$\sum_A \theta_A A_\mu \equiv \sum_\alpha \theta_\alpha A_\mu^\alpha+\theta_B B_\mu+\theta_C C_\mu .$$

By the use of Eqs. (8)–(10), we find

$$\theta_{W^i}=g\begin{pmatrix}\theta_i&&\\&\theta_i&\\&&\theta_i\end{pmatrix},\quad i=1,2$$

$$\theta_A=\frac{gg'}{(2g^2+g'^2)^{1/2}}\begin{pmatrix}\theta_3+\theta_0&&\\&\theta_3+\theta_0&\\&&\theta_3+\theta_0\end{pmatrix},$$

$$\theta_Z=\frac{1}{\sqrt{2}\,(2g^2+g'^2)^{1/2}}\begin{pmatrix}-2g^2\theta_3+g'^2\theta_0&&\\&-2g^2\theta_3+g'^2\theta_0&\\&&-2g^2\theta_3+g'^2\theta_0\end{pmatrix}, \tag{A2}$$

$$\theta_X=\frac{g'}{\sqrt{2}}\begin{pmatrix}\theta_0&&\\&-\theta_0&\\&&-\theta_0\end{pmatrix}.$$

Since $\langle \Psi^T \rangle = (0, 0, 0, \lambda; 0, 0, 0, \sigma; 0, 0, 0, \Sigma)$, we immediately find that the mass matrix is diagonal with masses given before, namely,

$$\begin{aligned} m_W^2 &= -(\theta_W \langle \Psi \rangle)_i (\theta_W \langle \Psi \rangle)_i \\ &= g^2(\lambda^2 + \sigma^2 + \Sigma^2) \\ &= 2g^2\lambda^2, \\ m_Z^2 &= -(\theta_Z \langle \Psi \rangle)_i (\theta_Z \langle \Psi \rangle)_i \\ &= (2g^2 + g'^2)\lambda^2 \\ m_X^2 &= -(\theta_X \langle \Psi \rangle)_i (\theta_X \langle \Psi \rangle)_i \\ &= g'^2\lambda^2, \\ m_A^2 &= 0, \end{aligned}$$

where we use $\sigma^2 + \Sigma^2 = \lambda^2$.

The last thing we need to know is the projection operator of the pseudo-Goldstone bosons, defined by

$$\pi^B_{\text{phys}} = \Psi_i (\theta_B \langle \Psi \rangle)_i / |\bar{\mu}_B|,$$

where $|\bar{\mu}_B| \equiv [-(\theta_B \langle \Psi \rangle)_i (\theta_B \langle \Psi \rangle)_i]^{1/2}$. From Eq. (12), we find that

$$\theta_B = \frac{1}{\sqrt{2}} \frac{1}{\lambda} \begin{pmatrix} -\theta^B & & \\ & \theta^B & \\ & & \theta^B \end{pmatrix}, \quad B = 1, 2, 3. \quad \text{(A3)}$$

It is clear that θ_B belongs to the group $SU^\phi(2) \otimes SU^{\psi, \xi}(2)$, but not the gauge group SU(2), because of the minus sign. From Eq. (11), one can easily check that the real Goldstone boson projection operators belong to the gauge group.

Using (A1)–(A3), Eq. (5) can be straightforwardly calculated. We find that the only nonvanishing coupling constant of (5a) is the $\pi^\pm W Z$ vertex, as given by (13). The nonvanishing four-point vertices (5b) and the tadpole contribution (5c) are given by (14) and (15), respectively.

APPENDIX B

In this appendix, we shall give an example in which the pions interact strongly with the color vector gluons, but remain as pseudo-Goldstone bosons. The model consists of two doublets ϕ and ψ which have no strong interaction as given before. In addition we have three scalar representations which transform as doublets with respect to $SU_L(2)$, and triplets with respect to the color SU(3) gauge group,

$$M^i = \begin{pmatrix} i(\Pi^+)_{i1}, & i(\Pi^+)_{i2}, & i(\Pi^+)_{i3} \\ (\Sigma - i\Pi^3)_{i1}, & (\Sigma - i\Pi^3)_{i2}, & (\Sigma - i\Pi^3)_{i3} \end{pmatrix}, \quad i = 1, 2, 3$$

where SU(2) indices are labeled by column and the color SU(3) indices are labeled by row. $M^{(i)}$ transforms as a triplet, $i = 1, 2, 3$, with respect to a global color SU′(3) group, where ϕ, ψ are singlet. The gauge-invariant Lagrangian is

$$\begin{aligned} \mathcal{L} = & \tfrac{1}{2} |(\partial_\mu - ig\tau_\alpha A^\alpha_\mu - ig' B_\mu)\phi|^2 + \tfrac{1}{2} |(\partial_\mu - ig\tau_\alpha A^\alpha_\mu - ig' C_\mu)\psi|^2 \\ & + \sum_{i=1}^{3} \tfrac{1}{2} \mathrm{Tr} |\partial_\mu M^i - ig\tau_\alpha A^\alpha_\mu M^i - ig' C_\mu M^i - if M^i (\lambda_\beta V^\beta_\mu)|^2 + \text{potential}, \end{aligned}$$

where $A^\alpha_\mu, B_\mu, C_\mu$ are the weak and electromagnetic gauge mesons, V_μ the color gluons, with g, g', and f the corresponding coupling constants. We see as before the gauge-covariant derivatives are invariant under the charge conjugation $\phi \xrightarrow{C} \tau_2 \phi^*$, $B_\mu \to -B_\mu$. Thus by imposing the above discrete symmetry, the potential has a pseudosymmetry,

$$\bar{G} = SU^\phi_L(2) \otimes SU^{\psi, M^i}_L(2) \otimes U(1) \otimes U(1) \otimes SU^C(3)$$

as compared with

$$G = SU_L(2) \otimes U(1) \otimes U(1) \otimes SU^C(3).$$

The color SU(3) is completely broken, by assigning vacuum expectation values $\langle \phi_i \rangle = \lambda$, $\langle \sigma \rangle = \sigma$, $\langle \Sigma_{ij} \rangle = \Sigma \delta_{ij}$, with $m_V^2 = f^2 \Sigma^2$. The masses of the weak gauge mesons are given as before by replacing Σ^2 by $3\Sigma^2$. The pseudo-Goldstone pions are again given by (12) with Π^i replaced by $(1/\sqrt{3})(\Pi^i_{11} + \Pi^i_{22} + \Pi^i_{33})$ and $\langle \Sigma \rangle$ replaced by $\sqrt{3} \langle \Sigma \rangle$ (note that $\lambda^2 = \sigma^2 + \Sigma^2$ is replaced by $\lambda^2 = \sigma^2 + 3\Sigma^2$). As discussed before, the color-gluon exchange does not contribute to the pion mass at the one-loop level. Indeed, one finds that the coupling constants for pseudo-Goldstone pions with the weak gauge mesons are the same as before.

We note that the pions are not singlet with respect to the color SU(3) group and thus interact strongly with the color gauge gluons. But the pions are singlet with respect to the color group SU″(3) consisting of the generators $\{\rho_\alpha \otimes 1 + 1 \otimes \rho'_\alpha\}$, where ρ_α and ρ'_α are generators of the color SU(3) and SU′(3) group. In this model, the classification of the hadrons should go by the SU″(3) group. The parity and isospin of the pions are classified by the chiral $SU(2) \otimes SU(2)$ group, which is an exact group of the strong interactions.

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We see L is also invariant under $\xi \leftrightarrows \psi$ (we can also impose reflection symmetry $\psi\to-\psi$ or $\xi\to-\xi$; then $d_1=0$). P is automatically chiral $SU(2)\otimes SU(2)$-invariant.
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