PHYSICAL REVIEW D VOLUME 10, NUMBER 4 15 AUGUST 1974

Invariant bilinear forms and the discrete symmetries for relativistic arbitrary-spin fields*

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(Received 23 October 1973)

The existence of a Hermitianizing matrix η is usually assumed in the study of first-order relativistic wave equations because it provides for an invariant scalar product, bilinear densities (e.g., Lagrangian), and parity realization in a canonical way. However, an η will exist only if the representation of SL(2, C) which governs the transformation of the wave function is self-conjugate. The drawbacks of this fact for theories with $s > 1$ are discussed and a class of relativistic wave equations which avoids these drawbacks and which does not allow for the existence of an η matrix is set aside for study. It is shown that a dual space may be defined (or, equivalently, a metric operator may be introduced) such that all of the above η-matrix benefits may be maintained without an η matrix. The discrete symmetries are defined for these equations and it is shown that the realization of parity in terms of an antilinear operator naturally emerges. The locality, positive-definite metric, and positive-definite energy of the second-quantized version of the formulation are described. These considerations apply to a class of wave equations which provide a simple and uniform description of a massive, spin-s relativistic particle and which remain consistent and causal in the presence of a minimally coupled external electromagnetic field.

I. INTRODUCTION

In the study of relativistic wave equations of the general form[1]

$$(i\beta\cdot\partial - m)_{\alpha\beta}\phi_\beta(x)=0, \quad \alpha, \beta=1, \ldots, N \tag{1.1}$$

it is usually taken as axiomatic that there exists a matrix η called a Hermitianizing matrix with the property that[2]

$$\eta\beta_\mu=\beta_\mu^\dagger\eta\,. \tag{1.2}$$

The existence of such a matrix has moreover been assumed for excellent reasons:

(1) It guarantees the existence of a relativistically invariant bilinear form on the solution space of Eq. (1.1),[3]

$$(\phi, \psi)=\int d\sigma_\mu(x)\phi^\dagger(x)\eta\beta^\mu\psi(x)\,, \tag{1.3}$$

where $\sigma(x)$ is a spacelike surface. The existence of such a form is of course essential if we wish to attribute a quantum-mechanical interpretation to the solutions of Eq. (1.1).

(2) It provides for the construction of invariant bilinear densities. For example, it ensures that the wave equation is derivable from a Lagrangian,

$$L(x)=\phi^\dagger(x)\eta(i\beta\cdot\partial-m)\phi(x)\,, \tag{1.4}$$

and guarantees it Hermiticity. This greatly facilitates the introduction of interactions into the formalism.

(3) It affords a simple realization of parity symmetry. For example, if β_0 is Hermitian and $\vec{\beta}$ is anti-Hermitian, then using Eq. (1.2) we see that the action of parity on $\phi(x)$ may be simply written as $[x'=(x_0, -\vec{x})]$

$$\phi_\alpha'(x')=\eta_{\alpha\beta}\phi_\beta(x), \quad \alpha, \beta=1, \ldots, N \tag{1.5}$$

thus guaranteeing the covariance of Eq. (1.1) under this transformation.

In order to see how η accomplishes all of these things consider the transformation properties of $\phi(x)$ under the homogeneous Lorentz group $(x'=\Lambda x)$:

$$\phi_\alpha'(x')=S_{\alpha\beta}(\Lambda)\phi_\beta(x), \quad \alpha, \beta=1, \ldots, N \tag{1.6}$$

where $S(\Lambda)$ is a direct sum of finite-dimensional (nonunitary) representations of SL(2, C),

$$S(\Lambda)=\bigoplus_{i=1}^{M}(n_i, m_i)\,, \tag{1.7}$$

where

$$S^\dagger(\Lambda)S(\Lambda)\neq I\,. \tag{1.8}$$

η's claim to fame is that

$$S^\dagger(\Lambda)\eta S(\Lambda)=\eta \text{ for all } \Lambda\,, \tag{1.9}$$

thus providing for the invariance of (1.3) and (1.4).

But η is a two-edged sword since it exists only for a restricted class of representations S. Indeed, η exists if and only if S is self-conjugate,[4,5] i.e., if and only if the direct sum (1.7) contains only self-conjugate $(n_i=m_i)$ or pairs of mutually conjugate $[(n_i, m_i)\oplus(m_i, n_i)]$ components. Some well-known examples are the Dirac equation $[S=(\frac{1}{2}, 0)\oplus(0, \frac{1}{2})]$ and the Petiau-Duffin-Kemmer (P-D-K) equations $[S=(0, 0)\oplus(\frac{1}{2}, \frac{1}{2})$ or $S=(1, 0)\oplus(\frac{1}{2}, \frac{1}{2})\oplus(0, 1)]$.

In order to see why conjugate representations are necessary we need only recall that while any finite-dimensional irreducible representation of

SL(2, C), $D^{(n,m)}(\Lambda)\equiv(n,m)$, is nonunitary,

$$D^{(n,m)}(\Lambda)^{\dagger}D^{(n,m)}(\Lambda)\neq I, \tag{1.10}$$

such representations have the additional property that

$$D^{(m,n)}(\Lambda)^{\dagger}D^{(n,m)}(\Lambda)=I. \tag{1.11}$$

η exploits this by coupling mutually conjugate representations and thus may achieve (1.9). Furthermore, since $D^{(n,m)}(\Lambda)$ is related to $D^{(m,n)}(\Lambda)$ by the parity operation we may also attain the transformation property (1.5).

In the examples cited above we have for the Dirac case

$$\eta_{\alpha\beta}=\begin{array}{c}(\tfrac{1}{2},0)\;(0,\tfrac{1}{2})\\ \begin{pmatrix}0&1\\1&0\end{pmatrix}_{\alpha\beta}\end{array}\begin{array}{l}(\tfrac{1}{2},0)\\(0,\tfrac{1}{2})\end{array},\quad \alpha,\beta=1,\ldots,4 \tag{1.12}$$

and for the P-D-K cases $\eta_{\alpha\beta}=\delta_{\alpha\beta}$, $\alpha,\beta=1,\ldots,5$ or

$$\eta_{\alpha\beta}=\begin{array}{c}(1,0)\;(\tfrac{1}{2},\tfrac{1}{2})\;(0,1)\\ \begin{pmatrix}&&1\\&1&\\1&&\end{pmatrix}_{\alpha\beta}\end{array}\begin{array}{l}(1,0)\\(\tfrac{1}{2},\tfrac{1}{2}),\\(0,1)\end{array}\quad \alpha,\beta=1,\ldots,10 \tag{1.13}$$

in the basis where $S(\Lambda)$ is completely reduced.

Although this requirement of conjugacy is readily implemented for the lower-spin cases, as evidenced by the above examples, difficulties develop if one attempts to maintain it in the description of higher-spin systems. Let us examine in a qualitative way why this happens.

Recall the following two facts: (1) In order to describe a spin-s particle the $(2s+1)$-dimensional, unitary, irreducible representation of SU(2), $D^{(s)}(R)$, must occur in $S(\Lambda)$ when Λ is restricted to the rotation subgroup.[6] This will happen for any representation (n,m) where s is contained in the set $\{n+m,\ldots,|n-m|\}$. (2) The irreducible SL(2, C) components of $S(\Lambda)$ may be coupled by the β matrices in Eq. (1.1) only if they interlock, i.e., if their labels differ by $\frac{1}{2}$ (see Ref. 7):

$$(n,m)\longleftarrow(n\pm\tfrac{1}{2},m\pm\tfrac{1}{2}). \tag{1.14}$$

Consider the simplest SL(2, C) representation which describes spin s, $(s,0)$. (This is the simplest because it describes no other spins.) The self-conjugacy requirement demands that $(0,s)$ must also be contained in the direct sum (1.7). At this point we may either (a) abandon equations of the form (1.1) and go to higher-order equations,[8,9] or (b) add further interlocking SL(2, C) components so that there may exist nontrivial β matrices. In case (a) one runs the risk of introducing unphysical masses into the theory which may not be dynamically independent of the physical masses,[10] and in case (b) there are two possibilities: (b1) The representations $(s,0)$ and $(0,s)$ are connected by intermediary interlocking representations which "bridge the gap," or (b2) there are representations which interlock with $(s,0)$ and $(0,s)$ separately, thus permitting the existence of β matrices, but these two sides do not interlock with each other ("open gap").

In case (b1) the β matrices may form an irreducible set, but the number of intermediary representations will grow with s as the gap widens. This will result in an increasing number of auxiliary spins among the independent components which must be projected out if we are to return to a pure spin-s theory. Moreover, upon the introduction of interactions this complex system of constraints leads in general to inconsistencies such as noncausal propagation,[11,12] indefinite metric,[13] losing constraint equations, etc.

In case (b2) the gap is not bridged and one is left with reducible β matrices with at least four $(2s+1)$ independent components; i.e., parity doubling occurs. A simple example of this is the representation[14-16]

$$S(\Lambda)=(s,0)\oplus(s-\tfrac{1}{2},\tfrac{1}{2})\oplus(\tfrac{1}{2},s-\tfrac{1}{2})\oplus(0,s),\quad s\geq\tfrac{3}{2} \tag{1.15}$$

whose wave equation suffers not only from parity doubling but also from negative-metric and negative-energy difficulties.[17] We may furthermore expect this to happen for all case-(b2) equations.

If we start with an SL(2, C) representation (n,m) where neither n nor m is zero, then we will again encounter auxiliary spins in the formalism and the resultant difficulties of case (b1).

In short, there seem to be pitfalls in every direction for self-conjugate representations when $s>1$.

On the other hand, if we give up the self-conjugacy requirement then there exists a family of wave equations which are simple and uniform no matter how high the spin. The β algebra is given by

$$(\beta\cdot p)^3=(\beta\cdot p)p^2 \tag{1.16}$$

for any spin, and the resultant wave equation has no secondary constraints, is consistent, and leads to causal propagation in a minimally coupled external electromagnetic field.[18] Thus many of the drawbacks of self-conjugacy can be avoided. Such equations, however, do not permit an η matrix when $s\geq1$.

An important question, therefore, is whether the benefits of an η matrix can be salvaged in its absence even for free fields. In the present report we shall study this question in detail and answer it in the affirmative.

Using a formulation analogous to that presented by Weinberg[8] and by Weaver, Hammer, and Good,[9] we shall argue that we do not need an η in order to enjoy its benefits (1), (2), and (3). We shall restrict our attention to wave equations whose β algebra is given by (1.16) and which describe noninteracting mass $m>0$, spin-s particles, since that is where these questions arise.

In Sec. II we shall define the system of wave equations which we wish to consider and construct their plane-wave solutions. A related system is defined in Sec. III and a mapping between the solutions of these equations is introduced and some of its properties are demonstrated in Sec. IV. After a study of the orthonormality properties of the plane-wave solutions in Sec. V, a bilinear form is defined in Sec. VI and its properties are studied in Sec. VII, where it is shown to provide a suitable scalar product for the solutions to the wave equation.

The Lagrangian and other bilinear densities are introduced and examined in Sec. VIII and the second-quantized formulation is discussed in Sec. IX. The theory is formally local, is realizable on a positive-definite metric Fock space, and has a nonnegative energy spectrum. The discrete symmetries are studies in Sec. X. It is shown there that, although it is not necessary, it is natural in the present formalism to realize parity in terms of an antiunitary operator. The action of the discrete symmetries on the commutation relations and field quantities is also described. Finally, in the Appendix, equivalent realizations of parity symmetry are briefly discussed in a general framework.

II. WAVE EQUATION AND PLANE WAVES

We consider wave equations of the general first-order form

$$(i\beta\cdot\partial-m)\phi(x)=0\,, \tag{2.1}$$

where m is taken to be a positive-definite multiple of the identity. $\phi(x)$ transforms under the action of the proper Poincaré group as

$$[U(a,\Lambda)\phi]_\alpha(x)=S_{\alpha\beta}(\Lambda)\phi_\beta(\Lambda^{-1}(x-a))\,, \tag{2.2}$$

where $\alpha,\beta=1,\ldots,N$; Λ is a homogeneous Lorentz transformation, a is a space-time translation, and $S(\Lambda)$ is a finite-dimensional representation of $SL(2,C)$. β_μ represents four $N\times N$ matrices with the property

$$S(\Lambda^{-1})\beta_\mu S(\Lambda)=(\Lambda\beta)_\mu\,, \tag{2.3}$$

thus guaranteeing the covariance of Eq. (2.1).

We shall assume that the solutions to Eq. (2.1) describe particles with a unique mass ($m>0$) (Ref. 19) and a unique spin s which may be determined from the nonzero components of the rest-frame solutions of Eq. (2.1) in momentum space. We assume that only $2(2s+1)$ of these components are nonzero corresponding to the positive- and negative-energy solutions of a spin-s particle. We shall further assume that the β_μ are irreducible and that β_0 may be chosen to be Hermitian and thus [as may be shown from the infinitesimal form of Eq. (2.3)] that β_i, $i=1,2,3$ is anti-Hermitian [see Eq. (3.9) below]:

$$\beta_0^\dagger=\beta_0,\quad \beta_i^\dagger=-\beta_i\,,\quad i=1,2,3\,. \tag{2.4}$$

These assumptions are very restrictive since they imply that the β matrices satisfy the algebra[12,20,21]

$$\sum_P(\beta^\mu\beta^\nu\beta^\lambda-g^{\mu\nu}\beta^\lambda)=0\,, \tag{2.5}$$

where P represents a sum over the six permutations of the vector indices. However, they do allow for the possibility of studying equations which do not permit the existence of an η matrix, and they exclude many peripheral questions which would only serve to complicate the issues at hand. These assumptions do, in fact, lead us to the simplest equations for which there is no η matrix and are thus ideal for our purposes.

Using a familiar technique[8,9] we expand the solutions to Eq. (2.1) as follows:

$$\phi_\alpha(x)=\sum_{\sigma=-s}^{s}\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E}\right)^{1/2}\times[u_\alpha(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x}+v_\alpha(\vec{p},\sigma)b^*(\vec{p},\sigma)e^{ip\cdot x}]\,, \tag{2.6}$$

where

$$u_\alpha(\vec{p},\sigma)=S_{\alpha\beta}(L(\vec{p}))u_\beta(0,\sigma) \tag{2.7}$$

and

$$v_\alpha(\vec{p},\sigma)=S_{\alpha\beta}(L(\vec{p}))v_\beta(0,\sigma')C^{(s)-1}{}_{\sigma'\sigma}\;; \tag{2.8}$$

$$\alpha,\beta=1,\ldots,N\,,$$

$$\sigma,\sigma'=-s,-s+1,\ldots,s\,,$$

$$E=E(\vec{p})=+(\vec{p}^2+m^2)^{1/2}\,,$$

$L(p)$ is the pure Lorentz transformation which boosts a particle of mass m from rest to momentum $\vec{p}$, and $u(0,\sigma)$ [$v(0,\sigma)$] are the positive-[negative-] energy rest-frame solutions of

$$(\beta\cdot p-m)\phi(\vec{p},\sigma)=0\,. \tag{2.9}$$

$u(0,\sigma)$ [$v(0,\sigma)$] are eigenstates of β_0 with eigenvalue

$+1$ $[-1]$ and they are also eigenstates of J_3, the generator of rotations about the 3 axis for the representation $S(\Lambda)$, with eigenvalue σ.

$C^{(s)}_{\sigma\sigma'}$ is defined as follows: For each unitary irreducible representation of SU(2), $D^{(s)}(R)$, there exists a unitary matrix $C^{(s)}$ which expresses the equivalence of $D^{(s)}(R)$ to its complex conjugate:

$$D^{(s)}(R)^* = C^{(s)} D^{(s)}(R) C^{(s)-1} . \tag{2.10}$$

$C^{(s)}$ may be chosen to have the properties[22]

$$C^{(s)*}C^{(s)} = (-)^{2s} , \tag{2.11}$$

$$C^{(s)\dagger}C^{(s)} = 1 , \tag{2.12}$$

$$C^{(s)}S_i^{(s)}C^{(s)-1} = -S_i^{(s)*} \tag{2.13}$$

and, for later use,

$$C^{(s-1)}K_i^{(s)}C^{(s)-1} = -K_i^{(s)*} , \tag{2.14}$$

where $S_i^{(s)}$ are the $(2s+1)$-dimensional generators of $D^{(s)}(R)$ (the "spin-s" matrices) and $K_i^{(s)}$ are three rectangular matrices with the properties[16]

$$D^{(s-1)}(R^{-1})\vec{K}^{(s)}D^{(s)}(R) = R\vec{K}^{(s)} \tag{2.15}$$

and

$$S_i^{(s)}S_j^{(s)} + K_i^{(s)\dagger}K_j^{(s)} = is\epsilon_{ijk}S_k^{(s)} + s^2\delta_{ij} . \tag{2.16}$$

Under the usual phase conventions $C^{(s)}$ may be written as

$$C^{(s)}_{\sigma\sigma'} = (-)^{s+\sigma}\delta_{\sigma',-\sigma} , \quad \sigma, \sigma' = -s, \ldots, s . \tag{2.17}$$

We may observe in this expansion some effects of our simplifying assumptions. The unique-mass assumption not only simplifies the expansion (2.6), but when coupled with assumption (2.4) leads to (2.5) and the resultant condition on β_0

$$\beta_0^3 = \beta_0 . \tag{2.18}$$

β_0 thus has eigenvalues $+1$, -1, and 0. The unique-spin assumption further demands that the $+1$ and -1 eigenvalues each correspond to a $(2s+1)$-dimensional subspace which carries the representation $D^{(s)}(R)$. Thus we need only sum over $(2s+1)$ values of σ in Eq. (2.6). The subspace corresponding to the $+1$ $[-1]$ eigenvalue of β_0 is assumed to be spanned by the $u(0,\sigma)$ $[v(0,\sigma)]$. If we did not make this assumption (and, correspondingly, the irreducibility assumption of β_μ) then we would in general be forced to allow for additional solutions corresponding to other quantum numbers, e.g., parity. The zero eigenvalues of β_0 correspond to dependent components which vanish in the rest frame. There are $N-2(2s+1)$ of these.

Although the class of equations under consideration is a restricted one, there do exist families of wave equations which satisfy all of the above restrictions. These equations may be listed according to the transformation properties of their solutions as follows[21]:

$$\begin{aligned} &(1)\ (n,0)\oplus(n-\tfrac{1}{2},\tfrac{1}{2}) ,\\ &(2)\ (n,0)\oplus(n+\tfrac{1}{2},\tfrac{1}{2}) ,\\ &(3)\ (n+\tfrac{1}{2},\tfrac{1}{2})\oplus(n,0)\oplus(n-\tfrac{1}{2},\tfrac{1}{2}) ,\\ &(4)\ (1,0)\oplus(\tfrac{1}{2},\tfrac{1}{2})\oplus(0,1) , \end{aligned} \tag{2.19}$$

and their conjugates.

III. ϕ

With every system defined by Eqs. (2.1), (2.2), and (2.3) there is a closely related system:

$$(i\tilde{\beta}\cdot\partial - m)\tilde{\phi}(x) = 0 , \tag{3.1}$$

where

$$[U(a,\Lambda)\tilde{\phi}](x) = \tilde{S}(\Lambda)\tilde{\phi}(\Lambda^{-1}(x-a)) \tag{3.2}$$

and

$$\tilde{S}^{-1}(\Lambda)\tilde{\beta}_\mu\tilde{S}(\Lambda) = (\Lambda\tilde{\beta})_\mu . \tag{3.3}$$

$\tilde{S}(\Lambda)$ is the representation of SL(2, C) which is conjugate to $S(\Lambda)$; i.e., if

$$S(\Lambda) = \bigoplus_{i=1}^{M} (n_i, m_i) \tag{3.4}$$

then

$$\tilde{S}(\Lambda) = \bigoplus_{i=1}^{M} (m_i, n_i) . \tag{3.5}$$

$S(\Lambda)$ and $\tilde{S}(\Lambda)$ act in the same N-dimensional space and are equivalent when Λ is restricted to the rotation subgroup. For pure Lorentz transformations both $\tilde{S}(\Lambda)$ and $S(\Lambda)$ are Hermitian and

$$\tilde{S}(L(\vec{p}))S(L(\vec{p})) = I . \tag{3.6}$$

This last relation indicates that the infinitesimal generators of pure Lorentz transformations (boosts) in the $S(\Lambda)$ and $\tilde{S}(\Lambda)$ representations differ only in sign.

In order to relate $\tilde{\beta}_\mu$ to β_μ consider the following relations which are equivalent to Eq. (2.3):

$$[J_i, \beta_0] = 0 , \tag{3.7}$$

$$[[N_3, \beta_0], N_3] = \beta_0 , \tag{3.8}$$

and

$$\beta_i = i[\beta_0, N_i] , \quad i = 1, 2, 3 \tag{3.9}$$

where J_i is the generator of spatial rotations about the ith axis and N_i generates boosts along the ith axis. Identical equations may also be written for Eq. (3.3), and since $\tilde{J}_i = J_i$ and $\tilde{N}_i = -N_i$, it is clear that we may write

$$\tilde{\beta}_0 = \beta_0 \text{ and } \tilde{\beta}_i = -\beta_i = \beta_i^\dagger , \tag{3.10}$$

where we have invoked Eq. (2.4) to get the last equality. We shall hereafter consider (3.10) as our definition of $\tilde{\beta}_\mu$.

We may expand the solutions to Eq. (3.1) in the same way as we did for the solutions of Eq. (2.1):

$$\tilde{\phi}_\alpha(x)=\sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E}\right)^{1/2} \times[\tilde{u}_\alpha(\vec{p},\sigma)\tilde{a}(\vec{p},\sigma)e^{-ip\cdot x} +\tilde{v}_\alpha(\vec{p},\sigma)\tilde{b}^*(\vec{p},\sigma)e^{ip\cdot x}], \quad (3.11)$$

where now

$$\tilde{u}_\alpha(\vec{p},\sigma)=\tilde{S}_{\alpha\beta}(L(\vec{p}))\tilde{u}_\beta(0,\sigma) \quad (3.12)$$

and

$$\tilde{v}_\alpha(\vec{p},\sigma)=\tilde{S}_{\alpha\beta}(L(\vec{p}))\,\tilde{v}_\beta(0,\sigma')C^{-1}{}_{\sigma'\sigma}; \quad (3.13)$$

$\alpha,\beta=1,\ldots,N$ and $\sigma,\sigma'=-s,\ldots,s$ as before.

Since $\tilde{u}(0,\sigma)$ and $\tilde{v}(0,\sigma)$ are the positive- and negative-energy rest-frame solutions of

$$(\tilde{\beta}\cdot p-m)\,\tilde{\phi}(\vec{p},\sigma)=0, \quad (3.14)$$

in light of Eq. (3.10) we have $\tilde{u}(0,\sigma)=u(0,\sigma)$ and $\tilde{v}(0,\sigma)=v(0,\sigma)$ and so we may hereafter omit the tilde over these solutions.

From (3.6) we see that we have

$$\tilde{S}(L(\vec{p}))=S^{-1}(L(\vec{p}))=S(L(-\vec{p})), \quad (3.15)$$

so we may rewrite the plane-wave solutions as

$$\tilde{u}_\alpha(\vec{p},\sigma)=S_{\alpha\beta}(L(-\vec{p}))u_\beta(0,\sigma) \quad (3.16)$$

and

$$\tilde{v}_\alpha(\vec{p},\sigma)=S_{\alpha\beta}(L(-\vec{p}))\,v_\beta(0,\sigma')C^{-1}{}_{\sigma'\sigma}; \quad (3.17)$$

hence, using Eqs. (3.11), (2.7), and (2.8) we get

$$\tilde{\phi}_\alpha(x)=\sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E}\right)^{1/2} \times[u_\alpha(-\vec{p},\sigma)\tilde{a}(\vec{p},\sigma)e^{-ip\cdot x} +v_\alpha(-\vec{p},\sigma)\tilde{b}^*(p,\sigma)e^{ip\cdot x}]. \quad (3.18)$$

IV. RELATION BETWEEN ϕ AND $\tilde{\phi}$

Consider the solutions of Eq. (2.1), $\phi(x)$, and the matrix corresponding to the Lorentz transformation

$$A(p)=A(\vec{p},p_0)=\{S^{-1}(L(\vec{p}))\}^2. \quad (4.1)$$

Form the matrix $A(i\partial)$ and operate on Eq. (2.6):

$$A_{\alpha\beta}(i\partial)\phi_\beta(x)=\sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E}\right)^{1/2}[A_{\alpha\beta}(p)u_\beta(\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x}+A_{\alpha\beta}(-p)v_\beta(\vec{p},\sigma)b^*(\vec{p},\sigma)e^{ip\cdot x}]. \quad (4.2)$$

Since [see Eq. (4.19)] $A(-p)=(-)^{2s}A(p)$, we may use Eqs. (4.1), (2.7), and (2.8) to get

$$A(i\partial)\phi(x)=\sum_\sigma \int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E}\right)^{1/2}[u(-\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip\cdot x}+(-)^{2s}v(-\vec{p},\sigma)b^*(\vec{p},\sigma)e^{ip\cdot x}]. \quad (4.3)$$

When we compare this equation with Eq. (3.18) we see that the operator $A(i\partial)$ maps every $\phi(x)$ to a $\tilde{\phi}(x)$, with the latter's coefficients given as

$$\tilde{a}(\vec{p},\sigma)=a(\vec{p},\sigma) \quad (4.4a)$$

and

$$\tilde{b}^*(\vec{p},\sigma)=(-)^{2s}b^*(\vec{p},\sigma). \quad (4.4b)$$

This mapping of the space of solutions of Eq. (2.1) to the space of solutions of Eq. (3.1) has an inverse ($A^{-1}(p)=\{S(L(\vec{p}))\}^2$), and associates a unique solution of Eq. (3.1) to each solution of Eq. (2.1) and vice versa; all solutions of Eq. (3.1) and Eq. (2.1) are so related. The mapping is thus one to one and onto, i.e., a bijection.

Let us redefine the tilde notation to mean that $\tilde{\phi}(x)$ is the particular solution of Eq. (3.1) associated with $\phi(x)$ [a solution of Eq. (2.1)] such that

$$\tilde{\phi}(x)=A(i\partial)\phi(x). \quad (4.5)$$

That is, $\tilde{\phi}(x)$ is given by the expansion (4.3) once $\phi(x)$ is given by the expansion (2.6).

Let us consider some properties of $A(i\partial)$:

(a) $S^\dagger(\Lambda^{-1})A(p)S(\Lambda^{-1})=A(\Lambda p)$. (4.6)

We start with the identity for a pure (Wigner) rotation,

$$(L^{-1}(\Lambda\vec{p})\Lambda L(\vec{p}))^\dagger L^{-1}(\Lambda\vec{p})\Lambda L(\vec{p})=I \quad (4.7)$$

if and only if

$$L(\vec{p})\Lambda^\dagger L^{-1}(\Lambda\vec{p})L^{-1}(\Lambda\vec{p})\Lambda L(\vec{p})=I, \quad (4.8)$$

where we have used the fact that $L^\dagger(\vec{p})=L(\vec{p})$. Equation (4.8) implies that

$$L^{-1}(\Lambda\vec{p})L^{-1}(\Lambda\vec{p})=\Lambda^{\dagger-1}L^{-1}(\vec{p})L^{-1}(\vec{p})\Lambda^{-1}, \quad (4.9)$$

and for the representation $S(\Lambda)$ we have

$$\{S(L^{-1}(\Lambda\vec{p}))\}^2=S^\dagger(\Lambda^{-1})\{S(L^{-1}(\vec{p}))\}^2S(\Lambda^{-1}), \quad (4.10)$$

which completes the proof. Note that the relation $\tilde{S}^\dagger(\Lambda)=S^{-1}(\Lambda)$ implies that we also have

$$\tilde{S}(\Lambda)A(p)S(\Lambda^{-1})=A(\Lambda p). \quad (4.11)$$

(b) $A(p)\beta\cdot pA^{-1}(p)=\tilde{\beta}\cdot p$. (4.12)

$$A(p)\beta\cdot pA^{-1}(p)=S^{-1}(L(\vec{p}))S^{-1}(L(\vec{p}))\beta\cdot p \times S(L(\vec{p}))S(L(\vec{p})) \quad (4.13)$$

and by (2.3)

$$A(p)\beta\cdot pA^{-1}(p)=S^{-1}(L(\vec{p}))(L(\vec{p})\beta)\cdot pS(L(\vec{p})) \quad (4.14)$$

$$=(L^2(\vec{p})\beta)\cdot p \quad (4.15)$$

$$=\beta\cdot(L^{-2}(\vec{p})p) \quad (4.16)$$

$$=\tilde{\beta}\cdot p\,, \quad (4.17)$$

since $(L^{-1}(\vec{p}))^2$ is precisely that transformation which brings a momentum p first to rest and then to a momentum with the same magnitude in 3-space but opposite direction. In configuration space Eq. (4.12) reads

$$A(i\partial)(i\beta\cdot\partial)=(i\tilde{\beta}\cdot\partial)A(i\partial)\,. \quad (4.18)$$

(c) $A(-p)=(-)^{2s}A(p)\,.$ (4.19)

Equation (4.11) implies that $A(p)$ is composed of constituents $a^{(i)}(p)$, each of which maps a function which transforms according to (n_i, m_i) to one which transforms according to (m_i, n_i):

$$A(p)\sim\bigoplus_{i=1}^{M} a^{(i)}(p)\,, \quad (4.20)$$

$$a^{(i)}(p):(n_i,m_i)\rightarrow(m_i,n_i)\,. \quad (4.21)$$

Since $a^{(i)}(p)$ is a function of p which transforms according to the $(\frac{1}{2},\frac{1}{2})$ representation, Eq. (4.21) restricts the order of p which may enter $a^{(i)}(p)$:

$$a^{(i)}(p)\sim\bigotimes_{j=1}^{N_i}(\tfrac{1}{2},\tfrac{1}{2})_j\,, \quad (4.22)$$

where $N_i=2|n_i-m_i|+2q_i$ and q_i may in general be a positive integer or (as is actually the case) zero. For half-integer spins (integer spins) $2|n_i-m_i|$ is odd (even) and this implies the desired result, Eq. (4.19).

V. PROPERTIES OF PLANE-WAVE SOLUTIONS

In this section we shall derive some orthonormality and completeness relations for the momentum-space solutions occurring in the expansion (2.6).

(a) $u^\dagger(-\vec{p},\sigma)\beta_\mu u(\vec{p},\sigma')=u^\dagger(-\vec{p},\sigma)\dfrac{p_\mu}{m}u(\vec{p},\sigma'),$ (5.1)

where the u's satisfy

$$(\beta\cdot p-m)u(\vec{p},\sigma)=0\,. \quad (5.2)$$

We prove the above relation using the algebraic property (1.16) which is equivalent to

$$\sum_P(\beta_\mu\beta_\nu\beta_\lambda-g_{\mu\nu}\beta_\lambda)=0\,, \quad (5.3)$$

where P represents a sum over the six permutations of the vector indices. We have

$$u^\dagger(-\vec{p},\sigma)(\beta_\mu\beta_\nu\beta_\lambda+\beta_\nu\beta_\mu\beta_\lambda+\beta_\nu\beta_\lambda\beta_\mu)p^\nu p^\lambda u(\vec{p},\sigma')$$

$$=u^\dagger(-\vec{p},\sigma)\frac{1}{2}\left(\sum_P\beta_\mu\beta_\nu\beta_\lambda\right)p^\nu p^\lambda u(\vec{p},\sigma') \quad (5.4)$$

$$=u^\dagger(-\vec{p},\sigma)\frac{1}{2}\left(\sum_P g_{\mu\nu}\beta_\lambda\right)p^\nu p^\lambda u(\vec{p},\sigma') \quad (5.5)$$

$$=u^\dagger(-\vec{p},\sigma)(2p_\mu\beta\cdot p+\beta_\mu p^2)u(\vec{p},\sigma') \quad (5.6)$$

$$=u^\dagger(-\vec{p},\sigma)(2mp_\mu+m^2\beta_\mu)u(\vec{p},\sigma')\,, \quad (5.7)$$

since $u(\vec{p},\sigma)$ is a solution of Eq. (5.2). But since Eqs. (5.2) and (2.4) imply that

$$u^\dagger(-\vec{p},\sigma)(\beta\cdot p-m)=0\,, \quad (5.8)$$

we find that the left-hand side of Eq. (5.4) is also equal to

$$u^\dagger(-\vec{p},\sigma)(3m^2\beta_\mu)u(\vec{p},\sigma)\,. \quad (5.9)$$

Equating (5.7) and (5.9) yields the desired result.

(b) $v^\dagger(-\vec{p},\sigma)\beta_\mu v(\vec{p},\sigma')=-v^\dagger(-\vec{p},\sigma)\dfrac{p_\mu}{m}v(\vec{p},\sigma')\,.$ (5.10)

This case is the same as the last except that $v(\vec{p},\sigma)$ satisfies the equation

$$(\beta\cdot p+m)v(\vec{p},\sigma)=0 \quad (5.11)$$

and hence also

$$v^\dagger(-\vec{p},\sigma)(\beta\cdot p+m)=0 \quad (5.12)$$

in contrast with Eqs. (5.2) and (5.8).

(c) $u^\dagger(-\vec{p},\sigma)u(\vec{p},\sigma')=\delta_{\sigma\sigma'}\,.$ (5.13)

This follows from (2.7) and the normalization $u^\dagger(0,\sigma)u(0,\sigma)=\delta_{\sigma\sigma'}$. Likewise we get

(d) $v^\dagger(-\vec{p},\sigma)v(\vec{p},\sigma')=\delta_{\sigma\sigma'}$ (5.14)

and

(e) $u^\dagger(-\vec{p},\sigma)v(\vec{p},\sigma')=0=v^\dagger(-\vec{p},\sigma)u(\vec{p},\sigma')\,.$ (5.15)

Using the above results (a) and (c) we have

(f) $u^\dagger(-\vec{p},\sigma)\beta_\mu u(\vec{p},\sigma')=\dfrac{p_\mu}{m}\delta_{\sigma\sigma'}$ (5.16)

and from (b) and (d) we get

(g) $v^\dagger(-\vec{p},\sigma)\beta_\mu v(\vec{p},\sigma')=-\dfrac{p_\mu}{m}\delta_{\sigma\sigma'}\,.$ (5.17)

Taking the 0th component of (f) and (g) we have

$$u^\dagger(-\vec{p},\sigma)\beta_0 u(\vec{p},\sigma')=\frac{E}{m}\delta_{\sigma\sigma'} \quad (5.18)$$

and

$$v^\dagger(-\vec{p},\sigma)\beta_0 v(\vec{p},\sigma')=-\frac{E}{m}\delta_{\sigma\sigma'}\,. \quad (5.19)$$

(h) $u^\dagger(\vec{p},\sigma)\beta_0 v(\vec{p},\sigma')=0$

$$=v^\dagger(\vec{p},\sigma)\beta_0 u(\vec{p},\sigma')\,. \tag{5.20}$$

Using (5.8) and (5.11) we have

$$u^\dagger(\vec{p},\sigma)(\beta_0 p_0+\vec{\beta}\cdot\vec{p}-m)=0 \tag{5.21}$$

and

$$(\beta_0 p_0-\vec{\beta}\cdot\vec{p}+m)v(\vec{p},\sigma')=0\,. \tag{5.22}$$

We thus find from (5.21)

$$u^\dagger(\vec{p},\sigma)\beta_0 v(\vec{p},\sigma')=\frac{1}{p_0}\,u^\dagger(\vec{p},\sigma)(-\vec{\beta}\cdot\vec{p}+m)\,v(\vec{p},\sigma') \tag{5.23}$$

and from (5.22)

$$u^\dagger(\vec{p},\sigma)\beta_0 v(\vec{p},\sigma')=-\frac{1}{p_0}\,u^\dagger(\vec{p},\sigma)(-\vec{\beta}\cdot\vec{p}+m)v(\vec{p},\sigma')\,, \tag{5.24}$$

which imply the desired result. Similar arguments apply to the right-hand side of (5.20).

(i) $$\sum_\sigma u_\alpha(\vec{p},\sigma)u_\beta^\dagger(-\vec{p},\sigma)=\frac{1}{2}\left[\frac{(\beta\cdot p)^2}{m^2}+\frac{\beta\cdot p}{m}\right]_{\alpha\beta}$$

$$\equiv\Lambda_+(p)\,. \tag{5.25}$$

The projection onto the positive-energy solutions in the rest frame may be written either as

$$\Lambda_+(0)=\sum_\sigma u(0,\sigma)u^\dagger(0,\sigma) \tag{5.26}$$

or

$$\Lambda_+(0)=\tfrac{1}{2}(\beta_0{}^2+\beta_0)\,. \tag{5.27}$$

The first form is clearly a projection onto the positive-energy rest-frame solutions. That the second form is also such a projection follows from

$$\beta_0 u(0,\sigma)=u(0,\sigma) \tag{5.28}$$

and

$$\beta_0 v(0,\sigma)=-v(0,\sigma)\,. \tag{5.29}$$

We now boost these rest-frame projectors to momentum p to get

$$\Lambda_+(p)=S(L(\vec{p}))\Lambda_+(0)S(L^{-1}(\vec{p})) \tag{5.30}$$

$$=\sum_\sigma u(\vec{p},\sigma)u^\dagger(-\vec{p},\sigma) \tag{5.31}$$

and

$$\Lambda_+(p)=S(L(\vec{p}))\tfrac{1}{2}(\beta_0{}^2+\beta_0)S(L^{-1}(\vec{p})) \tag{5.32}$$

$$=\frac{1}{2}\left[\frac{(\beta\cdot p)^2}{m^2}+\frac{\beta\cdot p}{m}\right]\,, \tag{5.33}$$

where the last equation follows from Eq. (2.3):

$$S^{-1}(L(\vec{p}))\beta\cdot pS(L(\vec{p}))=(L(\vec{p})\beta)\cdot p \tag{5.34}$$

$$=\beta\cdot(L^{-1}(\vec{p})p) \tag{5.35}$$

$$=\beta_0 m\,, \tag{5.36}$$

i.e.,

$$S(L(\vec{p}))\beta_0 S(L^{-1}(\vec{p}))=\frac{\beta\cdot p}{m}\,. \tag{5.37}$$

The desired result now follows from Eqs. (5.31) and (5.33).

(j) $$\sum_\sigma v_\alpha(\vec{p},\sigma)v_\beta^\dagger(-\vec{p},\sigma)=\frac{1}{2}\left[\frac{(\beta\cdot p)^2}{m^2}-\frac{\beta\cdot p}{m}\right]_{\alpha\beta}$$

$$\equiv\Lambda_-(p)\,. \tag{5.38}$$

We proceed similarly to the last case starting with the rest-frame projectors onto negative-energy states

$$\Lambda_-(0)=\sum_\sigma v(0,\sigma)v^\dagger(0,\sigma) \tag{5.39}$$

and

$$\Lambda_-(0)=\tfrac{1}{2}(\beta_0{}^2-\beta_0) \tag{5.40}$$

and boosting as before to get (5.38).

Note the generalization of the usual projectors for the Dirac equation for which $(\beta\cdot p)^2=p^2=m^2$.

VI. SCALAR PRODUCT

We are now in a position to discuss the central point of this paper. We assume that the states of a free, mass >0, spin-s, relativistic particle are described by the solutions $\phi(x)$ of the wave equation

$$(i\beta\cdot\partial-m)\phi(x)=0\,, \tag{2.1}$$

where $\phi(x)$ transforms according to (2.2). In order to obtain a quantum-mechanical interpretation for this description we must equip the solution space of Eq. (2.1) with a scalar product. To this end we therefore introduce the bilinear form for any two solutions ϕ and ψ of Eq. (2.1):

$$(\phi,\psi)\equiv\int d\sigma_\mu(x)\,\tilde{\phi}^\dagger(x)\beta^\mu\psi(x)\,, \tag{6.1}$$

where $\sigma(x)$ is a spacelike surface and $\tilde{\phi}(x)$ is defined as

$$\tilde{\phi}(x)=A(i\partial)\phi(x)\,. \tag{6.2}$$

Thus the dual to each solution of (2.1) is no longer the complex conjugate, transposed solution, but rather the solution must first be transformed by A and then the complex conjugate transpose is taken. So defined, $\tilde{\phi}(x)$ will satisfy Eqs. (3.1), (3.2), and (3.3) as can be seen from Eqs. (4.11) and (4.12). Equivalently, we could write the form

(6.1) in the usual fashion by explicitly displaying the $A(i\partial)$ as a metric operator. We shall do this whenever it is convenient.

In Sec. VII we shall consider some properties of the bilinear form (6.1) in both configuration space and (for the sake of exposition) in momentum space. We shall find that the form (6.1) does indeed provide a satisfactory scalar product on the solution space of Eq. (2.1).

Let us first establish the corresponding bilinear form in momentum space. To this end we retrace our steps a bit and reconsider the Fourier expansion.

Since Eq. (2.1) implies that $\phi(x)$ satisfies the Klein-Gordon equation componentwise, we may Fourier-expand the solutions

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int d^4p\, e^{-ip\cdot x}\delta(p^2-m^2)\phi(p)\,, \tag{6.3}$$

where $\phi(p)$ satisfies

$$(\beta\cdot p-m)\phi(p)=0 \tag{6.4}$$

on the mass shell.

Decomposing into positive- and negative frequency parts we have

$$\phi(x)=\phi^{(+)}(x)+\phi^{(-)}(x)\,, \tag{6.5}$$

with

$$\phi^{(\pm)}(x)=\frac{1}{(2\pi)^{3/2}}\int d^4p\, e^{-ip\cdot x}\delta(p^2-m^2)\theta(\pm p_0)\phi(p)\,. \tag{6.6}$$

Defining

$$\phi^{(+)}(\vec{p})=\frac{1}{\sqrt{2m}}\,\theta(p_0)\phi(p)$$
$$=\sqrt{2E}\,u(\vec{p},\sigma)a(\vec{p},\sigma) \tag{6.7}$$

and

$$\phi^{(-)}(\vec{p})\equiv\frac{1}{\sqrt{2m}}\,\theta(p_0)\phi(-p)$$
$$=\sqrt{2E}\;v(\vec{p},\sigma)b^*(\vec{p},\sigma) \tag{6.8}$$

we get

$$\phi(x)=\frac{\sqrt{2m}}{(2\pi)^{3/2}}\int\frac{d^3p}{2E}\left[\phi^{(+)}(\vec{p})\,e^{-ip\cdot x}+\phi^{(-)}(\vec{p})\,e^{ip\cdot x}\right]. \tag{6.9}$$

In (6.7) and (6.8) we have indicated the relationship to the previous expansion (2.6).

Using the definition of A we have

$$A(p)\phi^{(\pm)}(\vec{p})=\phi^{(\pm)}(-\vec{p})\,, \tag{6.10}$$

which in turn yields (6.2).

Assuming that the configuration-space integral is over a time slice [$\sigma(x)$ such that $t=$constant] the generality of which will be demonstrated below, we may derive the bilinear form in momentum space

$$(\phi,\psi)=\int d^3x\,\tilde{\phi}^\dagger(x)\beta_0\,\psi(x) \tag{6.11}$$

$$=\int\frac{d^3p}{2E}\left[\phi^{(+)\dagger}(-\vec{p})\,\psi^{(+)}(\vec{p})\right.$$
$$\left.-(-)^{2s}\,\phi^{(-)\dagger}(-\vec{p})\,\psi^{(-)}(\vec{p})\right], \tag{6.12}$$

where we have made use of the orthonormality relations of Sec. V.

We shall now examine the properties of (ϕ,ψ) in detail. In the above we wished only to point out that the dual operation in momentum space involves mapping the function to a reversed three-momentum.

VII. PROPERTIES OF (ϕ,ψ)

The bilinear form (6.1) on the solutions of Eq. (2.1) has the following properties.

(a) Unitarity. (ϕ,ψ) is invariant under the action of the proper Poincaré group.

For [see (2.2)]

$$[U(a,\Lambda)\phi](x)=S(\Lambda)\phi(\Lambda^{-1}(x-a)) \tag{7.1}$$

we have

$$(U(a,\Lambda)\phi,\,U(a,\Lambda)\psi)$$
$$=\int d\sigma_\mu(x)\left[A(i\partial)U(a,\Lambda)\phi\right]^\dagger(x)\beta^\mu\left[U(a,\Lambda)\psi\right](x) \tag{7.2}$$
$$=\int d\sigma_\mu(x)\left[A(i\partial)S(\Lambda)\phi(\Lambda^{-1}(x-a))\right]^\dagger$$
$$\times\beta^\mu S(\Lambda)\psi(\Lambda^{-1}(x-a))\,; \tag{7.3}$$

using Eq. (4.11) we get

$$A(i\partial)S(\Lambda)=\tilde{S}(\Lambda)A(i\Lambda^{-1}\partial)\,, \tag{7.4}$$

and so

$$(U(a,\Lambda)\phi,\,U(a,\Lambda)\psi)$$
$$=\int d\sigma_\mu(x)\,\tilde{\phi}^\dagger(\Lambda^{-1}(x-a))\tilde{S}^\dagger(\Lambda)S(\Lambda)(\Lambda\beta)^\mu$$
$$\times\,\psi(\Lambda^{-1}(x-a)) \tag{7.5}$$
$$=\int d\sigma_\mu(\Lambda^{-1}(x-a))\,\tilde{\phi}^\dagger(\Lambda^{-1}(x-a))$$
$$\times\beta^\mu\,\psi(\Lambda^{-1}(x-a)) \tag{7.6}$$
$$=\int d\sigma_\mu(x)\tilde{\phi}^\dagger(x)\beta^\mu\,\psi(x) \tag{7.7}$$
$$=(\phi,\psi)\,. \tag{7.8}$$

Consider for comparison the positive-energy solutions in momentum space which transform as

$$[U(a,\Lambda)\phi](p)=e^{ip\cdot a}S(\Lambda)\phi(\Lambda^{-1}p)\,. \tag{7.9}$$

We now have ($d\Omega=d^3p/2E$)

$$(U(a,\Lambda)\phi,U(a,\Lambda)\psi)$$
$$=\int d\Omega\{[A(p)U(a,\Lambda)\phi]^\dagger(p)U(a,\Lambda)\psi(p)\}, \tag{7.10}$$
$$=\int d\Omega[A(p)e^{ip\cdot a}S(\Lambda)\phi(\Lambda^{-1}p)]^\dagger e^{ip\cdot a}S(\Lambda)\psi(\Lambda^{-1}p) \tag{7.11}$$

and by (4.11) we have

$$(U(a,\Lambda)\phi,U(a,\Lambda)\psi)$$
$$=\int d\Omega[\tilde{S}(\Lambda)A(\Lambda^{-1}p)\phi(\Lambda^{-1}p)]^\dagger S(\Lambda)\psi(\Lambda^{-1}p) \tag{7.12}$$
$$=\int d\Omega\tilde{\phi}^\dagger(\Lambda^{-1}p)\psi(\Lambda^{-1}p) \tag{7.13}$$
$$=\int d\Omega\tilde{\phi}^\dagger(p)\psi(p) \tag{7.14}$$
$$=(\phi,\psi)\,, \tag{7.15}$$

where we have invoked the invariance of the volume element $d\Omega$.

(b) Conservation. (ϕ,ψ) is invariant with respect to variations in σ.

Using familiar techniques,[23] we have

$$\frac{\delta}{\delta\sigma(x)}(\phi,\psi)_\sigma=\frac{\delta}{\delta\sigma(x)}\int d\sigma^\mu(x)\tilde{\phi}^\dagger(x)\beta_\mu\psi(x) \tag{7.16}$$
$$=\partial^\mu[\tilde{\phi}^\dagger(x)\beta_\mu\psi(x)] \tag{7.17}$$
$$=\tilde{\phi}^\dagger(x)\beta_\mu\bar{\partial}^\mu\cdot\psi(x)+\tilde{\phi}^\dagger(x)\cdot\beta_\mu\partial^\mu\psi(x) \tag{7.18}$$
$$=im\tilde{\phi}^\dagger(x)\psi(x)-im\tilde{\phi}^\dagger(x)\psi(x) \tag{7.19}$$
$$=0\,,$$

where we have used Eqs. (2.1), (3.1), and (3.10). In particular we may conclude that (ϕ,ψ) will be conserved in time.

(c) $(\phi,\psi)=(\psi,\phi)^*$. In momentum space we have, e.g., for positive-energy solutions

$$(\phi,\psi)=\int d\Omega[A(p)\phi(\vec{p})]^\dagger\psi(\vec{p}) \tag{7.20}$$
$$=\int d\Omega\phi^\dagger(\vec{p})A(p)\psi(\vec{p}) \tag{7.21}$$
$$=\int d\Omega[A(p)\psi(\vec{p})]^{*\dagger}\phi^*(\vec{p}) \tag{7.22}$$
$$=(\psi,\phi)^*\,. \tag{7.23}$$

(d) $(\phi,\alpha\psi_1+\beta\psi_2)=\alpha(\phi,\psi_1)+\beta(\phi,\psi_2)$. This property is trivially true.

(e) $\|\phi\|^2\equiv(\phi,\phi)$. This is positive-definite for s equal to a half integer and positive- (negative-) definite for the positive- (negative-) energy solutions when s is an integer.

Into the scalar product (6.12) we insert the relations (6.7) and (6.8), and we use the orthonormality properties of Sec. V to get

$$(\phi,\phi)=\sum_\sigma\int d^3p[a^*(\vec{p},\sigma)a(\vec{p},\sigma)-(-)^{2s}b(\vec{p},\sigma)b^*(\vec{p},\sigma)]\,, \tag{7.24}$$

from which the assertion (e) follows.

In view of the above properties (a)–(e) we may conclude that the bilinear form (6.1) provides a suitable scalar product on the solution space of Eq. (2.1) for any spin, the negative-definite contributions in the integer-spin case being remedied as usual in the transition to the positive-definite-metric Fock space. See Sec. IX.

Thus even without an η matrix there exists a suitable scalar product for the formalism. However, this has entailed the introduction of a dual space which is more complicated than usual. In particular the Hermitian conjugation of operators with respect to this scalar product will now involve the mapping $A(i\partial)$:

$$O^{\text{H.c.}}=(AOA^{-1})^\dagger \tag{7.25}$$

or, equivalently, the metric operator $A(i\partial)$ must be considered in the taking of the Hermitian conjugate.[24]

VIII. LAGRANGIAN

Based upon the preceding discussion of bilinear forms we now consider the existence of bilinear densities such as the Lagrangian. We take the Lagrangian density for a mass-m, spin-s field to be

$$L(x)=\tfrac{1}{2}\tilde{\phi}^\dagger(x)(i\beta\cdot\partial-m)\phi(x)+\tfrac{1}{2}\tilde{\phi}^\dagger(x)(-i\beta\cdot\bar{\partial}-m)\phi(x)\,, \tag{8.1}$$

where we have explicitly symmetrized with respect to derivatives. That $L(x)$ is a scalar density under the transformations of the proper Poincaré group follows easily from the previously given transformation properties of ϕ and $\tilde{\phi}$.

In order to justify the independent variation of ϕ and $\tilde{\phi}$, consider, for a moment, the usual Dirac case. There the independent variation of Ψ and $\Psi^\dagger$ may be justified either by breaking Ψ up into its real and imaginary parts and varying them independently or by simply varying Ψ and $\Psi^\dagger$ and then determining whether or not the consequences of this independent variation, i.e., the Euler-Lagrange equations, are consistent. We shall follow this latter procedure.

Consider the more general density

$$L(x) = \tfrac{1}{2}\tilde{\psi}^{\dagger}(x)(i\beta\cdot\partial - m)\,\phi(x) + \tfrac{1}{2}\tilde{\psi}^{\dagger}(x)(-i\beta\cdot\overleftarrow{\partial} - m)\,\phi(x)\,. \tag{8.2}$$

Varying with respect to $\tilde{\psi}^{\dagger}(x)$ leads to the Euler-Lagrange equation

$$0 = \frac{\partial L}{\partial \tilde{\psi}^{\dagger}} - \partial_{\mu}\frac{\partial L}{\partial(\partial_{\mu}\tilde{\psi}^{\dagger})} \tag{8.3}$$

$$= \tfrac{1}{2}(i\beta\cdot\partial - m)\phi(x) - \tfrac{1}{2}m\phi(x) + \tfrac{1}{2}i\beta\cdot\partial\phi(x) \tag{8.4}$$

$$= (i\beta\cdot\partial - m)\phi(x)\,. \tag{8.5}$$

Similarly variation with respect to $\phi(x)$ leads to

$$\tilde{\psi}^{\dagger}(x)(-i\beta\cdot\overleftarrow{\partial} - m) = 0\,. \tag{8.6}$$

Now equations (8.5) and (8.6) are consistent with the choice $\phi(x) \equiv \psi(x)$, that is,

$$\tilde{\psi}^{\dagger}(x) = [A(i\partial)\phi(x)]^{\dagger}. \tag{8.7}$$

To see this operate on Eq. (8.5) with $A(i\partial)$, use Eqs. (4.18) and (3.10), and take the complex conjugate transpose:

$$0 = A(i\partial)(i\beta\cdot\partial - m)\phi(x) \tag{8.8}$$

$$= (i\beta^{\dagger}\cdot\partial - m)A(i\partial)\phi(x)\,, \tag{8.9}$$

which implies

$$\tilde{\phi}^{\dagger}(x)(-i\beta\cdot\overleftarrow{\partial} - m) = 0\,, \tag{8.10}$$

which is identical to Eq. (8.6) for $\phi(x) \equiv \psi(x)$. Thus it is consistent to independently vary $\phi(x)$ and $\tilde{\phi}^{\dagger}(x)$ in (8.1).

In view of the above discussion we see that the Lagrangian (8.1) leads to the equation of motion of ϕ, (2.1), and the associated equation for $\tilde{\phi}^{\dagger}$, (3.1). Thus the equations are derivable from a Lagrangian and another η-matrix benefit is achieved in the absence of an η.

Anticipating the second-quantized formulation, the Hermiticity of $L(x)$ may be seen as follows:

$$L^{\text{H.c.}}(x) = [\tilde{\phi}^{\dagger}(x)(i\beta\cdot\partial - m)\phi(x)]^{\text{H.c.}} \tag{8.11}$$

$$= \tilde{\phi}^{\dagger}(x)(i\beta\cdot\partial - m)^{\text{H.c.}}\,\phi(x)\,. \tag{8.12}$$

From Eqs. (7.25), (4.18), and (3.10) we have

$$(i\beta\cdot\partial - m)^{\text{H.c.}} = [A(i\partial)(i\beta\cdot\partial - m)A^{-1}(i\partial)]^{\dagger} \tag{8.13}$$

$$= (i\beta^{\dagger}\cdot\partial - m)^{\dagger} \tag{8.14}$$

$$= (-i\beta\cdot\overleftarrow{\partial} - m)\,, \tag{8.15}$$

so we get

$$L^{\text{H.c.}}(x) = L(x)\,. \tag{8.16}$$

Let us now consider some other bilinear densities. We may determine physically relevant quantities from the energy-momentum stress tensor

$$T^{\mu\nu}(x) = i\,\tilde{\phi}^{\dagger}(x)\beta^{\mu}\partial^{\nu}\phi(x)\,. \tag{8.17}$$

For example, the energy is given as

$$H = \int d^3x\, T^{00}(x) \tag{8.18}$$

$$= \int d^3x[i\,\tilde{\phi}^{\dagger}(x)\beta^0\partial^0\phi(x)] \tag{8.19}$$

$$= \int d^3x[\tilde{\phi}^{\dagger}(x)(-i\vec{\beta}\cdot\vec{\nabla} + m)\phi(x)]\,. \tag{8.20}$$

If we insert the expansion (2.6) and (4.3) and use the relations of Sec. V we get

$$H = \sum_{\sigma}\int d^3p\, E[a^*(\vec{p},\sigma)a(\vec{p},\sigma) + (-)^{2s}b(\vec{p},\sigma)b^*(\vec{p},\sigma)] \tag{8.21}$$

$$= \sum_{\sigma}\int d^3p\, E[|a(\vec{p},\sigma)|^2 + (-)^{2s}|b(\vec{p},\sigma)|^2]\,. \tag{8.22}$$

We see that H is positive-definite for all integer-spin particles and indefinite for half-integer-spin particles. This latter difficulty is remedied in the usual fashion in the second-quantized formalism by normal-ordering the operators in Eq. (8.21) and assigning Fermi-Dirac statistics to half-integer spins.

By virtue of the equation of motion [see Eqs. (7.17)–(7.19)] the current

$$j^{\mu}(x) = \tilde{\phi}^{\dagger}(x)\beta^{\mu}\phi(x) \tag{8.23}$$

is conserved:

$$\partial_{\mu}j^{\mu}(x) = 0\,. \tag{8.24}$$

The associated conserved charge is

$$Q = \int d^3x\, j^0(x) \tag{8.25}$$

$$= \int d^3x\, \tilde{\phi}^{\dagger}(x)\beta^0\phi(x)\,, \tag{8.26}$$

which upon inserting the expansions (2.6) and (4.3) becomes

$$Q = \sum_{\sigma}\int d^3p[a^*(\vec{p},\sigma)a(\vec{p},\sigma) - (-)^{2s}b(\vec{p},\sigma)b^*(\vec{p},\sigma)] \tag{8.27}$$

$$= \sum_{\sigma}\int d^3p[|a(\vec{p},\sigma)|^2 - (-)^{2s}|b(\vec{p},\sigma)|^2]\,, \tag{8.28}$$

which, as noted before, is positive-definite for half-integer spins and positive (negative) definite for positive- (negative-) energy solutions for integer spins. When second-quantized and normal-ordered using the usual spin-statistics relationship all spins lead to the latter case.

IX. SECOND QUANTIZATION

We formally second-quantize the present theory in the usual fashion. Let the a's and b's satisfy the canonical commutation (anticommutation) relations CCR (CAR) for integer (half-integer) spins, respectively:

$$[a(\vec{p},\sigma),\ a^*(\vec{p}',\sigma')]_\pm=\delta(\vec{p}-\vec{p}')\,\delta_{\sigma\sigma'}\ , \tag{9.1a}$$

$$[b(\vec{p},\sigma),\ b^*(\vec{p}',\sigma')]_\pm=\delta(\vec{p}-\vec{p}')\,\delta_{\sigma\sigma'}\ , \tag{9.1b}$$

and all other commutators (anticommutators) vanish. We assume[25] that the CCR hold when s is an integer and that the CAR hold when s is a half-integer. In both cases the relations (9.1) are realizable on a (positive-definite metric) Hilbert space which is the underlying Fock space for the formalism.

In configuration space we may determine the canonical conjugates to the independent components $[\alpha=1,\ldots,2(2s+1)]$ of the field $\phi_\alpha(x)$ from the Lagrangian (8.1). It is $\tilde{\phi}^\dagger_\beta(x)\,\beta^0_{\beta\alpha}$. We may calculate the CCR (CAR) from Eqs. (2.6), (4.3), and (9.1):

$$[\phi_\alpha(x),\ \tilde{\phi}^\dagger_\beta(x')]_\pm=\frac{1}{(2\pi)^3}\sum_\sigma\int d^3p\,\frac{m}{E}\,[u_\alpha(\vec{p},\sigma)\,u^\dagger_\beta(-\vec{p},\sigma)\,e^{-ip\cdot(x-x')}-v_\alpha(\vec{p},\sigma)\,v^\dagger_\beta(-\vec{p},\sigma)\,e^{ip\cdot(x-x')}] \tag{9.2}$$

and from Eqs. (5.25) and (5.38) we get

$$[\phi_\alpha(x),\ \tilde{\phi}^\dagger_\beta(x')]_\pm=\frac{1}{(2\pi)^3}\int\frac{d^3p}{2E}\left\{\left[\frac{(\beta\cdot p)^2}{m}+(\beta\cdot p)\right]e^{-ip\cdot(x-x')}-\left[\frac{(\beta\cdot p)^2}{m}-(\beta\cdot p)\right]e^{ip\cdot(x-x')}\right\}_{\alpha\beta} \tag{9.3}$$

$$=i\left[i\beta\cdot\partial-\frac{(\beta\cdot\partial)^2}{m}\right]_{\alpha\beta}\frac{-i}{(2\pi)^3}\int\frac{d^3p}{2E}\,[e^{-ip\cdot(x-x')}-e^{ip\cdot(x-x')}] \tag{9.4}$$

$$=i\left[i\beta\cdot\partial-\frac{(\beta\cdot\partial)^2}{m}\right]_{\alpha\beta}\Delta(x-x')\,, \tag{9.5}$$

which displays the covariance and causality of the commutator (anticommutator).

In order to see more clearly that the theory is indeed local, consider the projection of (9.5) onto the space of the independent components. The independent components are those which enter the time derivative term of Eq. (2.1) and so they correspond to the $2(2s+1)$-dimensional non-null subspace of β_0. Since Eq. (2.5) implies that $\beta_0(\beta_0^2-1)=0$ the projection onto the non-null subspace of β_0 is simply β_0^2.

In the following we will need the fact that for the independent components

$$[\beta_i\,,\beta_0]_{+\alpha\beta}=0,\quad \alpha,\beta=1,\ldots,2(2s+1)\ ,\quad i=1,2,3\ . \tag{9.6}$$

Proof. The algebraic relation (2.5) for the choice $\mu=0=\nu$, $\lambda=i=1,2,3$, becomes

$$\beta_0\beta_0\beta_i+\beta_0\beta_i\beta_0+\beta_i\beta_0\beta_0-\beta_i=0\,, \tag{9.7}$$

which implies

$$\beta_0^2\beta_i\beta_0+\beta_0\beta_i\beta_0^2+\beta_i\beta_0-\beta_i\beta_0=0\,, \tag{9.8}$$

where we have multiplied from the right by β_0 and used $\beta_0^3=\beta_0$. Again using this last relation we get

$$\beta_0^2\beta_i\beta_0^3+\beta_0^3\beta_i\beta_0^2=0 \tag{9.9}$$

or

$$\beta_0^2[\beta_i\,,\beta_0]_+\beta_0^2=0\,, \tag{9.10}$$

which is the desired result.

Now let us consider the commutator (anticommutator) for $x_0=x_0'$ on the space of independent components. We find

$$\beta_0^2[\phi(x),\ \tilde{\phi}^\dagger(x')\beta_0]_\pm|_{x_0=x_0'}\beta_0^2=i\beta_0^2\left[i\beta\cdot\partial\beta_0-\frac{(\beta\cdot\partial)^2}{m}\beta_0\right]\beta_0^2\Delta(x-x')\Big|_{x_0=x_0'}\ . \tag{9.11}$$

But purely spatial derivatives of

$$\Delta(x-x')|_{x_0=x_0'}=0$$

vanish, so we get

$$\beta_0^2[\phi(x),\ \tilde{\phi}^\dagger(x')\beta_0]_\pm|_{x_0=x_0'}\beta_0^2=i\beta_0^2\left[i\beta_0\partial_0-\frac{\beta_0^2\partial_0^2}{m}-\frac{[\beta_0,\beta_i]_+}{m}\partial_0\partial_i\right]\beta_0^2\beta_0\Delta(x-x')\Big|_{x_0=x_0'}\ . \tag{9.12}$$

From Eq. (9.10), $\beta_0^3=\beta_0$, and $(\Box^2+m^2)\Delta(x-x')=0$ we get

$$\beta_0^2[\phi(x),\ \tilde{\phi}^\dagger(x')\beta_0]_\pm|_{x_0=x_0'}\beta_0^2=-\beta_0^2\partial_0\Delta(x-x')|_{x_0=x_0'} \tag{9.13}$$

$$=\beta_0^2\delta^{(3)}(\vec{x}-\vec{x}') \tag{9.14}$$

and so

$$[\phi_\alpha(x),\tilde{\phi}^\dagger_\gamma(x')\beta_{0\,\gamma\beta}]_\pm|_{x_0=x_0'}=\delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{x}') \tag{9.15}$$

for $\alpha,\beta=1,\ldots,2(2s+1)$.

The energy and charge spectrum has already been mentioned in Sec. VIII and can be determined directly from Eqs. (8.21) and (8.27), respectively.

Once normally ordered, H is positive-definite for all spin values and Q is indefinite.

We see therefore that the difficulties of nonlocality, indefinite metric, and indefinite energy may be simultaneously avoided in the present formalism.

X. DISCRETE SYMMETRIES

As mentioned in the Introduction, in the absence of an η matrix we must abandon the usual implementation of a parity symmetry, P. In this section we shall first consider a more general parity realization in terms of a linear operator, and then we shall consider an antilinear realization appropriate to the present formalism in addition to the other discrete symmetries, time reversal T and charge conjugation C. We shall argue that the parity symmetry may still be realized even in the absence of an η matrix, i.e., in the absence of a self-conjugate representation.

First, let us consider the more customary parity realization on the solutions of Eq. (2.1). The action of P on these solutions may be given in terms of the coefficients of the expansion (2.6):

$$a_P(\vec{p},\sigma)=\eta_P a(-\vec{p},\sigma), \tag{10.1a}$$

$$b_P(\vec{p},\sigma)=(-)^{2s}\eta_P^* b(-\vec{p},\sigma), \tag{10.1b}$$

where η_P is an arbitrary phase.[26]

In configuration space we find

$$\phi^P(\vec{x},t)=\eta_P\sum_\sigma\int d^3p[u(\vec{p},\sigma)a(-\vec{p},\sigma)e^{-ip\cdot x}+(-)^{2s}v(\vec{p},\sigma)b^*(-\vec{p},\sigma)e^{ip\cdot x}] \tag{10.2}$$

$$=\eta_P\sum_\sigma\int d^3p[u(-\vec{p},\sigma)a(\vec{p},\sigma)e^{-ip_0x_0-i\vec{p}\cdot\vec{x}}+(-)^{2s}v(-\vec{p},\sigma)b^*(\vec{p},\sigma)e^{ip_0x_0+i\vec{p}\cdot\vec{x}}] \tag{10.3}$$

$$=\eta_P\tilde{\phi}(-\vec{x},t), \tag{10.4}$$

where we have used Eq. (4.3). Thus the usual parity matrix which acts only on the indices of ϕ is replaced by the matrix differential operator $A(i\partial)$, which has the effect of reversing the three-momenta of each of the Fourier components.

The covariance of the wave equation may be demonstrated as follows. In the primed frame, $x'=(-\vec{x},t)$, Eq. (2.1) reads

$$(i\beta\cdot\partial'-m)\phi'(x')=0 \tag{10.5}$$

if and only if

$$(i\tilde{\beta}\cdot\partial-m)\eta_P A(i\partial)\phi(x)=0, \tag{10.6}$$

where we have used Eqs. (3.10) and (10.4). Application of property (4.18) yields

$$\eta_P A(i\partial)(i\beta\cdot\partial-m)\phi(x)=0 \tag{10.7a}$$

if and only if

$$(i\beta\cdot\partial-m)\phi(x)=0, \tag{10.7b}$$

which tells us that the equations of motion in the unprimed frame and the primed frame are identical. Thus, with this definition of P, Eq. (2.1) is parity-covariant.

Since the operator involved in the parity operation is the same as that used in the dual transformation, it is natural in the present formalism to consider a parity realization which is antiunitary rather than unitary. We present a general discussion of the action of parity on the observables in a relativistic system in the Appendix and argue there that if parity is to be realized in terms of an antiunitary operator then it must also induce a reversal of operator products. We incorporate this in the following discussion of the discrete symmetries for the systems here under study.

Consider again the solutions to Eq. (2.1) as expanded in Eq. (2.6). Define the action of the discrete symmetries on the Fock-space operators now as follows:

$$a_T(\vec{p},\sigma)=Ta(\vec{p},\sigma)T^{-1}=\eta_T C^{(s)}_{\sigma\sigma'}a(-\vec{p},\sigma'), \tag{10.8a}$$

$$b_T(\vec{p},\sigma)=Tb(\vec{p},\sigma)T^{-1}=\eta_T^* C^{(s)}_{\sigma\sigma'}b(-\vec{p},\sigma'), \tag{10.8b}$$

$$a_C(\vec{p},\sigma)=\mathsf{C}a(\vec{p},\sigma)\mathsf{C}^{-1}=\eta_C b(\vec{p},\sigma), \tag{10.9a}$$

$$b_C(\vec{p},\sigma)=\mathsf{C}b(\vec{p},\sigma)\mathsf{C}^{-1}=\eta_C^* a(\vec{p},\sigma), \tag{10.9b}$$

$$a_P(\vec{p},\sigma)=Pa(\vec{p},\sigma)P^{-1}=\eta_P a^*(-\vec{p},\sigma), \tag{10.10a}$$

$$b_P(\vec{p},\sigma)=Pb(\vec{p},\sigma)P^{-1}=\eta_P^*(-)^{2s}b^*(-\vec{p},\sigma), \tag{10.10b}$$

where the η's are phases and $C^{(s)}$ is defined by Eq. (2.10). T and P are assumed to be antiunitary, and C to be unitary.

In configuration space we obtain the following transformation properties, again using the expansion (2.6).

A. Time reversal (antilinear)

We have the following equation:

$$\phi^T_\alpha(x) = T\phi_\alpha(x)T^{-1} \tag{10.11}$$

$$= \eta_T \frac{1}{(2\pi)^{3/2}} \sum_{\sigma,\sigma'} \int d^3p \left(\frac{m}{E}\right)^{1/2} [u^*_\alpha(\vec{p},\sigma)C^{(s)}_{\sigma\sigma'} a(-\vec{p},\sigma')e^{ip\cdot x} + v^*_\alpha(\vec{p},\sigma)C^{(s)*}_{\sigma\sigma'} b^*(-\vec{p},\sigma')e^{-ip\cdot x}] \tag{10.12}$$

$$= \eta_T \frac{1}{(2\pi)^{3/2}} \sum_{\sigma,\sigma'} \int d^3p \left(\frac{m}{E}\right)^{1/2} [u^*_\alpha(-\vec{p},\sigma)C^{(s)}_{\sigma\sigma'} a(\vec{p},\sigma')e^{+ip_0x_0+i\vec{p}\cdot\vec{x}} + v^*_\alpha(-\vec{p},\sigma)C^{(s)*}_{\sigma\sigma'} b^*(\vec{p},\sigma')e^{-ip_0x_0-i\vec{p}\cdot\vec{x}}] . \tag{10.13}$$

From Eq. (2.7) we have

$$u^*_\alpha(-\vec{p},\sigma)C^{(s)}_{\sigma\sigma'} = S^*_{\alpha\beta}(L(-\vec{p}))u_\beta(0,\sigma)C^{(s)}_{\sigma\sigma'} \tag{10.14}$$

$$= S^*_{\alpha\beta}(L(-\vec{p}))C_{\beta\gamma}u_\gamma(0,\sigma'), \tag{10.15}$$

where $C_{\beta\gamma}$ is given by

$$C = \bigoplus_{i=1}^{N} C^{(s_i)} \tag{10.16}$$

and s_i, $i=1,\ldots,N$, enumerates the N spins occurring in the completely reducible representation of the rotation subgroup, $S(R)$. For example, if $S(\Lambda)$ is the representation $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ then

$$C = \begin{vmatrix} C^{(s)} & 0 & 0 \\ 0 & C^{(s)} & 0 \\ 0 & 0 & C^{(s-1)} \end{vmatrix} \tag{10.17}$$

in the basis where $S(\Lambda)$ is completely reduced.

Now because of the properties (2.13) and (2.14) it follows that

$$C_{\alpha\beta}S_{\beta\gamma}(L(-\vec{p}))C^{-1}{}_{\gamma\delta} = S^*_{\alpha\delta}(L(\vec{p})) . \tag{10.18}$$

This is true because in any finite-dimensional representation the generators of pure Lorentz transformations may be constructed from submatrices whose nonzero contributions are proportional to $\vec{S}^{(s)}$, $\vec{K}^{(s)}$, or $\vec{K}^{(s)\dagger}$ and these alone.[27] Thus we get

$$u^*_\alpha(-\vec{p},\sigma)C^{(s)}_{\sigma\sigma'} = C_{\alpha\beta}u_\beta(\vec{p},\sigma') . \tag{10.19}$$

Similarly,

$$v^*_\alpha(-\vec{p},\sigma)C^{(s)*}_{\sigma\sigma'} = C_{\alpha\beta}v_\beta(\vec{p},\sigma') \tag{10.20}$$

and so we get

$$\phi^T_\alpha(\vec{x},t) = \eta_T C_{\alpha\beta}\phi_\beta(\vec{x},-t). \tag{10.21}$$

Similarly we have

$$\tilde{\phi}^{\dagger T}_\alpha(\vec{x},t) = \eta^*_T \tilde{\phi}^\dagger_\beta(\vec{x},-t)C^\dagger_{\beta\alpha} . \tag{10.22}$$

We note for later use that in addition to the property (10.18), C also has the property that

$$C\beta^*_\mu C^{-1} = \beta^\dagger_\mu . \tag{10.23}$$

This also follows from (2.13) and (2.14) along with the facts that, as a result of (2.3), (i) β_0 must be a multiple of the identity on each of its square submatrices in its SU(2) decomposition, and (ii) because of its 3-vector character, the nonvanishing submatrices of β_i, $i=1,2,3$, must be proportional to either $\vec{S}$, $\vec{K}$, or $\vec{K}^\dagger$ [21]

B. Charge conjugation (linear)

We have the following equation:

$$\phi^C_\alpha(x) = \mathrm{C}\phi_\alpha(x)\mathrm{C}^{-1} \tag{10.24}$$

$$= \eta_C \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int d^3p \left(\frac{m}{E}\right)^{1/2} [u_\alpha(\vec{p},\sigma)b(\vec{p},\sigma)e^{-ip\cdot x} + v_\alpha(\vec{p},\sigma)a^*(\vec{p},o)e^{ip\cdot x}] . \tag{10.25}$$

Introduce the matrix $\rho \neq I$, $\rho^2 = I$, which commutes with $S(\Lambda)$ and C and which has the further property that

$$\rho u(0,\sigma) = v(0,\sigma) \tag{10.26a}$$

and

$$\rho v(0,\sigma) = u(0,\sigma). \tag{10.26b}$$

For example, if $S(\Lambda) = (s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$, ρ would be the $[(2s+1)+(2s+1)+2s-1=6s+1]$-dimensional matrix

$$\rho = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \tag{10.27}$$

in the basis where $S(\Lambda)$ is completely reduced.

Note that Eqs. (10.26) imply that $[\rho,\beta_0]_+ = 0$, and hence since $[\rho, S(\Lambda)]_- = 0$ for all Λ, Eq. (2.3) yields

$$[\rho, \beta_\mu]_+ = 0 \; . \quad (10.28)$$

These properties allow us to use Eqs. (2.7) and (2.8) to establish

$$(\rho C^{-1})u^*(-\vec{p}, \sigma) = \rho C^{-1} S^*(L(-\vec{p}))\, u(0, \sigma) \quad (10.29)$$

$$= S(L(\vec{p}))\, C^{-1} \rho u(0, \sigma) \quad (10.30)$$

$$= S(L(\vec{p}))\, C^{-1} v(0, \sigma) \quad (10.31)$$

$$= v(\vec{p}, \sigma) \quad (10.32)$$

and

$$(\rho C^{-1}) v^*(-\vec{p}, \sigma) = (-)^{2s} u(\vec{p}, \sigma), \quad (10.33)$$

where we have made use of Eq. (2.11).

Using these relations in Eq. (10.25) we have

$$\phi_\alpha^C(x) = \eta_C \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int d^3p \left(\frac{m}{E}\right)^{1/2} [(-)^{2s} \rho C^{-1} v^*(-\vec{p}, \sigma) b(\vec{p}, \sigma) e^{-ip\cdot x} + \rho C^{-1} u^*(-\vec{p}, \sigma) a^*(\vec{p}, \sigma) e^{ip\cdot x}]_\alpha \quad (10.34)$$

$$= \eta_C (\rho C^{-1})_{\alpha\beta} \tilde{\phi}_\beta^*(x), \quad (10.35)$$

where to get the last equation we have used Eq. (4.3).

In a similar manner we find

$$\mathrm{C}\, \tilde{\phi}_\alpha^*(\vec{x}, t) \mathrm{C}^{-1} = (-)^{2s} (\rho C^{-1})_{\alpha\beta} \phi_\beta(x). \quad (10.36)$$

C. Parity (antilinear, reverses operator products)

We have the following equation:

$$\phi_\alpha^P(\vec{x}, t) = P \phi_\alpha(\vec{x}, t) P^{-1} \quad (10.37)$$

$$= \eta_P \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int d^3p \left(\frac{m}{E}\right)^{1/2} [u_\alpha^*(\vec{p}, \sigma) a^*(-\vec{p}, \sigma) e^{ip\cdot x} + v_\alpha^*(\vec{p}, \sigma)(-)^{2s} b(-\vec{p}, \sigma) e^{-ip\cdot x}]_\alpha \quad (10.38)$$

$$= \eta_P \frac{1}{(2\pi)^{3/2}} \sum_\sigma \int d^3p \left(\frac{m}{E}\right)^{1/2} [u_\alpha^*(-\vec{p}, \sigma) a^*(\vec{p}, \sigma) e^{ip_0 x_0 + i\vec{p}\cdot\vec{x}} + (-)^{2s} v_\alpha^*(-\vec{p}, \sigma) b(\vec{p}, \sigma) e^{-ip_0 x_0 - i\vec{p}\cdot\vec{x}}]_\alpha \quad (10.39)$$

$$= \eta_P \tilde{\phi}_\alpha^\dagger(-\vec{x}, t), \quad (10.40)$$

where we have used Eqs. (10.10), (2.6), (4.3), and the antilinear nature of P.

In a similar fashion we get

$$\tilde{\phi}_\alpha^{\dagger P}(\vec{x}, t) = P \tilde{\phi}_\alpha^\dagger(\vec{x}, t) P^{-1} = \eta_P^* \phi_\alpha(-\vec{x}, t). \quad (10.41)$$

So defined we see that T maps the solution space of Eq. (2.1) into itself ($\phi \to \phi$), while C and P involve the mapping $A(\phi \to \tilde{\phi})$.

We shall conclude with a consideration of the effects of the discrete symmetries on some important quantities.

Bearing in mind that T is taken to be antiunitary with no operator-product reversal, C is unitary with no operator-product reversal, and P is antiunitary and reverses operator products, it is easy to verify that these transformations defined by (10.8), (10.9), and (10.10) will preserve the Fock-space CCR (CAR) given in Eqs. (9.1).

Using the transformation properties of the fields under T, C, and P, the properties of C and ρ given in Eqs. (10.23) and (10.28), and, in the case of C, normal ordering of field operators, we may derive the following transformation properties:

$$T L(\vec{x}, t) T^{-1} = L(\vec{x}, -t), \quad (10.42)$$

$$T H T^{-1} = H, \quad (10.43)$$

$$T j_\mu T^{-1} = j^\mu(\vec{x}, -t); \quad (10.44)$$

$$\mathrm{C}\, L(\vec{x}, t) \mathrm{C}^{-1} = L(\vec{x}, t), \quad (10.45)$$

$$\mathrm{C} H \mathrm{C}^{-1} = H, \quad (10.46)$$

$$\mathrm{C} j^\mu(\vec{x}, t) \mathrm{C}^{-1} = -j^\mu(\vec{x}, t); \quad (10.47)$$

and

$$P L(\vec{x}, t) P^{-1} = L(-\vec{x}, t), \quad (10.48)$$

$$P H P^{-1} = H, \quad (10.49)$$

$$P j_\mu P^{-1} = j^\mu(-\vec{x}, t), \quad (10.50)$$

where for parity we have used the operator-reversal property

$$P(AB)P^{-1} = (PBP^{-1})(PAP^{-1}). \quad (10.51)$$

XI. DISCUSSION

In the present study we have examined certain relativistic descriptions of free, massive, spin-s particles which do not permit the existence of a Hermitianizing matrix, η, and we have seen that in the absence of such a matrix we may still have an invariant scalar product, bilinear densities, and a realization of parity invariance; in other words, all of the usual benefits of an η matrix. In order to achieve these ends we have seen that it was necessary for us to introduce a dual space (or, equivalently, a metric operator) which is more complicated than the usual one. Although the characteristic mapping A involves derivatives, we also saw in Sec. IX that the theory is still, at least formally, a local one.

It has been the purpose of this paper to present in detail the above results for free-particle theories, thus providing the framework for the introduction of interactions. The ultimate worth of the present formalism will of course depend upon its applications in this latter domain. In general we may expect that the mapping A will be interaction-dependent. However, given the difficulties of the traditional methods in forming operator products, we feel that the possibility of having a natural dependence upon the interaction of such products is not at all an unpleasant prospect.

The assumptions of the present work have limited its results to particle theories whose wave functions have the index-transformation properties of (2.19) and their conjugates.[28] The first class of the above wave equations is the simplest and has also been suggested by the consideration of the Galilei-invariant higher-spin wave equations.[16,18] This system and the others of (2.19) are consistent and causal in the presence of a minimally coupled external electromagnetic field, and they afford a simple and uniform description for any spin. Here we have seen that we may also define a scalar product, a Lagrangian, and a parity symmetry for these equations.

ACKNOWLEDGMENTS

The author is indebted to many of his friends and colleagues, especially at Syracuse University and at the University of Texas, for numerous discussions and correspondences containing invaluable questions, criticisms, suggestions, and encouragements. He is particularly grateful to Professor F. Rohrlich and Professor J. Weinberg for their constant advice and enthusiasm. He is indebted to Professor E. C. G. Sudarshan for discussions and for his critical reading of the manuscript. He wishes to thank Professor A. S. Wightman, who through discussions and correspondences has stimulated much of the material presented here.

APPENDIX

We wish to demonstrate that parity may be realized either as a linear (unitary) operator which does not reverse operator products (L, R) or as an antilinear (antiunitary) operator which does reverse operator products (A, R).[29]

Consider the invariance group of the system, the proper Poincaré group. The nonvanishing bracket relations of its Lie algebra are

$$[J_i, J_j] = i\epsilon_{ijk} J_k \,, \tag{A1a}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k \,, \tag{A1b}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k \,, \tag{A1c}$$

$$[K_i, P_j] = i\delta_{ij} P_0 \,, \tag{A1d}$$

$$[K_i, P_0] = iP_i \,, \tag{A1e}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \,, \tag{A1f}$$

where J_i, K_i, P_i, and P_0 generate space rotations, boosts, space translations, and time translations, respectively.

We may define the action of parity on the generators as

$$J_i^P = PJ_iP^{-1} = J_i \,, \tag{A2a}$$

$$K_i^P = PK_iP^{-1} = -K_i \,, \tag{A2b}$$

$$P_i^P = PP_iP^{-1} = -P_i \,, \tag{A2c}$$

$$P_0^P = PP_0P^{-1} = P_0 \,. \tag{A2d}$$

We now demand that the Lie algebra (A1) be preserved under the action of P given by (A2). This may be done in two ways, which we shall illustrate by means of (A1d).

1. P linear

If P is linear then we have

$$P[K_i, P_j]P^{-1} = i\delta_{ij} PP_0P^{-1} \tag{A3}$$

$$= i\delta_{ij} P_0 \tag{A4}$$

$$= [K_i, P_j] \tag{A5}$$

$$= [K_i^P, P_j^P] \tag{A6}$$

since both K_i and P_j change sign under P.

2. P antilinear

If P is antilinear then we have $P(\alpha O)P^{-1} = \alpha^* POP^{-1}$ and so

$$P[K_i, P_j]P^{-1} = -i\delta_{ij} PP_0P^{-1} \tag{A7}$$

$$= -i\delta_{ij} P_0 \tag{A8}$$

$$= -[K_i, P_j] \tag{A9}$$

$$= -[K_i^P, P_j^P] \tag{A10}$$

$$= [P_j^P, K_i^P] \,. \tag{A11}$$

Hence if P is antilinear then it must also reverse operator products if it is to preserve (A1d):

$$P(AB)P^{-1}=(PBP^{-1})(PAP^{-1})\,. \qquad \text{(A12)}$$

Identical results are obtained for the rest of the relations (A1).

It is this (A,R) type of realization which was presented in Sec. X of the text.

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*Work supported by the National Science Foundation and by the U. S. Atomic Energy Commission under Contract No. AEC(40-1) 3992.

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[1]We use the metric $g_{\mu\nu}=\text{diag}(1,-1,-1,-1)$; m is assumed to be a positive real number and β_μ are finite-dimensional square matrices with constant coefficients.

[2]See, for example, Harish-Chandra, Proc. R. Soc. A192, 195 (1947).

[3]We take * to be the complex conjugate of c-number quantities and the Hermitian conjugate of field operators. The † will further signify the transposition of matrix indices, $A^\dagger = A^{*T}$.

[4]See, for example, I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon, New York, 1963), p. 206.

[5]M. A. Naimark, *Linear Representations of the Lorentz Group* (Pergamon, New York, 1964), p. 390.

[6]For equations of the form (1.1), $D^{(s)}(R)$ must actually occur twice among the independent components.

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[19]This means that Eq. (2.1) implies that $\phi(x)$ satisfies the Klein-Gordon equation componentwise for a unique value of m.

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[21]See W. J. Hurley and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 85, 546 (1974) for a study of the representations of this algebra.

[22]See, for example, M. E. Rose, *Elementary Theory of Angular Momemtum* (Wiley, New York, 1957), p. 54.

[23]See, e.g., S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), p. 58.

[24]Putting a differential operator into the bilinear form is not a radical move. Aside from the usual Klein-Gordon spin-0 theory see, e.g., Refs. 8 and 9 for the representations considered there. See also, e.g., Y. Takahashi, *An Introduction to Field Quantization* (Pergamon, New York, 1969) for a general discussion. However, the usual treatments generally restrict their attention to self-conjugate representations and introduce differential operators only in the case of wave equations of order greater than 1. The reader who wishes to relate the present work to such general discussions is therefore cautioned to make the appropriate adjustments.

[25]We could have considered expansions which are more general than (2.6) and (4.3) and imposed causality in order to determine the necessity for the proper spin-statistics relation and crossing symmetry. However, such a procedure would essentially reproduce the elegant arguments presented by S. Weinberg (cf. Ref. 8, Sec. IV) and so we merely chose the coefficients with some foresight.

[26]For a discussion of the phases associated with the discrete symmetries see R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience, New York, 1961). See also G. Feinberg and S. Weinberg, Nuovo Cimento 14, 571 (1959).

[27]See Ref. 21, Appendix A.

[28]Note that the order of the β algebra is the same no matter how high the spin. The present theories therefore are outside of the domain of validity of the results of H. Umezawa and A. Visconti, Nucl. Phys. 1, 348 (1956).

[29]See, e.g., R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964), p. 17.