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New methods for the renormalization group

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The class of renormalization procedures with mass-independent counterterms is considered. It is shown to be nonempty. All these procedures, of which 't Hooft and Weinberg have given particular examples, lead to equivalent renormalization-group equations, which can be solved at arbitrary momenta, unlike the equations in the Gell-Mann–Low and Callan-Symanzik methods. Dimensional regularization is used throughout, as needed for non-Abelian gauge theories. It is shown how Weinberg's method using an ultraviolet cutoff extends to one using dimensional regularization. Finally, some calculations in quantum electrodynamics are exhibited.

I. INTRODUCTION

Weinberg[1] and 't Hooft[2] have recently produced new methods for the analysis of the large-momentum behavior of Green's functions. These methods are simpler and more powerful than the usual Gell-Mann–Low[3] and Callan-Symanzik[4,5] methods.

In all of these techniques, there is an equation that governs the behavior of the renormalized Green's functions under scaling of the external momenta. Now the equation in the Gell-Mann–Low approach[6] is homogeneous, but has coefficients that each depend on the coupling constant(s) and on the mass(es). On the other hand, the coefficients of the Callan-Symanzik equation each depend on the coupling constant only; however, the equation is inhomogeneous. So in either case, the only way to obtain an explicit solution is to approximate, and this is done by assuming a situation where the mass(es) can be neglected—at asymptotic nonexceptional spacelike momenta.

The ideas of 't Hooft and Weinberg depend on a closer analysis of the infinite parts of the renormalization constants. It is these parts that give the "noncanonical" scaling of the Green's functions. Their methods use the fact that these infinite renormalizations can be taken to be independent of mass. All other renormalizations (e.g., to mass-shell or to zero-momentum values) are irrelevant for the discussion of scaling.

Weinberg defines the renormalization constants at zero bare mass, while 't Hooft uses the power of the method of dimensional regularization to put (essentially) numerical values on the divergent parts of Feynman integrals. In either case, the result is an equation like that of Callan and Symanzik, but without the inhomogeneous term. Instead, there is a homogeneous term that gives an "effective mass" in the solution in addition to the usual "effective coupling constant." The solution applies at all momenta, including nonasymptotic timelike momenta.

The main aim of this paper is to show that these new methods are special cases (though probably the most convenient) of a general class of renormalization prescriptions each giving an equivalent renormalization-group equation. We show by a simple proof that the mass-independent procedures can indeed remove all the infinities of perturbation theory. Weinberg only sketches a proof,[1] and its completion would probably be complicated. 't Hooft's proof[2] requires the examination of a complicated set of differential equations, and the structure of the argument is not completely transparent.

In doing this, we show the close relation between 't Hooft's and Weinberg's procedures, despite the immense difference there seems to be when reading their expositions.

These general arguments occupy Secs. II, III, and VI.

The main application of the methods is likely to be to non-Abelian gauge theories, and there the most convenient method of regularization is that of continuation in the dimension of space-time. So we use this method throughout. We find in Sec. IV that Weinberg's procedure, which uses a momentum cutoff, translates naturally into one using dimensional regularization.

To show how all this works, we give in Secs. V and VII one-loop calculations for quantum electrodynamics in the two methods.

Owing to infrared divergences, it is not possible to use Weinberg's method in scalar theories. However, as 't Hooft notes,[2] these divergences do cancel. It thus follows that a mass-independent renormalization procedure does exist even for scalar theories. A detailed exposition of this is contained in the following paper.[7]

II. MASS-INDEPENDENT RENORMALIZATION

For simplicity, we will consider a theory with only one coupling constant, g, dimensionless in four dimensions, and one mass. The Lagrangian has unit normalization for the kinetic-energy terms and has bare parameters g_B, m_B. The dimension of space-time is n, and unrenormalized Green's functions are obtained in perturbation theory for each n by using the rules for evaluating n-dimensional integrals given for example in Refs. 8 and 9.

The Green's functions will have singularities at discrete values of n, in particular at $n=4$. By giving g_B and m_B a suitable n dependence, and by multiplying the (unrenormalized) Green's functions by suitable wave-function renormalizations, we obtain renormalized Green's functions that at $n=4$ are analytic and hence finite. The values at $n=4$ are the physical Green's functions.

To work this procedure in terms of finite quantities, the renormalized Green's functions are parametrized by a (finite) renormalized mass and coupling constant; the unrenormalized parameters are defined, according to some prescription, as functions of the renormalized parameters, of n, and possibly of an extra mass μ. For example, they could be defined so that the values of the 2- and 3-point functions at momentum squared $-\mu^2$ are equal to the renormalized parameters. Of course, the prescriptions we use will have the bare and renormalized parameters equal in the lowest order of the expansion in powers of g_R.

The renormalized coupling constant g_R and the renormalized mass m_R can be any functions of n analytic at $n=4$. Then the chosen renormalization prescription fixes g_B and m_B as functions of n, singular at $n=4$.

Note that when the S matrix is calculated by the usual method, the result depends only on g_B and m_B, and is independent of the choice of parametrization by renormalized quantities.

All this is rather abstract. To see what it means, consider the Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization[10] of quantum electrodynamics.

Let the full photon propagator be

$$D'_{F\mu\nu}=\frac{-g_{\mu\nu}}{q^2[1+\Pi(q^2, e_B, m_B, n)]} + \text{terms proportional to } q_\mu q_\nu . \tag{1}$$

Then $Z_3(e_B, m_B, n)$ is defined to be

$$[1+\Pi(0, e_B, m_B, n)]^{-1}, \tag{2}$$

which is a power series in e_B.

Let the full fermion propagator be

$$S'_F=\{\not{p}[1-\sigma_p(p^2, e_B, m_B, n)] - m_B-\sigma_m(p^2, e_B, m_B, n)\}^{-1}. \tag{3}$$

Then

$$Z_2=[1-\sigma_p(0, e_B, m_B, n)]^{-1}, \tag{4}$$

and

$$m_R=[m_B+\sigma_m(0, e_B, m_B, n)]Z_2. \tag{5}$$

Finally, let $\gamma_\mu\Gamma(p, p', e_B, m_B, n)$ be the part of the (unrenormalized) 3-point function proportional to γ_μ. Define

$$\begin{aligned} e_R &= Z_2 Z_3^{1/2}\Gamma(0, 0, e_B, m_B, n) \\ &= Z_2 Z_3^{1/2}[1-\sigma_p(0, e_B, m_B, n)]e_B \quad \text{by the Ward identity} \\ &= Z_3^{1/2} e_B . \end{aligned} \tag{6}$$

We can now express e_B, m_B, Z_2, and Z_3 in terms of e_R and m_R (at each n).

If Γ_u is an unrenormalized connected and amputated Green's function and if Z_Γ is the product of $Z^{1/2}$ for each of its external lines, then the renormalized Green's function is

$$\tilde{\Gamma}_R(n)\equiv Z_\Gamma\Gamma_u . \tag{7}$$

That quantum electrodynamics is renormalizable means that when $\tilde{\Gamma}_R$ is written in terms of m_R and e_R it is analytic at $n=4$ and so

$$\Gamma_R\equiv\lim_{n\to 4}\tilde{\Gamma}_R$$

is finite. Since for each $n\neq 4$ $\tilde{\Gamma}_R(n)$ is computed as a power series in $e_R(n)$ and as a (finite) function of $m_R(n)$, we see that its limit Γ_R as $n\to 4$ depends only on $e_R(4)$ and $m_R(4)$, and not, for example, on $de_R(n)/dn$. This last remark applies in any renormalization prescription.

In practice, when calculating the renormalized Green's functions, the values of the bare parameters will not be used—the calculation will be done purely in terms of the renormalized parameters, say by the usual subtraction procedure.[10] But for our purposes it will be necessary to consider the bare parameters. The renormalized coupling and mass are finite quantities which parametrize the theory according to some prescription, which in the case of 't Hooft's method has no direct relation to the standard subtraction techniques.

In the n-dimensional procedure, the bare coupling g_B has dimensions of mass to the power $(4-n)\rho$ for some constant ρ ($\rho=\frac{1}{2}$ in quantum electrodynamics; $\rho=1$ in ϕ^4 theory). The bare mass will have dimensions of mass. Our renormalization prescriptions will need an extra mass parameter μ, which could, for example, determine the subtraction points.

The wave-function renormalizations Z_e, the renormalized coupling g_R, and the mass renormalization factor $Z_m \equiv m_B/m_R$ (all of which we *define* to be dimensionless[11]) are functions of $g_B\mu^{(n-4)\rho}$ and n only: We assume some renormalization prescription that allows Z_e, g_R, and Z_m not to depend on m. Then, by dimensional analysis, μ and g_B can only appear in the combination $g_B\mu^{(n-4)\rho}$. That such a prescription exists is proved by 't Hooft[2] and by us in Sec. VI.

Once one such prescription is found, others can be constructed by

(a) defining a new renormalized coupling $\hat{g}_R$ as some given function of g_R and n (analytic at $n=4$ and $g_R=0$),

$$\hat{g}_R=\hat{g}_R(g_R(n),n)=\hat{g}_R(g_R(g_B\mu^{(n-4)\rho},n),n);$$

(b) for each Z_e (Z_m) defining $\hat{Z}_e$ ($\hat{Z}_m$) to be some analytic function $C_e(g_R,n)$ (C_m) multiplied by Z_e (Z_m);

(c) defining $\hat{m}_R=\hat{Z}_m^{-1}m_B$;

(d) defining new renormalized Green's functions

$$\hat{\Gamma}_R=\lim_{n\to 4}\hat{Z}_\Gamma\Gamma_u=C_\Gamma(g_R,4)\Gamma_R,$$

with Γ_u as before, and $\hat{\Gamma}_R$ expressed in terms of $\hat{m}_R$ and $\hat{g}_R$. Here, C_Γ is the product of factors $C_e^{1/2}$ for each external line of Γ.

In the lowest order of g_R, we will have $\hat{g}_R=g_R=g_B$, $\hat{m}_R=m_R=m_B$, and $Z_e=\hat{Z}_e=Z_m=\hat{Z}_m=1$.

Conversely, given any two mass-independent renormalization prescriptions, they can be obtained from each other by the above procedure. This is because a field renormalization Z_e when multiplied by the corresponding 2-point function gives a function analytic at $n=4$. Also, $Z_m^{-1}m_B$ is analytic. The corresponding renormalization factors in the second procedure each satisfy the same condition, so they can be different only by an analytic factor when the physical content of the theory is the same, i.e., when the bare parameters are the same. Of course, $\hat{g}_R$ in the second procedure must be an analytic function of g_R and n, since g_R and $\hat{g}_R$ are both functions of the same g_B and are both analytic as functions of n at $n=4$.

III. SCALING BEHAVIOR OF GREEN'S FUNCTIONS

We wish to investigate the behavior of a renormalized, amputated, and connected Green's function $\Gamma_R(p,g_R,m_R,\mu)$ as the set of external momenta p is scaled. We assume Γ_R is defined from the unrenormalized Green's function $\Gamma_u(p,g_B(n),m_B(n),n)$ according to a mass-independent renormalization prescription as discussed in Sec. II:

$$\Gamma_R(p,g_R,m_R,\mu)=\lim_{n\to 4}\tilde{\Gamma}_R(p,g_R(n),m_R(n),\mu,n),$$

where

$$\tilde{\Gamma}_R(p,g_R(n),m_R(n),\mu,n)=\tilde{\Gamma}_R(p,g_R(g_B\mu^{(n-4)\rho},n),m_B(n)Z_m^{-1}(g_B(n)\mu^{(n-4)\rho},n),\mu,n)$$
$$=Z_\Gamma(g_B(n)\mu^{(n-4)\rho},n)\Gamma_u(p,g_B(n),m_B(n),n). \tag{8}$$

Our derivation of the renormalization-group equation follows Weinberg's[1] in spirit, though not necessarily in actual choice of renormalization procedure.

Differentiate the second equation of (8) with respect to μ, keeping m_B and g_B fixed (and also p and n), to get

$$\mu\frac{\partial}{\partial\mu}\tilde{\Gamma}_R(p,g_R,m_R,\mu,n)\Big|_{\text{fixed } g_R\ m_R}+\mu\frac{\partial}{\partial\mu}g_R(g_B\mu^{(n-4)\rho},n)\frac{\partial\tilde{\Gamma}_R}{\partial g_R}+\mu\frac{\partial}{\partial\mu}m_R(m_B(n),g_B(n)\mu^{(n-4)\rho},n)\frac{\partial\tilde{\Gamma}_R}{\partial m_R}$$
$$=\left[\mu\frac{\partial}{\partial\mu}Z_\Gamma(g_B\mu^{(n-4)\rho},n)\right]\Gamma_u. \tag{9}$$

So

$$\left[\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial g_R}{\partial\mu}\frac{\partial}{\partial g_R}+Z_m m_R\mu\frac{\partial Z_m^{-1}}{\partial\mu}\frac{\partial}{\partial m_R}-\mu\frac{\partial Z_\Gamma}{\partial\mu}Z_\Gamma^{-1}\right]\tilde{\Gamma}_R=0, \tag{10}$$

where $Z_m = m_B/m_R$. Take the limit $n \to 4$ to get

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} - \gamma_m(g_R) m_R \frac{\partial}{\partial m_R} - \gamma_\Gamma(g_R)\right]\Gamma_R = 0, \tag{11}$$

where

$$\beta(g_R) = \lim_{n\to 4} \mu\frac{\partial}{\partial\mu} g_R(g_B(n)\mu^{(n-4)\rho}, n), \tag{12}$$

$$\gamma_m(g_R) = \lim_{n\to 4} \mu\frac{\partial}{\partial\mu} \ln Z_m(g_B(n)\mu^{(n-4)\rho}, n), \tag{13}$$

and

$$\gamma_\Gamma(g_R) = \lim_{n\to 4} \mu\frac{\partial}{\partial\mu} \ln Z_\Gamma(g_B(n)\mu^{(n-4)\rho}, n) = \tfrac{1}{2}\sum_e \lim_{n\to 4} \mu\frac{\partial}{\partial\mu} \ln Z_e(g_B(n)\mu^{(n-4)\rho}, n). \tag{14}$$

The sum in Eq. (14) is over the external lines of Γ. All the coefficients β, γ are finite as $n \to 4$ because they appear in a differential equation for a renormalized amplitude.

By dimensional analysis, we have

$$\left[\kappa\frac{\partial}{\partial\kappa} + \mu\frac{\partial}{\partial\mu} + m_R\frac{\partial}{\partial m_R} - D_\Gamma\right]\Gamma_R(\kappa p_0, g_R, m_R, \mu) = 0, \tag{15}$$

where p_0 is a set of fixed momenta, κ is a scale variable, and the dimensions of Γ are mass to the power D_Γ.

Combining Eqs. (11) and (15), we get our renormalization-group equation:

$$\left[\kappa\frac{\partial}{\partial\kappa} - \beta(g_R)\frac{\partial}{\partial g_R} + [1+\gamma_m(g_R)] m_R\frac{\partial}{\partial m_R} - D_\Gamma + \gamma_\Gamma(g_R)\right] \times \Gamma_R(\kappa p_0, g_R, m_R, \mu) = 0. \tag{16}$$

This is Weinberg's Eq. (3.6) (in Ref. 1), but it is now applicable with any mass-independent renormalization prescription using dimensional regularization. Thus, it can be applied[7] to scalar theories using 't Hooft's prescription.

As Weinberg shows, Eq. (16) is solved using an effective coupling constant and mass defined by

$$\kappa\frac{dg(\kappa)}{d\kappa} = \beta(g(\kappa)), \tag{17}$$

$$\kappa\frac{dm(\kappa)}{d\kappa} = -[1+\gamma_m(g(\kappa))] m(\kappa), \tag{18}$$

$$g(1) = g_R \text{ and } m(1) = m_R. \tag{19}$$

Then the solution of Eq. (16) is

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) = \kappa^{D_\Gamma}\exp\left[-\int_1^\kappa \gamma_\Gamma(g(\kappa'))\frac{d\kappa'}{\kappa'}\right] \times \Gamma_R(p_0, g(\kappa), m(\kappa), \mu). \tag{20}$$

All the renormalization methods are for the same theory, so the renormalization-group equations should give the same results. The difficulty or ease of solution should not vary much from method to method.

Suppose the renormalized coupling and mass and the field renormalizations in a second mass-independent renormalization are $\hat{g}_R, \hat{m}_R, \hat{Z}_e$. Then there exist functions $A(g_R, n), C_m(g_R, n), C_e(g_R, n)$ analytic at $n = 4$ such that

$$\hat{g}_R = g_R A, \quad \hat{m}_R = m_R C_m, \quad \text{and} \quad \hat{Z}_e = Z_e C_e, \tag{21}$$

as the bare couplings are the same. (Note that the second procedure produces the same renormalized Green's functions apart from a factor $C_\Gamma = \prod C_e^{1/2}$, and the same S matrix.)

Then

$$\hat{\beta}(\hat{g}_R) = \lim_{n\to 4} \mu\frac{\partial\hat{g}_R}{\partial\mu} = \lim_{n\to 4} \mu\frac{\partial g_R}{\partial\mu}\frac{\partial\hat{g}_R}{\partial g_R}\bigg|_{\text{fixed } n} = \beta(g_R)\left(\frac{\partial\hat{g}_R}{\partial g_R}\right)_n, \tag{22}$$

and

$$\hat{\gamma}(\hat{g}_R) = \lim_{n\to 4} \mu\frac{\partial}{\partial\mu}\ln\hat{Z} = \gamma(g_R) + \beta(g_R)\frac{\partial}{\partial g_R}\ln C(g_R, 4). \tag{23}$$

It is easy to verify that the renormalization-group equation for $\hat{\Gamma}_R$ in the second approach does give the equation in the first approach for Γ_R when the substitutions (22) and (23) are made.

We note that since $A = C_m = C_e = 1$ in the lowest order of g_R, Eqs. (22) and (23) show that the renormalization-group equations in the two approaches are the same in lowest order.

IV. WEINBERG'S PROCEDURE

In this section, we give a version of Weinberg's zero-mass renormalization prescription in which dimensional regularization is used rather than an ultraviolet cutoff.

The renormalization constants are defined by

Weinberg in terms of Green's functions evaluated with $m_B = 0$, and with external momenta squared $-\mu^2$. However, the method is more transparent if we treat the mass as a coupling constant. As an example, we consider quantum electrodynamics, whose Lagrangian in the Landau gauge is[12]

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu A^\mu)^2/(2\alpha_B) + i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_B\bar{\psi}\psi - e_B\bar{\psi}\gamma^\mu\psi A_\mu, \tag{24}$$

with $\alpha_B \to 0$.

Now, the standard treatment would use as the free electron Lagrangian

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_B\bar{\psi}\psi, \tag{25}$$

with propagator $(\not{p} - m_B)^{-1}$. However, we will take the free electron Lagrangian to be

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi \tag{26}$$

with propagator $\not{p}^{-1}$. Then the interaction Lagrangian is

$$\mathcal{L}_I = -e_B\bar{\psi}\gamma_\mu A^\mu\psi - m_B\bar{\psi}\psi. \tag{27}$$

From the standpoint of calculating the finite Green's functions, the perturbation theory based on this splitting is clumsy, the second term having to be treated to all orders in m_B to get the usual results. This corresponds to picking up the successive terms of

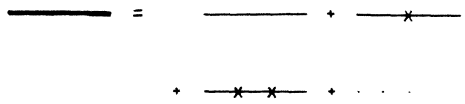

FIG. 1. Relation of massive to massless propagators. Here, the heavy line describes the massive and the light line the massless propagator, and the cross is where the term $-m_B\bar{\psi}\psi$ acts.

$$\frac{1}{\not{p} - m_B} = \frac{1}{\not{p}} + \frac{1}{\not{p}}m_B\frac{1}{\not{p}} + \frac{1}{\not{p}}m_B\frac{1}{\not{p}}m_B\frac{1}{\not{p}} + \cdots \tag{28}$$

in perturbation theory in the way Fig. 1 depicts. However, the procedure is convenient from the standpoint of renormalization. One easily sees by power counting that the only superficially divergent diagrams are of orders 0 and 1 in m_B; moreover, in order m_B only the electron self-energy is divergent.

First, we can write the photon propagator as

$$D'_{F\mu\nu}(q, e_B, m_B, n) = \frac{-g_{\mu\nu}}{q^2[1 + \Pi(q^2, e_B, 0, n)] + R(q^2, e_B, m_B, n)} + \text{terms in } q_\mu q_\nu \tag{29}$$

(using notation similar to that of Sec. II). Here, R is the sum of the self-energy diagrams of order $m_B, m_B^2, \ldots$; it is not over-all divergent. Next, we isolate the divergence in $\Pi(q^2, e_B, 0, n)$ by subtracting at $q^2 = -\mu^2$:

$$\begin{aligned} 1 + \Pi(q^2, e_B, 0, n) + R(q^2, e_B, m_B, n) &= 1 + \Pi(-\mu^2, e_B, 0, n) + \Pi^f \\ &= Z_3^{-1}(1 + Z_3\Pi^f), \end{aligned} \tag{30}$$

where

$$Z_3(e_B\mu^{n/2-2}, n)^{-1} = 1 + \Pi(-\mu^2, e_B, 0, n) \tag{31}$$

and

$$\Pi^f = \Pi(p^2, e_B, m_B, n) - \Pi(-\mu^2, e_B, 0, n). \tag{32}$$

The Ward identity now allows us to write

$$e_R = Z_3^{1/2}e_B\mu^{n/2-2}, \tag{33}$$

where the factor $\mu^{n/2-2}$ is to keep e_R dimensionless, as explained in Sec. II.

Note that in Eq. (30) Π^f is not over-all divergent. The factor Z_3 multiplying it is only used to remove any subdivergences, as in the usual renormalization procedures.

Next, we write the inverse electron propagator as

$$S_F'^{-1} = \not{p} - \not{p}\sigma(p^2, e_B, 0, n) - m_B - m_B N(p^2, e_B, n) + \cdots. \tag{34}$$

Here, N is independent of m_B, and the dots indicate the terms with more insertions of the mass vertex. The divergences in the self-energy are isolated by subtraction at $p^2 = -\mu^2$:

$$S_F'^{-1} = \not{p}[1 - \sigma(-\mu^2, e_B, 0, n)] - m_B[1 + N(-\mu^2, e_B, n)] + \Sigma^f, \tag{35}$$

where Σ^f has no over-all divergence.

Hence, we can write

$$S_F'^{-1} = Z_2^{-1}(\not{p} - m_R + \text{finite}), \tag{36}$$

with

$$Z_2^{-1} = 1 - \sigma(-\mu^2, e_B, 0, n), \tag{37}$$

$$m_R = m_B Z_m^{-1}, \tag{38}$$

and

$$Z_m^{-1} = [1 + N(-\mu^2, e_B, n)]Z_2. \tag{39}$$

The procedure set out above gives exactly the same renormalizations as Weinberg's. For example, Weinberg defines Z_2 to be $-\mu^2$ times the coefficient of $\not{p}$ in the electron propagator when evaluated with $m_B = 0$ and $p^2 = -\mu^2$. This gives

$$Z_2 = (-\mu^2)\left\{\frac{1-\sigma(-\mu^2, e_B, 0, n)+O(m_B{}^2)}{-\mu^2[1-\sigma(-\mu^2, e_B, 0, n)]^2 - m_B{}^2[1+N(-\mu^2, e_B, n)]^2 + O(m_B{}^3)}\right\}_{m_B=0}$$

$$= [1-\sigma(-\mu^2, e_B, 0, n)]^{-1},$$

which is the same as Z_2 given by Eq. (37).

V. QUANTUM-ELECTRODYNAMICS CALCULATION

To show how the method works, we calculate the coefficients of Eq. (16) in the one-loop approximation to quantum electrodynamics. Since the calculations are done with zero-mass propagators, all the integrals are simple and can be worked out explicitly in terms of B and Γ functions using the formulas listed in the Appendix.

By Eq. (37), Z_2 is defined to be

$$[1-\sigma(-\mu^2, e_B, 0, n)]^{-1},$$

where $\not{p}\sigma(p^2, e_B, 0, n)$ is the term of order $m_B{}^0$ in the electron self-energy. Now in the one-loop approximation (Fig. 2)

$$\begin{aligned}
\not{p}\sigma(p^2, e_B, 0, n) &= \frac{1}{(2\pi)^n}\int d^n k \frac{\gamma^\mu i(\not{p}+\not{k})\gamma^\nu(-ie_B)^2}{k^2(p+k)^2}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\\
&= \frac{-ie_B{}^2}{(2\pi)^n}\int d^n k\left[\frac{\not{p}(2-n)+\not{k}(1-n)}{k^2(p+k)^2} - \frac{\not{k}\not{p}\not{k}}{k^4(p+k)^2}\right] \quad \text{by (A1) and (A2)}\\
&= \frac{-ie_B{}^2}{(2\pi)^n}\int_0^1 dx \int d^n k\left[\frac{\not{p}(2-n)+\not{k}(1-n)}{(p^2x+2p\cdot kx+k^2)^2} - \frac{2\not{k}\not{p}\not{k}(1-x)}{(p^2x+2p\cdot kx+k^2)^3}\right] \quad \text{by (A10)}\\
&= \frac{e_B{}^2\not{p}}{(4\pi)^{n/2}}\int_0^1 dx\{[-p^2x(1-x)]^{n/2-2}\Gamma(2-\tfrac{1}{2}n)[2-n-x(1-n)-\tfrac{1}{2}(1-x)(2-n)]\\
&\qquad + [-p^2x(1-x)]^{n/2-3}\Gamma(3-\tfrac{1}{2}n)p^2x^2(1-x)\} \quad \text{by (A6)–(A8)}\\
&= \frac{e_B{}^2(-p^2)^{n/2}\not{p}\Gamma(2-\tfrac{1}{2}n)}{(4\pi)^{n/2}}\int_0^1 dx[x(1-x)]^{n/2-2}(1-\tfrac{1}{2}n)(1-2x)\\
&= 0 .
\end{aligned} \tag{40}$$

So to this order

$$Z_2(e_B\mu^{n/2}, n) = 1 . \tag{41}$$

Next, Z_3 is defined by Eq. (31):

$$Z_3(e_B\mu^{n/2-2}, n) = [1+\Pi(-\mu^2, e_B, 0, n)]^{-1},$$

where Π appears in the photon self-energy:

$$\Pi_{\kappa\lambda} = (q_\kappa q_\lambda - g_{\kappa\lambda}q^2)\Pi . \tag{42}$$

The one-loop contribution to $\Pi_{\kappa\lambda}$ is (Fig. 3)

$$\begin{aligned}
i\,\mathrm{Tr}\int \frac{d^n k}{(2\pi)^n}(-ie_B)^2\gamma_\kappa\frac{i\not{k}}{k^2}\gamma_\lambda\frac{i(\not{k}+\not{q})}{(k+q)^2} &= \frac{ie_B{}^2 2^{n/2}}{(2\pi)^n}\int_0^1 dx\int d^n k\frac{[-g_{\kappa\lambda}k\cdot(k+q)+2k_\kappa k_\lambda + k_\kappa q_\lambda + k_\lambda q_\kappa]}{(k^2+2q\cdot kx+q^2x)^2} \quad \text{by (A5)}\\
&= \frac{2e_B{}^2}{(2\pi)^{n/2}}(-q^2)^{n/2-2}(q_\kappa q_\lambda - g_{\kappa\lambda}q^2)\Gamma(2-\tfrac{1}{2}n)\int_0^1 dx[x(1-x)]^{n/2-1}\\
&= 2(2\pi)^{-n/2}e_B{}^2(-q^2)^{n/2-2}(q_\kappa q_\lambda - g_{\kappa\lambda}q^2)\Gamma(2-\tfrac{1}{2}n)B(\tfrac{1}{2}n, \tfrac{1}{2}n) .
\end{aligned} \tag{43}$$

So

$$Z_3 = 1 - 2(2\pi)^{-n/2}e_B{}^2\mu^{n-4}\Gamma(2-\tfrac{1}{2}n)B(\tfrac{1}{2}n, \tfrac{1}{2}n) \tag{44}$$

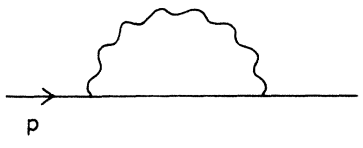

FIG. 2. Fermion self-energy.

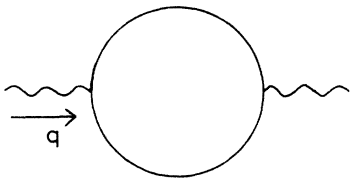

FIG. 3. Photon self-energy.

to this order, and e_R is given by Eq. (33):

$$e_R = Z_3^{1/2} e_B \mu^{n/2-2}.$$

Hence, by Eq. (12)

$$\begin{aligned}\beta(e_R) &= \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} Z_3^{1/2} e_B \mu^{n/2-2} \\ &= \lim_{n\to 4} \left[(\tfrac{1}{2}n - 2) Z_3^{1/2} e_B \mu^{n/2-2} + \mu^{n/2-2} e_B \mu \frac{\partial}{\partial \mu} Z_3^{1/2} \right] \\ &= \lim_{n\to 4} [(\tfrac{1}{2}n - 2) e_R - \mu^{n/2-2} e_B (2\pi)^{-n/2} e_B^2 \mu^{n-4} (n-4) \Gamma(2 - \tfrac{1}{2}n) B(\tfrac{1}{2}n, \tfrac{1}{2}n)] + O(e_B^5) \\ &= \lim_{n\to 4} 2(2\pi)^{-n/2} e_R^3 \Gamma(3 - \tfrac{1}{2}n) B(\tfrac{1}{2}n, \tfrac{1}{2}n) + O(e_R^5) \\ &= e_R^3 (12\pi^2)^{-1} + O(e_R^5), \end{aligned} \tag{45}$$

which is the standard result.

Next, we evaluate Z_m using the diagram of Fig. 4 and using the definition in Eq. (39):

$$\begin{aligned} Z_m^{-1} &= \left[1 - \frac{i e_B^2}{(2\pi)^n} \int d^n k \frac{\gamma_\mu (\not{p} + \not{k})^2 \gamma_\nu}{(p+k)^4 k^2} \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \right] Z_2 \Big|_{p^2 = -\mu^2} \\ &= 1 - \frac{i e_B^2}{(2\pi)^n} \int d^n k \frac{(n-1)}{(p+k)^2 k^2} \Big|_{p^2 = -\mu^2} \quad \text{by (A1), (A2), and (41)} \\ &= 1 + \frac{e_B^2 \mu^{n-4}}{(4\pi)^{n/2}} \Gamma(2 - \tfrac{1}{2}n)(n-1) B(\tfrac{1}{2}n - 1, \tfrac{1}{2}n - 1). \end{aligned} \tag{46}$$

So by Eq. (13)

$$\begin{aligned}\gamma_m(e_R) &= \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} \ln Z_m(e_B \mu^{n/2-2}, n) \\ &= \lim_{n\to 4} [-(4\pi)^{-n/2} e_B^2 (n-4) \mu^{n-4} \Gamma(2 - \tfrac{1}{2}n)(n-1) B(\tfrac{1}{2}n - 1, \tfrac{1}{2}n - 1)] + O(e_R^4) \\ &= 3 e_R^2 / (8\pi^2) + O(e_R^4). \end{aligned} \tag{47}$$

Finally,

$$\begin{aligned}\gamma_\Gamma(e_R) &= \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} \ln Z_\Gamma \\ &= \tfrac{1}{2} \left[n_f^\Gamma \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} \ln Z_2 + n_b^\Gamma \lim_{n\to 4} \mu \frac{\partial}{\partial \mu} \ln Z_3 \right], \end{aligned}$$

where n_f^Γ and n_b^Γ are the numbers of external electron and photon lines of Γ. Evaluating the derivatives in the same way as above gives

$$\gamma_\Gamma(e_R) = n_b^\Gamma e_R^2 / (12\pi^2) + O(e_R^4). \tag{48}$$

VI. 't HOOFT'S METHOD

Instead of taking μ to be a renormalization point, 't Hooft[2] defines μ to be a "unit of mass" (with the dimensions of mass). Its only use is to absorb the n-dependent part of the dimensions of the bare parameters in such a way that a renormalized parameter has the same dimension for any n as the corresponding bare parameter has at $n = 4$.

For example, e_B in quantum electrodynamics has mass dimension $2 - \frac{1}{2}n$, so we write e_B as $\mu^{2-n/2}$ times a dimensionless quantity, and choose e_R to be dimensionless for all n.

In this, our procedure differs from 't Hooft's, for he absorbs all the dimensions of the parameters into powers of μ so that m_R, for example, would be dimensionless. However, our procedure seems more convenient, and yields the same final results.

The bare coupling constant, g_B, and mass, m_B, and the field renormalization(s), Z, are expressed in terms of μ and the renormalized parameters, g_R and m_R. Although we shall allow later that g_R and m_R may depend on n (analytically at $n = 4$), initially we assume they are constants. Then g_B, m_B, and Z are written in the form

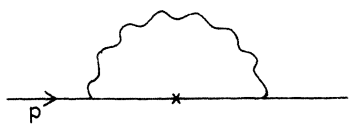

FIG. 4. Contribution to mass renormalization.

$$g_B \mu^{(n-4)\rho} = g_R + \sum_{\nu=1}^{\infty} \frac{a_\nu(g_R, m_R, \mu)}{(n-4)^\nu}, \tag{49}$$

$$m_B = m_R + \sum_{\nu=1}^{\infty} \frac{b_\nu(g_R, m_R, \mu)}{(n-4)^\nu} \equiv m_R Z_m, \tag{50}$$

$$Z = 1 + \sum_{\nu=1}^{\infty} \frac{c_\nu(g_R, m_R, \mu)}{(n-4)^\nu}. \tag{51}$$

So these quantities are just the renormalized quantities (g_R, m_R, and, in the case of Z, 1) plus whatever poles are needed to cancel the poles in the Feynman integrals (all multiplied by a power of μ to get the dimensions correct, if necessary).

The coefficients a_ν, b_ν, and c_ν are unique given g_R, m_R, and μ.

Next, we prove a restriction on the dependence of a_ν, b_ν, and c_ν on m_R and μ.

Theorem. a_ν, b_ν, and c_ν do not depend on μ; a_ν and c_ν do not depend on m_R (assuming g_R is dimensionless). Thus, Z_m does not depend on μ or on m_R.

Proof. The poles in Eqs. (49), (50), and (51) (to a given order in g_R) are defined to be precisely those needed to subtract the poles in the corresponding Feynman integrals (and thus give a finite result).

μ only appears as a power of μ^{4-n} (multiplying coupling constants). When the Laurent expansion of each Feynman integral is made (to find the poles which we need to cancel), logarithms of μ will appear.

On the other hand, in the residues of the poles m_R only appears in polynomials,[2] so, in the residues, it only appears in positive powers, independent of n.

But a_ν and c_ν are dimensionless, while b_ν, μ, and m_R have dimension 1 (all independent of n). The only way a_ν, b_ν, and c_ν can have the correct dimensions is the way stated in the theorem.[13,14]

Thus, we have proved that the (dimensionless) renormalization factors $g_B \mu^{(n-4)\rho}/g_R$, Z_m, and Z are all independent of mass in this particular renormalization prescription, justifying everything said earlier.

The proof clearly extends to the cases where

(a) there are several couplings, masses and fields [then there is a collection of mass renormalization factors Z_{ij} (independent of the masses) and in an obvious notation $m_{Bi} = \sum_j Z_{ij} m_{Rj}$],

(b) there is a coupling constant (or constants) f with dimension (at $n=4$) (dimension 1 is the only case in a *renormalizable* theory; f appears in a power series, and so in positive powers only; the above proof applies if we treat $f_B \mu^{(4-n)\rho}$ and f_R as masses were treated above), and

(c) the mass is of a boson and the convenient parameters are m_B^2, m_R^2 (then the counterterms depend quadratically on the massive parameters[15]).

To derive the renormalization-group equation, we may invert Eqs. (49)–(51) to obtain g_R, m_R, and Z as functions of g_B, m_B, μ, and n, and then use the methods of Sec. III. In practice, it is more convenient not actually to do the inversion, but to proceed as follows.

Differentiation of (49) with respect to μ yields

$$\rho\left[(n-4)g_R + a_1 + \frac{a_2}{n-4} + \cdots\right] = \left(\mu \frac{\partial g_R}{\partial \mu}\right)_{g_B} \left[1 + \frac{a_1'}{n-4} + \frac{a_2'}{(n-4)^2} + \cdots\right], \tag{52}$$

where $a_\nu' = da_\nu/dg_R$. Knowing $(\mu \partial g_R/\partial \mu)_{g_B}$ to be analytic, from its position in Eq. (10) for the quantity $\tilde{\Gamma}_R$ which is analytic at $n=4$, we may write

$$\left(\mu \frac{\partial g_R}{\partial \mu}\right)_{g_B} = x_0 + x_1(n-4) + x_2(n-4)^2 + \cdots. \tag{53}$$

We may now solve (52) for the x_ν, obtaining

$$\begin{aligned} x_0 &= (a_1 - g_R a_1')\rho, \\ x_1 &= g_R \rho, \\ x_\nu &= 0 \text{ for } \nu \geq 2, \end{aligned} \tag{54}$$

and in addition the following family of identities satisfied by the a_ν:

$$a_{\nu-1}' x_0 + a_\nu' x_1 = \rho a_\nu, \quad \nu = 2, 3, \ldots. \tag{55}$$

These identities were also obtained by 't Hooft, but by a different method. From (12), (53), and (54), we now obtain

$$\beta(g_R) = (a_1 - g_R a_1')\rho. \tag{56}$$

Similarly, we may treat Eqs. (50) and (51), being led to the results

$$\mu \frac{\partial}{\partial \mu} \ln Z_m = \frac{\rho g_R b_1'}{m_R}, \tag{57}$$

$$\mu \frac{\partial}{\partial \mu} \ln Z_\Gamma = \rho g_R c_1', \tag{58}$$

and families of identities involving the b_ν and c_ν,

$$b_{\nu-1}' x_0 + b_\nu' x_1 = \rho g_R b_{\nu-1} b_1'/m_R, \quad \nu = 2, 3, \ldots, \tag{59}$$

$$c_{\nu-1}' x_0 + c_\nu' x_1 = \rho g_R c_{\nu-1} c_1', \quad \nu = 2, 3, \ldots, \tag{60}$$

also available from 't Hooft's approach. Now, from (13), (14), (57), and (58), we get

$$\gamma_m(g_R) = \rho g_R b_1'/m_R, \tag{61}$$

$$\gamma_\Gamma(g_R) = \rho g_R c_1'. \tag{62}$$

Since a_1, b_1, c_1 are determined uniquely by the 't Hooft renormalization procedure, the renormalization-group equation (16) is fully specified.

We may alternatively give a derivation of the

solution (20) of (16) similar in spirit to that of 't Hooft. An attractive feature of this derivation is the way it brings the effective coupling constant and the effective mass directly into the analysis. Returning to Eqs. (49)–(51), we see that they define functions

$$g_B(\mu, g_R, n), \quad m_B \equiv m_R Z_m(g_R, n), \quad Z_\Gamma(g_R, n)$$

for constant m_R and g_R with coefficients a_ν, b_ν, and c_ν determined by the 't Hooft renormalization procedure. With these functions, we can define

$$g_B(\hat{\mu}, \hat{g}_R, n), \quad \hat{m}_R Z_m(\hat{g}_R, n), \quad Z_\Gamma(\hat{g}_R, n)$$

for $\hat{\mu} = \mu(1+\epsilon)$, where ϵ is small, and $\hat{g}_R = g_R + \epsilon\delta$, where $\delta = \alpha_0 + \alpha_1(n-4) + \cdots$ is analytic at $n=4$, and $\hat{m}_R = m_R(1+\epsilon\xi)$, where $\xi = \eta_0 + \eta_1(n-4) + \cdots$ also is analytic at $n=4$.[16] We now try to choose δ and ξ so that

$$g_B(\hat{\mu}, \hat{g}_R, n) = g_B(\mu, g_R, n) \tag{63}$$

and

$$\hat{m}_R Z_m(\hat{g}_R, n) = m_R Z_m(g_R, n), \tag{64}$$

and then we are clearly comparing two ('t Hooft) renormalizations of the same theory. Equation (63) requires

$$\mu \frac{\partial g_B}{\partial \mu} + \delta \frac{\partial g_B}{\partial g_R} = 0,$$

and so can be solved by

$$\delta = \rho[a_1 - g_R a_1' + g_R(n-4)], \tag{65}$$

and hence (64) leads to

$$\xi = -g_R b_1' \rho / m_R. \tag{66}$$

Since we are comparing two different renormalizations of the same theory, it follows from the discussion in Sec. II that

$$Z_\Gamma(g_R, n) = Z_\Gamma(\hat{g}_R, n) z(\epsilon, g_R, n) \tag{67}$$

defines a function z analytic at $n=4$. In the same way in which we obtained (65), (66) from (63), (64), we obtain easily

$$z = 1 - \epsilon g_R c_1' \rho. \tag{68}$$

We write Eq. (8) for each of the two renormalizations:

$$\Gamma_R(p, g_R, m_R, \mu) = \lim_{n\to 4} Z_\Gamma(g_B \mu^{(n-4)\rho}, n) \times \Gamma_u(p, g_B, m_B, n), \tag{69}$$

$$\Gamma_R(p, \hat{g}_R(4), \hat{m}_R, \mu) = \lim_{n\to 4} Z_\Gamma(g_B \hat{\mu}^{(n-4)\rho}, n) \times \Gamma_u(p, g_B, m_B, n). \tag{70}$$

As the bare quantities are the same in each case, we can combine Eqs. (67), (69), and (70) to give

$$\Gamma_R(p, g_R, m_R, \mu) = z(\epsilon, g_R, 4) \times \Gamma_R(p, \hat{g}_R(4), \hat{m}_R, \hat{\mu}), \tag{71}$$

with

$$\hat{g}_R(4) = g_R + \epsilon\rho(a_1 - g_R a_1') \tag{72}$$

and

$$\hat{m}_R = m_R - \epsilon g_R b_1' \rho. \tag{73}$$

The values of $\hat{g}_R$ at $n \neq 4$ are irrelevant since everything in Eq. (71) is finite.

Our intention is to derive Eq. (20) by comparing the renormalized Green's functions Γ_R arising from the two sets g_R, m_R, μ and $\hat{g}_R(4), \hat{m}_R, \hat{\mu}$. So we now pass to the results of a finite change $\mu \to \hat{\mu} = \mu\kappa$ in the "unit of mass" μ. Instead of $\hat{g}_R(4)$, as given by (72), we will have $g(\kappa)$ which obeys

$$\kappa \frac{dg(\kappa)}{d\kappa} = \rho(a_1 - g(\kappa)a_1') \tag{74}$$

and the boundary condition $g(1) = g_R$. Similarly, instead of $\hat{m}_R$, we will have $\kappa m(\kappa)$ which obeys

$$\kappa \frac{\partial \kappa m(\kappa)}{\partial \kappa} = -g(\kappa) b_1'(g(\kappa), \kappa m(\kappa))\rho \tag{75}$$

and $m(1) = m_R$. [The extra κ multiplying $m(\kappa)$ is to make the $m(\kappa)$ used here the same as in Eq. (18).] Further, instead of z, we will have $z(\kappa)$ which obeys

$$\kappa \frac{\partial z(\kappa)}{\partial \kappa} = -g(\kappa) c_1'(g(\kappa))\rho z(\kappa), \tag{76}$$

and $z(1) = 1$. Thus, (71) for the finite change of unit of mass reads as

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) = z(\kappa)\Gamma_R(\kappa p_0, g(\kappa), \kappa m(\kappa), \mu\kappa) \tag{77}$$

upon renaming p as κp_0. Now, if D_Γ is the mass dimension of Γ_R in the usual sense, we may use standard dimensional analysis to express (77) as

$$\Gamma_R(\kappa p_0, g_R, m_R, \mu) = z(\kappa)\kappa^{D_\Gamma} \Gamma_R(p_0, g(\kappa), m(\kappa), \mu). \tag{78}$$

This equation coincides with (20) as derived by the previous method. Equations (74) and (56) yield (17); (75) and (61) yield (18); and, upon use of (62), integration of (76) yields the second factor of (20). In this approach, the κ-dependent effective coupling constant and effective mass enter as the finite parameters of a renormalization different from the one based on g_R and m_R, there being a κ-dependent finite renormalization involved in the comparison of the two sets of renormalized Green's functions.

Since in lowest order Eq. (16) is the same when derived by any mass-independent renormalization procedure, we have in particular shown that the methods of Weinberg and of 't Hooft give the same results in lowest order (when both methods are applicable).

VII. QUANTUM ELECTRODYNAMICS BY 't HOOFT'S METHOD

We now repeat the calculations of Sec. V using 't Hooft's prescription. Let

$$e_B = \mu^{2-n/2}\left[e_R + a_{13}e_R{}^3/(n-4) + O(e_R{}^5)\right],$$
$$m_B = m_R + b_{12}m_R e_R{}^2/(n-4) + O(e_R{}^4),$$
$$Z_2 = 1 + c_{12}e_R{}^2/(n-4) + O(e_R{}^4), \tag{79}$$

and

$$Z_3 = 1 + d_{12}e_R{}^2/(n-4) + O(e_R{}^4).$$

Instead of (26), we now choose the free electron Lagrangian to be

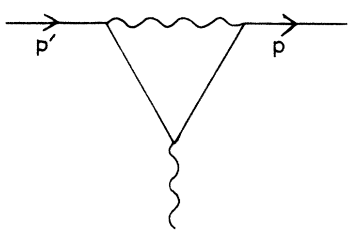

FIG. 5. Vertex correction.

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi - m_R\bar{\psi}\psi, \tag{80}$$

with propagator

$$(\not{p} - m_R)^{-1}. \tag{81}$$

The interaction Lagrangian is

$$-(m_B - m_R)\bar{\psi}\psi - e_B\bar{\psi}A^\mu\gamma_\mu\psi. \tag{82}$$

Then the fermion self-energy (Fig. 2) is to order $e_R{}^2$

$$\Sigma = \frac{1}{(2\pi)^n}\int d^nk\,\frac{\gamma_\mu i(\not{p}+\not{k}+m_R)\gamma_\nu}{k^2[(p+k)^2-m_R{}^2]}(-ie_B)^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$
$$= \frac{-ie_B{}^2}{(2\pi)^n}\int d^nk\,\frac{\not{p}(2-n)+\not{k}(1-n)-\not{k}\not{p}\not{k}/k^2 + m_R(n-1)}{k^2[(p+k)^2-m_R{}^2]}$$
$$= \frac{e_B{}^2}{(4\pi)^{n/2}}\int_0^1 dx\left\{\frac{(2-n)\not{p}-(1-n)\not{p}x - \frac{1}{2}(2-n)\not{p}(1-x) + m_R(n-1)}{[m_R{}^2x - p^2x(1-x)]^{2-n/2}}\Gamma(2-\tfrac{1}{2}n) + \frac{\not{p}p^2x^2(1-x)}{[m_R{}^2x - p^2x(1-x)]^{3-n/2}}\Gamma(3-\tfrac{1}{2}n)\right\}$$
$$= -\frac{3e_R{}^2m_R}{8\pi^2}\frac{1}{(n-4)} + \text{regular} + O(e_R{}^4). \tag{83}$$

We must also add the mass counterterm $m_B - m_R$ to Σ.

Next, Z_2 and m_B are defined by requiring

$$Z_2 S_F'^{-1} = Z_2(\not{p} - m_B - \Sigma)$$

to be regular at $n=4$. So

$$Z_2 = 1, \tag{84}$$

and

$$m_B = m_R\{1 + 3e_R{}^2/[8\pi^2(n-4)]\} + O(e_R{}^4). \tag{85}$$

The photon self-energy is (Fig. 3)

$$\Pi_{\kappa\lambda} = \frac{-1}{i}\,\mathrm{Tr}\int\frac{d^nk}{(2\pi)^n}\,\frac{(-ie_B)^2\gamma_\kappa i(\not{k}+m_R)\gamma_\lambda i(\not{k}+\not{q}+m_R)}{[k^2-m_R{}^2][(q+k)^2-m_R{}^2]}$$
$$= \frac{ie_B{}^2 2^{n/2}}{(2\pi)^n}\int_0^1 dx\int d^nk\,\frac{\{g_{\kappa\lambda}[m_R{}^2 - k\cdot(k+q)] + 2k_\kappa k_\lambda + k_\kappa q_\lambda + k_\lambda q_\kappa\}}{(m_R{}^2 - q^2x - 2q\cdot kx - k^2)^2}$$
$$= \frac{-e_B{}^2}{(2\pi)^{n/2}}\int_0^1 dx[m_R{}^2 - q^2x(1-x)]^{n/2-2}\left(\Gamma(2-\tfrac{1}{2}n)\{g_{\kappa\lambda}[m_R{}^2 + q^2x(1-x)] - 2q_\kappa q_\lambda x(1-x)\}\right.$$
$$\left. + \Gamma(1-\tfrac{1}{2}n)[m_R{}^2 - q^2x(1-x)]\,(\tfrac{1}{2}n-1)g_{\kappa\lambda}\right)$$
$$= \frac{2e_B{}^2}{(2\pi)^{n/2}}\Gamma(2-\tfrac{1}{2}n)\int_0^1 dx\,[m_R{}^2 - q^2x(1-x)]^{n/2-2}x(1-x)\,(q_\kappa q_\lambda - g_{\kappa\lambda}q^2)$$
$$= (q_\kappa q_\lambda - g_{\kappa\lambda}q^2)\left(\frac{-e_R{}^2}{6\pi^2}\frac{1}{n-4} + \text{regular} + O(e_R{}^4)\right). \tag{86}$$

So

$$1+\Pi = 1 - e_R^2/[6\pi^2(n-4)] + O(e_R^4). \tag{87}$$

Z_3 is defined to be such that $(1+\Pi)Z_3$ is regular, so

$$Z_3 = 1 + e_R^2/[6\pi^2(n-4)] + O(e_R^4). \tag{88}$$

To evaluate e_B we find the pole part of the vertex correction Γ_μ to lowest order (Fig. 5):

$$\Gamma_\mu = -\frac{ie_B^3}{(2\pi)^n}\int d^n k \frac{\gamma_\kappa(\not k+\not p+m_R)\gamma_\mu(\not k+\not p'+m_R)\gamma_\lambda}{[(k+p)^2-m_R^2][(k+p')^2-m_R^2]k^2}\left(g^{\kappa\lambda}-\frac{k^\kappa k^\lambda}{k^2}\right)$$

$$= \frac{2ie_B^3}{(2\pi)^n}\int_0^1 dx\int_0^{1-x} dy\int d^n k \frac{(n-3)k^2\gamma_\mu + (4-2n)\not k k_\mu}{[m_R^2(x+y)-p^2x-p'^2y-2k\cdot(px+p'y)-k^2]^3} + \text{terms regular at } n=4.$$

Here we have picked out the terms in the numerator that are quadratic in k. These are the only terms contributing to the pole. We have also used Eqs. (A1) and (A2). So (again picking out only the contribution to the pole) we have, using (A8) and (A9),

$$\Gamma_\mu = \frac{\frac{1}{2}e_B^3}{(4\pi)^{n/2}}\Gamma(2-\tfrac{1}{2}n) \times \int_0^1 dx\int_0^{1-x} dy\,[n(n-3)+4-2n]\gamma_\mu + \text{regular terms}$$

$$= \text{regular terms} + O(e_R^5). \tag{89}$$

We define e_B by requiring $Z_2 Z_3^{1/2}(\Gamma_\mu + e_B\gamma_\mu)$ to be finite at $n=4$. So we have, ignoring terms of higher order than e_R^3,

$$a_{13} = -1/(12\pi^2),$$

and hence

$$e_B\mu^{n/2-2} = e_R - \frac{e_R^3}{12\pi^2}\frac{1}{n-4} + O(e_R^5). \tag{90}$$

Note that

$$e_B\mu^{n/2-2} = e_R Z_3^{-1/2},$$

as we could have obtained from the Ward identity.

Thus we find, using Eqs. (56), (61), (62), and (14), that

$$\beta(e_R) = \tfrac{1}{2}\left(1 - e_R\frac{d}{de_R}\right)\left(\frac{-e_R^3}{12\pi^2}\right)$$

$$= e_R^3/(12\pi^2), \tag{91}$$

$$\gamma_m(e_R) = \tfrac{1}{2}e_R\frac{d}{de_R}\frac{3e_R^2}{8\pi^2}$$

$$= 3e_R^2/(8\pi^2), \tag{92}$$

and

$$\gamma_\Gamma(e_R) = \tfrac{1}{2}[n_f^\Gamma\gamma_2(e_R) + n_b^\Gamma\gamma_3(e_R)], \tag{93}$$

with

$$\gamma_2(e_R) = 0$$

and

$$\gamma_3(e_R^2) = e_R^2/(6\pi^2).$$

These results agree with those previously derived in Sec. V. We note that since e_B, Z_2, and Z_3 are independent of mass, they are most easily derived by setting $m_R = m_B = 0$ before working out the integrals.

APPENDIX

The formulas (given in Appendix A of Ref. 8) needed for evaluating the n-dimensional integrals are

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \tag{A1}$$

$$g^\mu_\mu = n, \tag{A2}$$

$$\mathrm{Tr}\gamma^\mu = 0 = \mathrm{Tr}\gamma^\lambda\gamma^\mu\gamma^\nu, \tag{A3}$$

$$\mathrm{Tr}\,1 = 2^{n/2}, \tag{A4}$$

$$\mathrm{Tr}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = 2^{n/2}(g^{\kappa\lambda}g^{\mu\nu} - g^{\kappa\mu}g^{\lambda\nu} + g^{\kappa\nu}g^{\lambda\mu}), \tag{A5}$$

$$\int d^n k \frac{1}{(m^2-2p\cdot k-k^2)^\alpha} = \frac{i\pi^{n/2}\Gamma(\alpha-\frac{1}{2}n)}{(m^2+p^2)^{\alpha-n/2}\Gamma(\alpha)}, \tag{A6}$$

$$\int d^n k \frac{k_\mu}{(m^2-2p\cdot k-k^2)^\alpha} = \frac{i\pi^{n/2}\Gamma(\alpha-\frac{1}{2}n)}{(m^2+p^2)^{\alpha-n/2}\Gamma(\alpha)}(-p_\mu), \tag{A7}$$

$$\int d^n k \frac{k^2}{(m^2-2p\cdot k-k^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2+p^2)^{\alpha-n/2}}\frac{1}{\Gamma(\alpha)}[\Gamma(\alpha-\tfrac{1}{2}n)p^2 - \Gamma(\alpha-1-\tfrac{1}{2}n)\tfrac{1}{2}n(m^2+p^2)], \tag{A8}$$

$$\int d^n k \frac{k_\mu k_\nu}{(m^2 - 2p\cdot k - k^2)^\alpha} = \frac{i\pi^{n/2}}{(m^2+p^2)^{\alpha-n/2}} \frac{1}{\Gamma(\alpha)} [\Gamma(\alpha - \tfrac{1}{2}n)p_\mu p_\nu - \Gamma(\alpha - 1 - \tfrac{1}{2}n)\tfrac{1}{2}g_{\mu\nu}(m^2+p^2)] , \tag{A9}$$

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[ax+b(1-x)]^{\alpha+\beta}} . \tag{A10}$$

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[11]Remove the dimensions of g_R by extracting a suitable power of μ. In the discussion of quantum electrodynamics above this was not done, but for our purposes it is more convenient to make g_R dimensionless. Since in the discussion of scaling we will want to consider the zero-mass theory, μ must be used for this purpose rather than m_B or m_R.

[12]Our metric and Dirac matrices are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965). In order to obtain finite Green's functions, it is necessary to renormalize the gauge parameter by writing $\alpha_B = \alpha_R Z_3$. So the renormalization-group equation must include an extra term involving $\partial\Gamma_R/\partial\alpha_R$. Thus, in the solution, there will be an effective gauge parameter $\alpha(\kappa)$ in addition to the effective coupling and mass. Since the dependence of e_B on μ and e_R does not involve α, this creates no difficulty in actually solving the equation. But, since the Green's functions are gauge-dependent, it does add some complication. However, if $\alpha = 0$, the free-photon propagator is purely transverse. Since by the Ward identity the photon self-energy is also transverse, no renormalization of α is now needed. Moreover, the derivation of the renormalization-group equation (16) works with no extra term needed, and so $\alpha(\kappa) = 0$ identically. Therefore we use the Landau gauge $\alpha = 0$.

[13]Strictly speaking, this proof is inductive. It is done order by order in g_R. We have proved the theorem true of the counterterms of a given order in g_R if it is true in the next lowest order—since these counterterms are to subtract singularities of Feynman integrals which only contain the bare parameters to a lower order. The theorem is certainly true in lowest order: $g_B\mu^{(n-4)\rho} = g_R + O(g_R^2)$, $m_B = m_R[1 + O(g_R)]$, $Z = 1 + O(g_R)$. So by induction it is true in all orders.

[14]Notice that we have assumed in the proof that the renormalization is done purely by counterterms in the Lagrangian (so the kinetic energy terms are like $Z_2 i\bar{\psi}\gamma^\mu\partial_\mu\psi$). However, the Lagrangian can be transformed to one of the type considered elsewhere in the paper in the usual way by dividing each coupling constant by the appropriate product of $Z^{1/2}$'s. If the bare parameters defined one way are of the forms (49) to (51) and satisfy the theorem, then the same is true of the bare parameters defined the other way.

[15]We now have a new proof that if ϕ^3 theory is renormalizable, then the only infinite renormalization is of the mass, by the one-loop self-energy. Since the coupling constant has dimension 1, $f_B = \mu^{2-n/2} f_R$, $m_B^2 = m_R^2 + Bf_R^2/(n-4)$, $Z = 1$.

[16]δ and ξ should by previous arguments be functions only of g_R.