Semiclassical representation of nonrelativistic quantum electrodynamics

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It has been shown that the solution of the quantum-electrodynamic problem for a strong field can be determined according to a known solution of the corresponding semiclassical problem. Series expansions for the operators in the Heisenberg representation according to deviation of semiclassical results from quantum-electrodynamic results have been written.

I. INTRODUCTION

This paper is the conclusion of the investigation of the connection between nonrelativistic quantum electrodynamics and the semiclassical approximation begun in Refs. 1 and 2, in which the derivation of the semiclassical approximation from quantum electrodynamics has been described. It follows from these papers that there is a possibility in principle of recording the operators in the Heisenberg representation, in the case of strong fields, in terms of the fields determined by semiclassical electrodynamics. In this paper we want to show how one can find an explicit form for such a record for the operators of a medium and a field in the Heisenberg representation and for the density matrix of the medium interacting with a quantum field. For these values we obtain the series expansions according to the deviation of semiclassical results from the exact ones that allow one to obtain the solutions of quantum electrodynamics at large field occupation numbers $n_K \gg 1$ according to solutions of the semiclassical problem.

In Sec. II we obtain equations determining the evolution operator $g(t)$ (the symbols coincide with those of Refs. 1 and 2 in terms of the semiclassical evolution operator $\tilde{G}$). The equation for the unitary operator U is written in which all quantum corrections for the operator $\tilde{G}$ are taken into account.

In Sec. III the operators of a medium and a field in the Heisenberg representation are written in terms of solutions of semiclassical electrodynamics. A semiclassical estimate of the quantum-electrodynamic density matrix $R(t)$ and reduced density matrices of a medium and a field is given.

II. EVOLUTION OPERATOR

Previously,[1,2] it has been shown that the quantum-electrodynamic evolution operator $g(t)$ satisfying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} g(t) = H_{\mathrm{OR}}(t) g(t), \quad g(0) = 1$$
$$H_{\mathrm{OR}}(t) = -\sum_j \frac{e_j}{m_j c} \vec{\mathrm{P}}(t)\cdot \vec{\mathrm{A}}^{\mathrm{op}}(\vec{\mathrm{r}}_j \cdot t) \equiv \vec{\mathrm{P}}(t)\cdot[\vec{\mathrm{A}}_0^{\mathrm{op}}(t) + \Delta\vec{\mathrm{A}}(t)] \tag{1}$$

can be represented as $g(t) = G(t)Q(t)$, where $G(t)$ depends only on the operators of the classical field $\vec{\mathrm{A}}_0^{\mathrm{op}}$ and is determined by Eqs. (22) from Ref. 2, and $Q(t)$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t} Q(t) = G^{-1}\vec{\mathrm{P}}G\cdot\Delta\vec{\mathrm{A}}(t)Q(t). \tag{2}$$

It has been shown that there is a one-to-one correspondence between definite terms G and the semiclassical evolution operator $\tilde{G}$, and in fulfilling the requirements determined in Ref. 2, $G = \tilde{G}$, where $\tilde{G}$ satisfies the equation

$$i\hbar \frac{\partial}{\partial t}\tilde{G} = \vec{\mathrm{P}}\cdot\langle \vec{\mathrm{A}}_M^{\mathrm{op}}(t)\rangle_x \tilde{G},$$
$$\langle \vec{\mathrm{A}}_M^{\mathrm{op}}(t)\rangle_x = \vec{\mathrm{A}}_0^{\mathrm{op}}(t) + \frac{i}{\hbar}\int_0^t \langle \tilde{G}(\tau)[H_{\mathrm{OR}}(\tau), \vec{\mathrm{A}}^{\mathrm{op}}(t)]\tilde{G}(\tau)\rangle_x \, d\tau, \tag{3}$$

where $\langle \cdots \rangle_x$ denotes the averaging over the initial density matrix of medium T. The eigenvalue (3) in the basis of eigenfunctions of the vector potential gives essentially semiclassical electrodynamics.

From Eqs. (1) and (2) it follows that the nonunitary operator satisfies the equation

$$i\hbar \frac{\partial}{\partial t} G = \vec{\mathrm{P}}(t)\cdot[\vec{\mathrm{A}}_0^{\mathrm{op}}(t) + \Delta\vec{\mathrm{A}}(t) - G(t)\Delta\vec{\mathrm{A}}(t)G^{-1}(t)]G(t). \tag{4}$$

It can be seen from Eqs. (3) and (4) that the operator G can be written as $G = \tilde{G}\theta$, where θ describes evolution of "classical" noise corrections and is defined by the equation

$$i\hbar \frac{\partial}{\partial t}\theta = \tilde{G}^{-1}(t)\vec{P}\tilde{G}(t)\cdot[\vec{A}_0^{op}(t)+\tilde{G}^{-1}\Delta\vec{A}(t)\tilde{G} - \langle\vec{A}_M^{op}(t)\rangle_x - \theta\Delta\vec{A}(t)\theta^{-1}]\theta(t). \quad (5)$$

Now we can introduce the operator $U=\theta Q$, determined by the evolution of all quantum corrections for the semiclassical approximation. The equation for U follows from the definition and Eqs. (3) and (5):

$$i\hbar\frac{\partial}{\partial t}U(t)=\tilde{G}^{-1}\vec{P}\tilde{G}\cdot[\vec{A}_0^{op}(t)+\tilde{G}^{-1}(t)\Delta\vec{A}(t)\tilde{G}(t) - \langle\vec{A}_M^{op}(t)\rangle_x]U. \quad (6)$$

Using Eq. (3) one can write Eq. (6) in the form

$$i\hbar\frac{\partial}{\partial t}U(t)=\tilde{G}^{-1}\vec{P}\tilde{G}\cdot[\vec{A}_M^{op}(t)-\langle\vec{A}_M^{op}(t)\rangle_x+\vec{N}(t)+\Delta\vec{A}(t)]U(t),$$
$$\vec{N}(t)=\left(\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau\sum_{\alpha=1}^{3}\tilde{G}^{-1}(\tau)P_\alpha\tilde{G}(\tau)[\langle\tilde{G}^{-1}(\tau_1)[H_{OR}(\tau_1),A_\alpha^{op}(\tau)]\tilde{G}(\tau_1)\rangle_x,\Delta\vec{A}(t)]\tilde{G}(\tau)\,d\tau\,d\tau_1, \quad (7)$$

where $\alpha=1,2,3$ numbers the orthogonal unit vectors of the coordinate system. Using Eqs. (6) and (7), one can calculate the evolution operator by use of the solution of an appropriate semiclassical problem and interpreting classical fields as eigenvalues of operators of classical field.

III. OPERATORS IN THE HEISENBERG REPRESENTATION

The operator of a quantum medium in the Heisenberg representation is

$$B_H(t)=U^{-1}(t)\tilde{G}^{-1}(t)B(t)\tilde{G}(t)U(t),$$

whence and from Eq. (7) follows the expansion

$$B_H(t)=B_H^0(t)+\frac{i}{\hbar}\int_0^t[\vec{P}_H^0(\tau)\cdot(\vec{A}_M^{op}(\tau)-\langle\vec{A}_M^{op}(\tau)\rangle_x+\vec{N}(\tau)+\Delta\vec{A}(\tau)),B_H^0(t)]\,d\tau+\cdots,\quad B_H^0(t)=\tilde{G}^{-1}B(t)\tilde{G}(t). \quad (8)$$

For $n_K\gg 1$ the terms with $\Delta\vec{A}$ carried out to the right may be neglected[2] and Eq. (8) is rewritten

$$B_H(t)_{n_K\gg1}=B_H^0(t)+\frac{i}{\hbar}\int_0^t[\vec{P}_H^0(\tau),B_H^0(t)]\cdot[\vec{A}_M^{op}(\tau)-\langle\vec{A}_M^{op}(\tau)\rangle_x+\vec{N}(\tau)]\,d\tau$$
$$-\left(\frac{i}{\hbar}\right)^2\int_0^t\int_\tau^t\vec{P}_H^0(\tau)\cdot\sum_{\alpha=1}^{3}[\langle A_{\alpha M}^{op}(\tau_1)\rangle_x,\Delta\vec{A}(\tau)][P_{\alpha H}^0(\tau_1),B_H^0(t)]\,d\tau\,d\tau_1+\cdots. \quad (9)$$

The field operator $\vec{F}_H(t)$ can be similarly represented:

$$\vec{F}_H(t)_{n_K\gg1}=\vec{F}_0^{op}(t)+\frac{i}{\hbar}\int_0^t\sum_{\alpha=1}^{3}P_{\alpha H}^0(\tau)[A_\alpha^{op}(\tau),\vec{F}^{op}(t)]\,d\tau$$
$$+\left(\frac{i}{\hbar}\right)^2\int_0^t\int_0^\tau\sum_{\alpha,\beta=1}^{3}[P_{\alpha H}^0(\tau_1),P_{\beta H}^0(\tau)][A_\beta^{op}(\tau),\vec{F}^{op}(t)][A_{\alpha M}^{op}(\tau_1)-\langle A_{\alpha M}^{op}(\tau_1)\rangle_x+N_\alpha(\tau_1)]\,d\tau_1\,d\tau$$
$$-\left(\frac{i}{\hbar}\right)^3\int_0^t\int_0^\tau\int_{\tau_1}^{\tau}\sum_{\alpha,\beta,\gamma=1}^{3}P_{\alpha H}^0(\tau_1)[P_{\beta H}^0(\tau_2),P_{\gamma H}^0(\tau)][\langle A_{\beta M}^{op}(\tau_2)\rangle_x,\Delta A_\alpha(\tau_1)][A_\gamma^{op}(\tau),\vec{F}^{op}(t)]\,d\tau_2\,d\tau_1\,d\tau+\cdots.$$

Averaging Eqs. (9) and (10) over the initial density matrix we obtain the possibility of determining appropriate physical values according to the solution of the semiclassical problem.

Likewise we can write the density matrix

$$R(t)=g(t)\rho(t)T(t)g^{-1}(t),$$

where $g(t)$ is the evolution operator satisfying Eq. (1), and $\rho(t)$, $T(t)$ are the initial matrices of a field and a medium in the interaction representation:

$$R(t)=\tilde{G}(t)\left\{\rho(t)T(t)-\frac{i}{\hbar}\int_0^t[\vec{P}_H^0(\tau)\cdot(\vec{A}_M^{op}(\tau)-\langle\vec{A}_M^{op}(\tau)\rangle_x+\vec{N}(\tau)+\Delta\vec{A}(\tau)),\rho(\tau)T(\tau)]\,d\tau+\cdots\right\}\tilde{G}^{-1}(t). \quad (11)$$

Averaging Eq. (11) over the states of medium or field, one can obtain the density matrix of a field and a medium, respectively.

Formulas (9)–(11) allow the results of quantum electrodynamics to be obtained according to a known solution of the semiclassical problem. Interpreting semiclassical fields as eigenvalues of operators of classical field and substituting them into formulas (9)–(11), we can find the values of interest without solving the quantum-electrodynamic Schrödinger equation (1).

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[2]E. P. Gordov and S. D. Tvorogov, Phys. Rev. D 9. 1711 (1974).