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Wave functions for free, massless particles and nonintegrable representations of sl(2,c)

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A class of representations of the Lie algebra sl(2, c) is discussed. These representations are operator-irreducible and su(2)-integrable, but are not all integrable to representations of the group SL(2, c). Free, massless fields and wave functions belonging to such index-space representations are considered, and the possible invariant helicities are given in each case. It is shown that the nonintegrability of the sl(2, c) representations does not preclude the possibility of realizing the appropriate unitary representations of the Poincaré group ISL(2, c). This permits the removal of discrepancies between the results of Bender, Frishman *et al.*, and Simon *et al.* In particular, it is concluded that the free electromagnetic potential, in the radiation gauge, belongs to an infinite-dimensional representation of sl(2, c) which is not integrable. The relationships between free, massless fields and wave functions, having the same invariant helicity but belonging to different representations of sl(2, c), are discussed, and the corresponding generalization of a result due to Weinberg is obtained.

I. INTRODUCTION

Elementary particles are commonly described in terms of multicomponent fields or wave functions $\psi(x)$, which belong to finite-dimensional representations of the group SL(2, c). In the "index space" of those ψ belonging to a given such representation there acts a set of operators (matrices) $S_{\mu\nu}=-S_{\nu\mu}$, satisfying the commutation relations characteristic of sl(2, c), the Lie algebra of SL(2, c), viz.[1]

$$i[S_{\mu\nu}, S_{\rho\sigma}]=g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\nu\rho}S_{\mu\sigma}-g_{\mu\sigma}S_{\nu\rho}. \tag{1}$$

These $S_{\mu\nu}$ generate in the index space the corresponding representation of SL(2, c).

Attention has been focussed in recent years on infinite-component fields and wave functions.[2] In such cases, one must be careful to distinguish between a representation of the Lie algebra, and one of the group, in the index space. One can conceive of functions belonging to a representation of sl(2, c) which is not integrable to a representation of SL(2, c). The index space might still be a Hilbert space, and contain a dense subspace which is in the domain of definition of each of a set of operators $S_{\mu\nu}$, which is invariant under the action of those $S_{\mu\nu}$, and on which the commutation relations (1) are satisfied. The existence of such a common, invariant, dense domain is not sufficient to guarantee that the $S_{\mu\nu}$ be integrable.[3]

Can fields and wave functions belonging to nonintegrable representations of sl(2, c) be used to describe elementary particles? For such an application, one seeks a Hilbert space H of wave functions, carrying some particular unitary representation of the Poincaré group ISL(2, c), generated by the operators

$$P_\mu=i\partial/\partial x^\mu,$$
$$J_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu+S_{\mu\nu}. \tag{2}$$

These operators must form a representation in H of isl(2, c), integrable to the particular unitary representation of ISL(2, c) of interest. However, if such a space H can be found, the wave functions in it (or at least in a dense subspace of it) will satisfy one or more wave equations, which may be thought of as projecting onto H a much larger space, the direct product of the index space and some space of (scalar) functions of the space-time coordinates. Then it is conceivable that the integrability of P_μ and $J_{\mu\nu}$ in H does *not* always require the integrability of the $S_{\mu\nu}$ in the index space.

We shall show that this observation provides the clue to the resolution of an apparent paradox which has arisen as a result of recent studies, by several people, of the description of free, massless particles in terms of infinite-component fields and wave functions.

Let us suppose that we are given wave functions ψ belonging to an index-space representation of sl(2, c), not necessarily integrable, and are required to describe a free, massless particle with invariant helicity λ, where 2λ is integral. Then we seek a Hilbert space of such functions, carrying the irreducible, unitary representation[4] O_{λ^+} of ISL(2, c). As Bargmann and Wigner[5] have shown, the wave equations to be satisfied in this case are

$$P_\mu P^\mu\psi=0,$$
$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma\psi=\lambda P_\mu\psi. \tag{3}$$

The second equation reduces to

$$\tilde{S}_{\mu\nu} P^{\nu} \psi = \lambda P_{\mu} \psi , \tag{4}$$

where $\tilde{S}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} S^{\rho\sigma}$. We look for (positive energy) solutions in the form

$$\psi(x) = (2\pi)^{-3/2} \int d^3p\, E^{-1/2} e^{ip\cdot x}\, \phi(\vec{p}), \tag{5}$$

where $p^{\mu} = (E, \vec{p})$, $E = |\vec{p}|$, and $p\cdot x = p_{\mu}x^{\mu}$, and we find that Eqs. (4) are satisfied provided

$$\tilde{S}_{\mu\nu} p^{\nu} \phi = \lambda p_{\mu} \phi . \tag{6}$$

For a given value of λ, one cannot proceed further if the index-space representation of sl(2, c) is such that there are no (nontrivial) solutions of Eqs. (6). One may ask for which representations there *are* Hilbert spaces of solutions, carrying the representation O_{λ^+}, but we prefer to tackle the converse questions:

(Q1) Given an index-space representation of sl(2, c), for which values of λ are there solutions to Eqs. (6)?

(Q2) For which of these are there corresponding Hilbert spaces carrying the representation O_{λ^+} of ISL(2, c)?

We shall tackle these questions for a certain class C of Hilbert-space representations of sl(2, c), which we may call operator-irreducible and su(2)-integrable. Let R be a typical element of C, and $S_{\mu\nu}$ the operators of R in a corresponding Hilbert space K_R. Then we require that there is a subspace D_R, dense in K_R, lying in the domain of each $S_{\mu\nu}$, and invariant under the action of the $S_{\mu\nu}$. Secondly, we require that the only continuous operators on K_R which commute on D_R with all $S_{\mu\nu}$ are the scalar multiples of the identity operator I. Finally, we require that the (reducible) representation of su(2) formed by the operators $\vec{S} = (S_{23}, S_{31}, S_{12})$ is integrable to a representation of SU(2), and that every (finite-dimensional) SU(2)-irreducible subspace of K_R is contained in D_R.

Then it can be deduced that[6] the Casimir invariants

$$\begin{aligned} G_1 &= \tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = (k_0{}^2 + c^2 - 1) I , \\ G_2 &= \tfrac{1}{4} S_{\mu\nu} \tilde{S}^{\mu\nu} = i k_0 c I , \end{aligned} \tag{7}$$

are (on D_R) multiples of I, with $2k_0$ a non-negative integer, and c a complex number. Furthermore, the representation of SU(2) is of the form

$$(k_0) \oplus (k_0 + 1) \oplus \cdots , \tag{8}$$

where (s) denotes the $(2s+1)$-dimensional irreducible representation of SU(2), on which $\vec{S}^2 = s(s+1)I$. Thus

$$D_R \supset D(s), \quad s = k_0, k_0 + 1, \ldots , \tag{9}$$

where $D(s)$ denotes the representation space for (s). It can also be deduced that, if $k_0 - c$ is nonintegral, or if $|c| \le k_0$, the series (8) is infinite. Then R may be *partially* labeled $\{k_0, c\}$. If $k_0 - c$ is integral, with $|c| > k_0$, any one of three cases can occur. In the first, the series is finite, terminating with $(|c| - 1)$. Then R, which may be *completely* specified by $\{k_0, c\}$ to within equivalence, is finite-dimensional and, necessarily, integrable.

In the second case, the series is infinite, with the subspace

$$D(k_0) \oplus D(k_0 + 1) \oplus \cdots \oplus D(|c| - 1) ,$$

but not its complement, being invariant under the action of the $S_{\mu\nu}$. Then we *partially* label R by $\{k_0 \leftarrow c\}$, to indicate that, under the action of the algebra, a vector in $D(s)$, $s \ge |c|$, can be carried into $D(s')$, $s' < |c|$, but not vice versa. In the third case, the series is again infinite, but now the linear span of the subspaces $D(|c|)$, $D(|c|+1), \ldots$ forms the *(unclosed)* invariant subspace of D_R, and R may be *partially* labeled by $\{k_0 \rightarrow c\}$.

It is well known[6] that there are, for the group SL(2, c), irreducible representations $[k_0, c]$, and operator-irreducible (or "integer point") representations $[k_0 \leftarrow c], [k_0 \rightarrow c]$, whose associated representations of sl(2, c) have all the properties of the representations $\{k_0, c\}$, $\{k_0 \leftarrow c\}$, and $\{k_0 \rightarrow c\}$, respectively, as described so far. However, it must be stressed that the infinite-dimensional representations of sl(2, c) cannot be specified completely by the values of the Casimir invariants (7). The operators $S_{\mu\nu}$ are not completely defined until one specifies their domains, and the property (9) does not do that.

For given k_0 and c, with $k_0 - c$ nonintegral, for example, there are many representations $\{k_0, c\}$ with all the properties described above, but distinguished by the variation in the domains of the $S_{\mu\nu}$. For that representation of sl(2, c) defined by the representation $[k_0, c]$ of SL(2, c), these domains will be determined, and any $\{k_0, c\}$ for which the domains differ from these, must be nonintegrable.

In Sec. II, we complete the definition of the representations of sl(2, c) in C. For the rest of this section, we shall refer to representations in C "of the type" $\{k_0, c\}$, $\{k_0 \leftarrow c\}$, and $\{k_0 \rightarrow c\}$.

To return to the questions (Q1), (Q2), we note that Weinberg[7] has answered (Q1) for all finite-dimensional representations of sl(2, c). For fields or wave functions belonging to the finite-dimensional irreducible representation $\{k_0, c\}$, he found that the only possible invariant helicity is

$$\lambda = k_0\, \mathrm{sgn}(c) . \tag{10}$$

Furthermore, it is not difficult to see that a corresponding Hilbert space, carrying the representation O_{λ^+}, can be found in every such case. It follows that in order to describe free, massless particles with any given invariant helicity, it is *possible* to restrict one's attention to finite-dimensional representations.

On the other hand, Bender[8] claims to have shown that certain "radiation gauge" fields, and in particular the free electromagnetic potential in the radiation gauge, belong to infinite-dimensional irreducible representations of SL(2, c). He has described fields with integral helicity $\lambda \neq 0$, belonging to $[\,|\lambda|, \mathrm{sgn}(\lambda)]$; and with half-odd-integral helicity λ ($|\lambda| \geq \frac{3}{2}$) belonging to $[\,|\lambda|, \frac{3}{2}\mathrm{sgn}(\lambda)]$.

Following upon his work, Frishman *et al.*[9] have considered question (Q1) for an arbitrary infinite-dimensional representation $[k_0, c]$, and have concluded that the possible values of λ are k_0 and $-k_0$ in every case. They have described fields with these invariant helicities, in particular reproducing the results of Bender, but have not considered the second question (Q2) for the corresponding wave functions.

More recently, Simon *et al.*[10] have reconsidered (Q1) for wave functions belonging to an arbitrary infinite-dimensional representation $[k_0, c]$, obtaining results which disagree with those of Frishman *et al.* They have concluded that, if the solutions to Eqs. (6) are required to belong to the corresponding representation space for $[k_0, c]$, then the possibilities are (at least for $\lambda \neq 0$)

$$\lambda = \pm k_0, \quad \text{provided } \mathrm{Re}(\pm c) > 1 \tag{11}$$

and

$$\lambda = c, \quad \text{provided } c - k_0 \text{ is integral}$$
$$k_0 \geq |c|, \quad k_0 > 1. \tag{12}$$

A somewhat paradoxical situation arises when one attempts to reconcile these results with those of Bender, since they imply that there are no wave functions with integral helicity $\lambda \neq 0$ in the representation space for $[\,|\lambda|, \mathrm{sgn}(\lambda)]$. However, it is well known in particular that a perfectly good realization of $O_{1^+} \oplus O_{-1^+}$ can be obtained in terms of the radiation-gauge electromagnetic potential, which, according to Bender, belongs to the $s = 1$ component of an element of $[1, 1] \oplus [1, -1]$.

In attempting to understand this rather puzzling situation, we find that Simon *et al.* and Bender have all worked with representations of sl(2, c) rather than SL(2, c). Our analysis in what follows shows that the free electromagnetic potential, for example, may properly be regarded as belonging to the $s = 1$ component of an element of a representation in C of the type $\{1, 1\} \oplus \{1, -1\}$. This representation is inequivalent to that discussed by Simon *et al.*, whose results therefore do not apply, and it is not integrable to the representation $[1, 1] \oplus [1, -1]$. However, this nonintegrability does not remove the possibility of realizing the representation $O_{1^+} \oplus O_{-1^+}$ in a Hilbert space of such wave functions, as we shall show.

In Sec. III, we answer the question (Q1) for the representations of sl(2, c) in the class C, and find that

(a) for $k_0 - c$ nonintegral, representations of the type $\{k_0, c\}$ can be found such that solutions to Eqs. (6) exist for $\lambda = k_0$ and $\lambda = -k_0$;

(b) for $k_0 - c$ integral, with $|c| \leq k_0$, representations $\{k_0, c\}$ can be found such that the possibilities are[11] $\lambda = k_0$, $-k_0$, and $-c$;

(c) for $k_0 - c$ integral, with $|c| > k_0$, (i) representations $\{k_0 \leftarrow c\}$ can be found such that the possibilities are $\lambda = k_0$ and $-k_0$, and (ii) representations $\{k_0 \rightarrow c\}$ can be found such that the possibilities are $\lambda = k_0$, $-k_0$, c, and $-c$.

As regards the second question (Q2), we show in Sec. IV that for each value of k_0, c, and λ in (a)–(c) above, a Hilbert space of the corresponding wave functions can be found, carrying the representation O_{λ^+}. [There is one possibly exceptional case, arising when the representation of sl(2, c) is of the type $\{0, 0\}$.]

Weinberg[7] has shown that a field or wave function $\psi(x)$ with invariant helicity λ, belonging to the finite-dimensional representation $\{\,|\lambda|, (|\lambda|+n+1) \times \mathrm{sgn}(\lambda)\}$, where n is a positive integer, can be regarded as the nth space-time derivative of a $\psi'(x)$ belonging to $\{\,|\lambda|, (|\lambda|+1)\,\mathrm{sgn}(\lambda)\}$, and having the same helicity. In Sec. V, we generalize this result to obtain the relationships between fields or wave functions with the same helicity, but belonging to various index space representations of sl(2, c).

II. REPRESENTATIONS OF sl(2,c)

We shall now complete the definition of the infinite-dimensional representations of the type $\{k_0, c\}$, $\{k_0 \leftarrow c\}$, and $\{k_0 \rightarrow c\}$ in the class C. (The finite-dimensional ones are well known.)

In accordance with (8), we introduce a Hilbert space $K(k_0)$, and in it, a complete orthonormal set of SU(2) basis vectors $\xi_{s,m}$, where m runs over the values $s, s-1, \ldots, -s$ for each value of s in the set $\{k_0, k_0+1, \ldots\}$. The action of the operators $\vec{S}$ in such a basis is well known. We have

$$\begin{aligned} \vec{S}^2 \xi_{s,m} &= s(s+1)\xi_{s,m}, \\ S_3 \xi_{s,m} &= m\xi_{s,m}, \\ S_\pm \xi_{s,m} &= [(s \pm m + 1)(s \mp m)]^{1/2} \xi_{s,m\pm 1}, \end{aligned} \tag{13}$$

where $S_\pm = S_1 \pm iS_2$. From the requirement that the commutation relations (1) be satisfied on D_R, we deduce that the action of the remaining operators $T_3=S_{30}$, $T_\pm=S_{10}\pm iS_{20}$, is of the form[6]

$$\begin{aligned} T_3\xi_{s,m} &= D_s^{(-)}[s^2-m^2]^{1/2}\xi_{s-1,m}+mD_s^{(0)}\xi_{s,m} \\ &\quad -D_s^{(+)}[(s+1)^2-m^2]^{1/2}\xi_{s+1,m}, \\ T_\pm\xi_{s,m} &= \pm D_s^{(-)}[(s\mp m)(s\mp m-1)]^{1/2}\xi_{s-1,m\pm1} \\ &\quad +D_s^{(0)}[(s\mp m)(s\pm m+1)]^{1/2}\xi_{s,m\pm1} \\ &\quad \pm D_s^{(+)}[(s\pm m+1)(s\pm m+2)]^{1/2}\xi_{s+1,m\pm1}, \end{aligned} \tag{14}$$

where

$$D_s^{(0)}=ik_0c/s(s+1) \tag{15}$$

and the coefficients $D_s^{(-)}$ and $D_s^{(+)}$ satisfy

$$D_s^{(-)}D_{s-1}^{(+)}=(s^2-k_0^2)(c^2-s^2)/s^2(4s^2-1). \tag{16}$$

Suppose we take, in the case when either k_0-c is nonintegral, or $|c|\le k_0$,

$$\begin{aligned} D_s^{(-)} &= \frac{i[(s^2-k_0^2)(s^2-c^2)]^{1/2}}{s\tau(s)(4s^2-1)^{1/2}}, \\ D_{s-1}^{(+)} &= \frac{i\tau(s)[(s^2-k_0^2)(s^2-c^2)]^{1/2}}{s(4s^2-1)^{1/2}}, \end{aligned} \tag{17}$$

where $\tau(s)$ is a finite nonzero complex number for each $s\in\{k_0,k_0+1,\ldots\}$. Then it is readily checked that a representation of sl(2, c) satisfying Eqs. (13)–(17) has all the properties of a representation in C of the type $\{k_0,c\}$, as described in the Introduction. However, Eqs. (13)–(17) do not specify a representation completely, since they do not completely define the domains of $\vec{S}$ and $\vec{T}$. According to the known theory of SL(2, c), it is always possible, for given k_0 and c, and *arbitrary* $\tau(s)$, to choose these domains in such a way that the representation of sl(2, c) so defined is integrable to the representation $[k_0,c]$ of SL(2, c). However, we shall define the representation $\{k_0,c,\tau\}$ of sl(2, c) in C by *specifying* the domains of $\vec{S}$ and $\vec{T}$ to be the largest possible,[12] consistent with Eqs. (13)–(17). Then if we write Eqs. (13)–(14) in the form

$$S_{\mu\nu}\xi_{s,m}=\sum_{s',m'}(\alpha_{\mu\nu})_{sm;s'm'}\xi_{s',m'},$$

and if

$$\phi=\sum_{s,m}A_{s,m}\xi_{s,m}, \quad \sum_{s,m}|A_{s,m}|^2<\infty$$

we can say that ϕ is in the domain of $S_{\mu\nu}$ if and only if

$$\sum_{s',m'}\left|\sum_{s,m}A_{s,m}(\alpha_{\mu\nu})_{s,m;s',m'}\right|^2<\infty.$$

With this definition, representations $\{k_0,c,\tau\}$ with the same k_0 and c, but different $\tau(s)$, cannot all be regarded as equivalent, and so cannot all be integrable to the representation $[k_0,c]$ of SL(2, c). For example, consider the representations $\{0,0,\tau\}$. The vector

$$\phi=\sum_{s=0}^{\infty}A_s\xi_{s,0},$$

$$A_s=(-1)^s(2s+1)^{1/2}\tau(s)\tau(s-1)\cdots\tau(0),$$

belongs to $K(0)$ for *some* $\tau(s)$, say $\tau(s)=(s+1)^2/(s+2)^2$. Then it is an eigenvector of T_3, corresponding to the eigenvalue $-i$. But there is no such eigenvector in $K(0)$ for, say, $\tau(s)=1$, when T_3 is self-adjoint, as it must be if $\{0,0,\tau\}$ is to be integrable to the *unitary* representation $[0,0]$.

It is natural then to ask which, if any, of the $\{k_0,c,\tau\}$ are integrable to $[k_0,c]$. In the cases with c pure imaginary, or with $k_0=0$, $0\le c\le 1$, it would seem that one can answer this question. Then $[k_0,c]$ is unitary, and the $S_{\mu\nu}$ in $\{k_0,c,\tau\}$ are all self-adjoint only if

$$\tau(s)^*\tau(s)=1, \quad s\in\{k_0,k_0+1,\ldots\}. \tag{18}$$

What is more, if $S_{\mu\nu}$ and $S'_{\mu\nu}$ are the operators in $\{k_0,c,\tau\}$ and $\{k_0,c,\tau'\}$, where τ and τ' both satisfy (18), then it is easily seen that $S_{\mu\nu}$ and $S'_{\mu\nu}$ are related by a unitary transformation. Then one may suppose that, if $[k_0,c]$ is unitary, the representation $\{k_0,c,\tau\}$ is integrable provided the condition (18) is satisfied. In the cases when $[k_0,c]$ is nonunitary, it is plausible that again $\{k_0,c,\tau\}$ is integrable for some $\tau(s)$ but not others. However, it is not clear that $\{k_0,c,1\}$, in particular, is integrable, as Simon *et al.*[10] have assumed. We shall not attempt to determine which, if any, of the $\{k_0,c,\tau\}$ are integrable. However, from what we have said so far, we can assert that the domains of $\vec{S}$ and $\vec{T}$ of *any* $\{k_0,c,\tau\}$ are at least as large as (i.e., include) those of a representation integrable to $[k_0,c]$. Therefore, if we find an eigenvector of T_3, say, in $\{k_0,c,\tau_1\}$, for which there is no corresponding eigenvector in *every* $\{k_0,c,\tau_2\}$, we can conclude that $\{k_0,c,\tau_1\}$ is not integrable. We use this argument in Sec. IV when deducing that Bender's integer-spin radiation-gauge potentials belong to nonintegrable representations of sl(2, c).

We shall henceforth retain the labeling $\{k_0,c,\tau\}$ for the representation of sl(2, c) defined by Eqs. (13)–(17), with the described assumption about the domains of $\vec{S}$ and $\vec{T}$. This labeling is clumsy to the extent that it introduces an unimportant distinction between representations which should properly be regarded as equivalent, but at least it distinguishes ones which are clearly inequivalent.

So far we have not considered the representations in C of the type $\{k_0\text{-}c\}$ and $\{k_0\text{-}c\}$, which arise when k_0-c is integral, with $|c|>k_0$. In the case of one of the type $\{k_0\text{-}c\}$, the action of the

$S_{\mu\nu}$ in $K(k_0)$ is as in Eqs. (13)–(16), but now

$$D_s^{(-)} = \frac{i(s+|c|)(s^2-k_0^2)^{1/2}}{s(2s-1)\tau(s)}\,, \qquad D_{s-1}^{(+)} = \frac{i(s-|c|)(s^2-k_0^2)^{1/2}\tau(s)}{s(2s+1)}\,, \tag{19}$$

with $\tau(s)$ as before. Then, as required, the subspace

$$D(k_0)\oplus D(k_0+1)\oplus\cdots\oplus D(|c|-1),$$

but not its complement, is invariant. Again, the domains of $\vec{S}$ and $\vec{T}$ are taken to be the largest possible, consistent with Eqs. (13)–(16), (19), and the representation is labeled $\{k_0 \leftarrow c, \tau\}$. Not all such representations, for given k_0 and c, can be regarded as equivalent, nor can they all be integrable to $[k_0 \leftarrow c]$.

Similar remarks apply in the case of the representation $\{k_0 \rightarrow c, \tau\}$, where the action of the $S_{\mu\nu}$ is as in Eqs. (13)–(16), with

$$D_s^{(-)} = \frac{i(s-|c|)(s^2-k_0^2)^{1/2}}{s(2s-1)\tau(s)}\,, \qquad D_{s-1}^{(+)} = \frac{i(s+|c|)(s^2-k_0^2)^{1/2}\tau(s)}{s(2s+1)}\,. \tag{20}$$

In Sec. IV, we shall need to know the matrix elements of $S_{\mu\nu}$ in a different basis, for each of the representations $\{k_0, c, \tau\}$, $\{k_0 \leftarrow c, \tau\}$, and $\{k_0 \rightarrow c, \tau\}$. With the adoption of this basis, a general element χ of the space $K(k_0)$ is written as an infinite-component object

$$\chi = (\chi^{(k_0)}, \chi^{(k_0+1)}, \ldots). \tag{21}$$

If k_0 is integral, $\chi^{(s)} \equiv \chi^{(s)}_{ij\cdots k}$ is completely symmetric and traceless in its s three-vector subscripts $i, j, \ldots, k$. If k_0 is half-odd-integral, $\chi^{(s)} \equiv \chi^{(s)}_{ij\cdots k}$ is completely symmetric and traceless in its $s-\frac{1}{2}$ three-vector subscripts $i, j, \ldots, k$ and in addition is a spinor, satisfying

$$\sigma_i \chi^{(s)}_{ij\cdots k} = 0\,, \tag{22}$$

where σ_i are a set of Pauli matrices, acting in the usual way in the two-dimensional spinor space. The scalar product of two elements χ, χ' of $K(k_0)$ is given by

$$\chi^\dagger\chi' = \sum_{s=k_0}^{\infty} \chi^{(s)*}_{ij\cdots k}\chi'^{(s)}_{ij\cdots k}\,, \tag{23}$$

where $\chi^{(s)*}$ is to be interpreted as the Hermitian conjugate of $\chi^{(s)}$, if $2s$ is an odd integer.

The action of the su(2) operators $\vec{S}$ is now given by

$$(S_r\chi)^{(s)}_{ij\cdots k} = i\epsilon_{rmi}\chi^{(s)}_{mj\cdots k} + i\epsilon_{rmj}\chi^{(s)}_{im\cdots k} + \cdots + i\epsilon_{rmk}\chi^{(s)}_{ij\cdots m} + \tfrac{1}{2}\sigma_r\chi^{(s)}_{ij\cdots k}\,, \tag{24}$$

where the last term is to be omitted if s is integral. For the remaining operators $\vec{T}$, one finds

$$(T_r\chi)^{(s)}_{ij\cdots k} = D^{(-)}_{s+1}[(s+1)(2s+1)]^{1/2}\chi^{(s+1)}_{rij\cdots k} + D^{(0)}_s (S_r\chi)^{(s)}_{ij\cdots k} - D^{(+)}_{s-1}[s(2s-1)]^{-1/2}\rho^{(s-1)}_{rij\cdots k}\,. \tag{25}$$

Here the coefficients $D^{(0)}_s$, $D^{(+)}_s$, and $D^{(-)}_s$ are given by Eqs. (15) and (17), (19), or (20), according as the representation is $\{k_0, c, \tau\}$, $\{k_0 \leftarrow c, \tau\}$, or $\{k_0 \rightarrow c, \tau\}$. The quantity $\rho^{(s-1)}$ is given by

$$\begin{aligned}\rho^{(s-1)}_{rijm\cdots nk} = {} & (2s-1)(\delta_{ri}\chi^{(s-1)}_{jm\cdots nk} + \delta_{rj}\chi^{(s-1)}_{im\cdots nk} + \cdots + \delta_{rk}\chi^{(s-1)}_{ijm\cdots n}) \\ & - 2(\delta_{ij}\chi^{(s-1)}_{rm\cdots nk} + \delta_{im}\chi^{(s-1)}_{rj\cdots nk} + \cdots + \delta_{nk}\chi^{(s-1)}_{rijm\cdots}) \\ & - i\sigma_u(\epsilon_{rui}\chi^{(s-1)}_{jm\cdots nk} + \epsilon_{ruj}\chi^{(s-1)}_{im\cdots nk} + \cdots + \epsilon_{ruk}\chi^{(s-1)}_{ijm\cdots n}),\end{aligned} \tag{26}$$

where again the terms involving Pauli matrices are to be omitted if s is integral.

III. POSSIBLE HELICITIES

In this section, we shall answer the question (Q1), posed in the Introduction, for each representation of sl(2, c) of the type $\{k_0, c, \tau\}$, $\{k_0 \leftarrow c, \tau\}$, and $\{k_0 \rightarrow c, \tau\}$. Note first that, on D_R,

$$(\tilde{S}_{\mu\nu} - k_0 g_{\mu\nu})(\tilde{S}^{\nu\rho} + k_0 g^{\nu\rho})\,(\tilde{S}_{\rho\sigma} - c g_{\rho\sigma})(\tilde{S}^{\sigma\tau} + c g^{\sigma\tau}) \equiv 0\,. \tag{27}$$

This is a consequence[13] of the commutation relations (1), and the assumed form (7) for the invariants G_1 and G_2. It follows at once that nontrivial solutions to Eqs. (6), with ϕ belonging to $\{k_0, c, \tau\}$, $\{k_0 \leftarrow c, \tau\}$, or $\{k_0 \rightarrow c, \tau\}$, cannot be found unless

$$(\lambda^2 - k_0^2)(\lambda^2 - c^2) = 0\,. \tag{28}$$

Furthermore, $\lambda^2 = c^2$ is not possible unless $k_0 - c$ is integral. This is easily seen by considering Eqs. (6) "in the frame"

$$p^\mu = (p, 0, 0, p), \quad p > 0\,, \tag{29}$$

where they are

$$S_{12}\phi = \lambda\phi\,, \qquad (S_{23} + S_{20})\phi = (S_{13} + S_{10})\phi = 0\,. \tag{6$'$}$$

The spectrum of $S_{12} = S_3$ in $K(k_0)$ does not include c or $-c$ unless $k_0 - c$ is integral.

It is convenient in what follows to work with Eqs.

(6′) rather than (6). The integrability of the representation of su(2) provided by $\vec{S}$ ensures that nontrivial solutions can be found for the former set, if and only if they can be found for the latter set, for any $p^\mu = (E, \vec{p})$, with $\vec{p}$ real. The operators S_{12}, $S_{23}+S_{20}$, and $S_{13}+S_{10}$ form a representation of the e(2) subalgebra of sl(2, c), and Eqs. (6′) indicate that ϕ belongs to a one-dimensional representation of this subalgebra. The problem here is therefore that of finding all the one-dimensional representations of e(2) in each of the representations $\{k_0, c, \tau\}$, $\{k_0 - c, \tau\}$, and $\{k_0 - c, \tau\}$. We shall see that the solution depends in each case on the nature of $\tau(s)$, and this again emphasizes the inequivalence of these representations of sl(2, c), for various τ. Note that a one-dimensional representation of e(2) is trivially integrable to a one-dimensional unitary representation of E(2), whether or not the representation of sl(2, c) containing it is integrable.

Following Frishman *et al.*,[9] we note that the first of Eqs. (6′) can be satisfied only if ϕ is of the form

$$\phi = \sum_{s=k_0}^{\infty} A_s \xi_{s,\lambda} \tag{30}$$

(where of course $A_s = 0$ for $s < |\lambda|$). Then the remaining equations in that set can be used to determine the coefficients A_s. These remaining equations can be put in the form

$$(S_\pm \mp iT_\pm)\phi = 0,$$

and, on substitution for $S_\pm$, $T_\pm$ from the formulas (13), (14), we obtain two equations to be satisfied by the coefficients:

$$\begin{aligned} A_{s+1} D_{s+1}^{(-)}[(s \mp \lambda + 1)(s \mp \lambda)]^{1/2} \\ + A_s[\, i \pm D_s^{(0)}][(s \pm \lambda + 1)(s \mp \lambda)]^{1/2} \\ + A_{s-1} D_{s-1}^{(+)}[(s \pm \lambda + 1)(s \pm \lambda)]^{1/2} = 0\,. \end{aligned} \tag{31}$$

Multiplying by $[(s \mp \lambda + 1)(s \mp \lambda)]^{1/2}$, and subtracting one resultant equation from the other, we obtain

$$\begin{aligned} A_{s+1}(2s+1)\lambda D_{s+1}^{(-)} = A_s[\, s D_s^{(0)} - \lambda i] \\ \times [(s+1-\lambda)(s+1+\lambda)]^{1/2}\,. \end{aligned} \tag{32}$$

With the help of this equation, we readily find all the solutions of Eqs. (31) and hence of (6′) in the form (30). We list the results for the various representations of sl(2, c) discussed in Sec. II.

(A) $\{k_0, c, \tau\}$, *$k_0 - c$ nonintegral.*

(i) $\lambda = k_0$.

$$A_{s+1} = -\frac{A_s \tau(s+1)[(2s+3)(s+1-c)]^{1/2}}{[(2s+1)(s+1+c)]^{1/2}}\,, \quad s \geq k_0\,;$$

A_{k_0} arbitrary.

(ii) $\lambda = -k_0$.

$$A_{s+1} = -\frac{A_s \tau(s+1)[(2s+3)(s+1+c)]^{1/2}}{[(2s+1)(s+1-c)]^{1/2}}\,, \quad s \geq k_0\,;$$

A_{k_0} arbitrary.

Note that there are two distinct solutions for $\lambda = 0$, in the cases with $k_0 = 0$. These may be distinguished in the frame (29) by the eigenvalues of iT_3, which are $(1-c)$ on the first, and $(1+c)$ on the second solution, whether or not $k_0 = 0$. In this connection, we note that, in general when $\lambda \neq 0$, Eqs. (6) imply that

$$-iS_{\mu\nu}p^\nu \phi = (1 - k_0 c/\lambda)p_\mu \phi\,, \tag{33}$$

which in the frame (29) yields the equation

$$iT_3\phi = (1 - k_0 c/\lambda)\phi\,. \tag{34}$$

(B) $\{k_0, c, \tau\}$, *$k_0 - c$ integral,* $|c| \leq k_0$, $k_0 \neq 0$.

(i), (ii) as for (A) above.

(iii) $\lambda = -c$.

$$A_{s+1} = -\frac{A_s \tau(s+1)[(2s+3)(s+1+k_0)]^{1/2}}{[(2s+1)(s+1-k_0)]^{1/2}}\,, \quad s \geq k_0\,;$$

A_{k_0} arbitrary.

Note that there are only two solutions, one with $\lambda = k_0$, the other with $\lambda = -k_0$, in the cases when $k_0^2 = c^2 \neq 0$.

Note also that there is no solution with $\lambda = c$, contrary to the claims of Simon *et al.*[10] Their solution with

$$A_{s+1} = -\frac{A_s \tau(s+1)[(2s+3)(s+1-k_0)]^{1/2}}{[(2s+1)(s+1+k_0)]^{1/2}}\,, \quad s \geq k_0\,;$$

A_{k_0} arbitrary,

while it satisfies Eq. (32) with $\lambda = c$, does not satisfy Eqs. (31) at $s = k_0$.

(C) $\{0, 0, \tau\}$. In this case there are two solutions with $\lambda = 0$. Set

$$A_s = b_s(-1)^s(2s+1)^{1/2}\tau(s)\tau(s-1)\cdots\tau(0)\,.$$

Then b_s must satisfy

$$(s+2)b_{s+2} - (2s+3)b_{s+1} + (s+1)b_s = 0\,, \quad s \geq 0\,.$$

The solutions are as follows.

(i) $b_s = 1$. The corresponding ϕ of Eq. (30) is an eigenvector of T_3, with eigenvalue $-i$, as discussed in Sec. II.

(ii) $b_s = \sum_{r=1}^{s}(1/r)$. On the corresponding ϕ, $(T_3 + i)^2$ vanishes, but $(T_3 + i)$ does not. Thus $(T_3 + i)$ transforms this ϕ into a multiple of that defined in *(i)*.

(D) $\{k_0 \leftarrow c, \tau\}$, *$k_0 - c$ integral,* $|c| > k_0$.

(i) $\lambda = k_0\,\mathrm{sgn}(c)$.

$$A_{s+1} = -A_s \tau(s+1)(s+1-|c|)/(s+1+|c|), \quad k_0 \le s < |c|;$$

$A_s = 0, \quad s \ge |c|;$

A_{k_0} arbitrary.

(ii) $\lambda = -k_0\,\mathrm{sgn}(c)$.

$$A_{s+1} = -A_s \tau(s+1), \quad s \ge k_0;$$

A_{k_0} arbitrary.

Here the solution ϕ corresponding to *(i)* lies in the invariant subspace

$$D(k_0) \oplus D(k_0+1) \oplus \cdots \oplus D(|c|-1).$$

The $S_{\mu\nu}$ of $\{k_0 \leftarrow c, \tau\}$ realize in this subspace a finite-dimensional representation, integrable to the representation $[k_0, c]$ of SL(2, c). The solution *(i)* is therefore the one solution for this finite-dimensional representation, in accordance with Weinberg's result (10). Note that iT_3 has the eigenvalue $(1-|c|)$ on the first, and $(1+|c|)$ on the second solution. In the cases with $k_0 = 0$, there are still two distinct solutions with $\lambda = 0$, labeled by these two eigenvalues of iT_3.

(E) $\{k_0 \rightarrow c, \tau\}$, *$k_0 - c$ integral,* $|c| > k_0$.

(i) $\lambda = c$.

$$A_{s+1} = -\frac{A_s \tau(s+1)[(s+1-k_0)(s+1+|c|)]^{1/2}}{[(s+1+k_0)(s+1-|c|)]^{1/2}}, \quad s \ge |c|;$$

$A_s = 0, \quad k_0 \le s < |c|;$

$A_{|c|}$ arbitrary.

(ii) $\lambda = -c$.

$$A_{s+1} = -\frac{A_s \tau(s+1)[(s+1+k_0)(s+1+|c|)]^{1/2}}{[(s+1-k_0)(s+1-|c|)]^{1/2}}, \quad s \ge |c|;$$

$A_s = 0, \quad k_0 \le s < |c|;$

$A_{|c|}$ arbitrary.

(iii) $\lambda = -k_0\,\mathrm{sgn}(c)$.

$$A_{s+1} = -A_s \tau(s+1)(s+1+|c|)/(s+1-|c|), \quad s \ge |c|;$$

$A_s = 0, \quad k_0 \le s < |c|;$

$A_{|c|}$ arbitrary.

(iv) $\lambda = k_0\,\mathrm{sgn}(c)$.

$$A_{s+1} = -A_s \tau(s+1), \quad s \ge k_0;$$

A_{k_0} arbitrary.

Here the solutions ϕ corresponding to *(i)–(iii)* lie in the subspace

$$D(|c|) \oplus D(|c|+1) \oplus \cdots$$

in which the $S_{\mu\nu}$ of $\{k_0 \rightarrow c, \tau\}$ may be seen to realize the representation $\{|c|, k_0\,\mathrm{sgn}(c), \tau'\}$ of sl(2, c), where

$$\tau'(s) = \frac{\tau(s)[(s+|c|)(2s-1)]^{1/2}}{[(s-|c|)(2s+1)]^{1/2}}.$$

The solutions *(i)–(iii)* are therefore the three solutions for $\{|c|, k_0\,\mathrm{sgn}(c), \tau'\}$, as described in *(B)*. Note that again there are two solutions with $\lambda = 0$ in the cases with $k_0 = 0$, and that again these can be distinguished by the corresponding eigenvalues of iT_3.

Up to this stage we have not discussed whether or not ϕ, as in Eq. (30), actually lies in $K(k_0)$ for any of the sequences of coefficients A_s defined in *(A)–(E)* above. For this to be so, we require in each case that

$$\sum_{s=k_0}^{\infty} |A_s|^2 < \infty. \tag{35}$$

It was the imposition of this requirement, in the cases $\{k_0, c, 1\}$, which led Simon *et al.* to dismiss many of the solutions in *(A)–(E)* as merely "formal," and to arrive at the results (11), (12). However, whether or not the condition (35) holds in a given case, depends critically on the nature of $\tau(s)$, as is clearly shown by the formulas defining A_s in each of the cases in *(A)–(E)*. Indeed, it is possible to find $\tau(s)$, e.g., $\tau(s) = e^s$, such that *none* of these solutions belongs to $K(k_0)$ [excepting the solution corresponding to *(D) (i)*, which is always normalizable]. For our purposes, it is sufficient to note that $\tau(s)$ can be found, e.g., $\tau(s) = e^{-s}$, such that in *every* case the corresponding ϕ belongs to $K(k_0)$.

It is appropriate at this stage to compare the above approach to the solution of Eqs. (6′) with the second approach considered by Frishman *et al.*,[9] since they failed to find all the solutions by either method. In this second approach, they used the realization of the representations $[k_0, c]$, $[k_0 \leftarrow c]$, and $[k_0 \rightarrow c]$ of SL(2, c), in Hilbert spaces of functions $f(z, z^*)$, where z^* denotes the complex conjugate of z.[6] The generators $S_{\mu\nu}$ appear there in the form

$$\begin{aligned}
S_3 &= z\partial/\partial z - z^*\partial/\partial z^* + \tfrac{1}{2}(n_2 - n_1),\\
S_+ &= -z^2\partial/\partial z - \partial/\partial z^* + (n_1 - 1)z,\\
S_- &= \partial/\partial z + z^{*2}\partial/\partial z^* - (n_2 - 1)z^*,\\
T_3 &= iz\partial/\partial z + iz^*\partial/\partial z^* - \tfrac{1}{2}i(n_1 + n_2 - 2),\\
T_+ &= -iz^2\partial/\partial z + i\partial/\partial z^* + i(n_1 - 1)z,\\
T_- &= i\partial/\partial z - iz^{*2}\partial/\partial z^* + i(n_2 - 1)z^*.
\end{aligned} \tag{36}$$

If $n_1 = k_0 + c$, $n_2 = c - k_0$, where either $k_0 - c$ is nonintegral, or $|c| \le k_0$, then these $S_{\mu\nu}$ generate, in an appropriate Hilbert space, the representation $[k_0, c]$ of SL(2, c). If $k_0 - c$ is integral, and $|c| > k_0$, one may take $n_1 = (k_0 + c)\,\mathrm{sgn}(c)$, $n_2 = (c - k_0)\,\mathrm{sgn}(c)$, corresponding to $[k_0 \leftarrow c]$; or $n_1 = -(k_0 + c)\,\mathrm{sgn}(c)$,

$n_2=(k_0-c)\,\mathrm{sgn}(c)$, corresponding to $[k_0\!\rightarrow c]$. However, it seems plausible that if we consider Hilbert spaces of functions $f(z, z^*)$ *other* than those associated with $[k_0, c]$, $[k_0\!-c]$, and $[k_0\!\rightarrow c]$, then these $S_{\mu\nu}$, with the same choices of n_1 and n_2, can provide realizations of $\{k_0, c, \tau\}$, $\{k_0\!-c, \tau\}$, and $\{k_0\!\rightarrow c, \tau\}$, with $\tau(s)$ not everywhere equal to 1. At any rate, all the solutions described in *(A)–(E)* above reappear here, if we substitute from Eqs. (36) for $S_{\mu\nu}$ in Eqs. (6′), and solve the resultant partial-differential equations for $\phi(z, z^*)$.

We then have

$$z\partial\phi/\partial z - z^*\partial\phi/\partial z^* = [\lambda + \tfrac{1}{2}(n_1-n_2)]\phi\,, \qquad (37)$$

$$z[z\partial\phi/\partial z - (n_1-1)\phi] = z^*[z^*\partial\phi/\partial z^* - (n_2-1)\phi] = 0\,. \qquad (38)$$

It would appear at first glance that Eqs. (38) imply that

$$z\partial\phi/\partial z - (n_1-1)\phi = z^*\partial\phi/\partial z^* - (n_2-1)\phi = 0\,,$$

which, taken with Eq. (37), imply that

$$\phi = z^{n_1-1}z^{*n_2-1}\,, \quad \lambda = \tfrac{1}{2}(n_1-n_2)\,.$$

This is the only solution found by Frishman *et al.* However, there are other solutions. For example, the generalized function[14] $\phi=\delta(z, z^*)$ satisfies Eqs. (37), (38) with $\lambda=\frac{1}{2}(n_2-n_1)$ since, in that case,

$$z\partial\phi/\partial z = z^*\partial\phi/\partial z^* = -\phi\,, \quad z\phi = z^*\phi = 0\,.$$

We shall merely list the solutions we have found for Eqs. (37), (38). In checking that ϕ is a solution in each case, one must be careful with the homogeneous functions of degree (α, β),

$$z^\alpha z^{*\beta}\,,$$

which appear. The structure of these is described in Ref. 14, and they are to be interpreted as generalized functions. In particular, if both α and β are negative integers, then

$$z^\alpha z^{*\beta} \sim (\partial/\partial z)^{-(\alpha+1)}(\partial/\partial z^*)^{-(\beta+1)}\delta(z, z^*)$$

is concentrated at $z=z^*=0$. The function $z^{-1}z^{*-1}\ln(zz^*)$, which also appears below, is an associated generalized function[14] of order 1 and degree (−1, −1). Note the correspondence of the following solutions with those in *(A)–(E)* above.

(A′) k_0-c *nonintegral*, $n_1=k_0+c$, $n_2=c-k_0$.

(i) $\lambda=k_0=\frac{1}{2}(n_1-n_2)$, $\phi=z^{n_1-1}z^{*n_2-1}$.

(ii) $\lambda=-k_0=\frac{1}{2}(n_2-n_1)$, $\phi=z^{-1}z^{*-1}$.

(B′) k_0-c *integral*, $|c|\le k_0$, $k_0\ne 0$, $n_1=k_0+c$, $n_2=c-k_0$. Here n_1 is a non-negative, and n_2 a nonpositive integer.

(i), *(ii) as above.*

(iii) $\lambda=-c=-\frac{1}{2}(n_1+n_2)$, $\phi=z^{-1}z^{*n_2-1}$.

(C′) $k_0=c=n_1=n_2=0$.

(i) $\lambda=0$, $\phi=z^{-1}z^{*-1}$.

(ii) $\lambda=0$, $\phi=z^{-1}z^{*-1}\ln(zz^*)$.

(D′) k_0-c *integral*, $|c|>k_0$, $n_1=(k_0+c)\,\mathrm{sgn}(c)$, $n_2=(c-k_0)\,\mathrm{sgn}(c)$. Here n_1 and n_2 are positive integers.

(i) $\lambda=k_0\,\mathrm{sgn}(c)=\frac{1}{2}(n_1-n_2)$, $\phi=z^{n_1-1}z^{*n_2-1}$.

(ii) $\lambda=-k_0\,\mathrm{sgn}(c)=\frac{1}{2}(n_2-n_1)$, $\phi=z^{-1}z^{*-1}$.

(E′) k_0-c *integral*, $|c|>k_0$, $n_1=-(k_0+c)\,\mathrm{sgn}(c)$, $n_2=(k_0-c)\,\mathrm{sgn}(c)$. Here n_1 and n_2 are negative integers.

(i) $\lambda=|c|=-\frac{1}{2}(n_1+n_2)$, $\phi=z^{-1}z^{*n_2-1}$.

(ii) $\lambda=-|c|=\frac{1}{2}(n_1+n_2)$, $\phi=z^{n_1-1}z^{*-1}$.

(iii) $\lambda=-k_0\,\mathrm{sgn}(c)=\frac{1}{2}(n_1-n_2)$, $\phi=z^{n_1-1}z^{*n_2-1}$.

(iv) $\lambda=k_0\,\mathrm{sgn}(c)=\frac{1}{2}(n_2-n_1)$, $\phi=z^{-1}z^{*-1}$.

IV. REPRESENTATIONS OF ISL(2,c)

We now consider the second question (Q2) posed in the Introduction for those representations of sl(2, c), and corresponding values of λ, described above. It seems sure that approaches could be used which are more direct than the one adopted.[15] We prefer the method used here because it reveals much of the structure of wave functions satisfying Eqs. (6), so leading us to the results of Sec. V. Moreover, it involves a generalization of techniques used by Bender in his original paper,[8] and enables us to justify our claim that he described fields belonging to representations of sl(2, c), and not necessarily of SL(2, c).

We begin by considering a Hilbert space of functions $\phi^{(s)}_{ij\cdots k}(\vec{p})$, for some fixed non-negative integer $2s$. These functions are tensors or tensor-spinors of the type described in Sec. II. The Hilbert space is defined with the scalar product

$$(\phi^{(s)}, \phi^{(s)\prime}) = \int d^3p\, E^{-1}\phi^{(s)*}_{ij\cdots k}(\vec{p})\phi^{(s)\prime}_{ij\cdots k}(\vec{p})\,. \qquad (39)$$

It is clear from the work of Chakrabarti[16] that this space carries the representation

$$O_{s^+}\oplus O_{s-1^+}\oplus\cdots\oplus O_{-s^+}$$

of ISL(2, c), generated by p^μ and $J_{\mu\nu}$, with

$$\vec{J}=(J_{23}, J_{31}, J_{12})=\vec{L}+\vec{S}^{(s)}\,, \qquad \vec{K}=(J_{10}, J_{20}, J_{30})=\vec{N}+\vec{S}^{(s)}\times\vec{n}\,, \qquad (40)$$

where

$$\vec{L}=-i\vec{p}\times\partial/\partial\vec{p}\,, \quad \vec{N}=-iE\partial/\partial\vec{p}\,,$$

$$\vec{n}=\vec{p}/E\,,$$

and $\vec{S}^{(s)}$ are the generators of the $(2s+1)$-dimensional representation of SU(2), acting in the space of the tensor and spinor indices in the manner of Eq. (24). We project onto the subspace carrying the representation O_{λ^+} (where $\lambda-s$ is integral and $|\lambda|\le s$), by requiring that

$$\vec{S}^{(s)} \cdot \vec{n} \phi^{(s)} = \lambda \phi^{(s)} . \tag{41}$$

(Note that $\vec{S}^{(s)} \cdot \vec{n}$ commutes with p^μ, $\vec{J}$, and $\vec{K}$—it is a Poincaré invariant.)

Generalizing somewhat, we consider the Hilbert spaces $G(s, \lambda, \alpha)$ (where α is an arbitrary complex number) of functions $\phi^{(s)}(\vec{p})$ satisfying Eq. (41), with scalar product

$$(\phi^{(s)}, \phi^{(s)\prime}) = \int d^3p\, E^{(1-\alpha-\alpha^*)} \phi^{(s)*}_{ij\cdots k}(\vec{p}) \phi^{(s)\prime}_{ij\cdots k}(\vec{p}) . \tag{42}$$

For each value of α, $G(s, \lambda, \alpha)$ carries the representation $O_{\lambda+}$, with generators p^μ and $J_{\mu\nu}$, where now

$$\begin{aligned} \vec{J} &= \vec{L} + \vec{S}^{(s)} , \\ \vec{K} &= \vec{N} + \vec{S}^{(s)} \times \vec{n} + i(\alpha - 1)\vec{n} . \end{aligned} \tag{43}$$

The equivalence of the representations in $G(s, \lambda, 1)$ and $G(s, \lambda, \alpha)$, $\alpha \neq 1$, is easily established using the mapping

$$\phi^{(s)}(\vec{p}) \to E^{(\alpha-1)} \phi^{(s)}(\vec{p}) \tag{44}$$

from $G(s, \lambda, 1)$ into $G(s, \lambda, \alpha)$. This mapping carries the operators (40) into (43).

The radiation-gauge potentials considered by Bender are of the type $\phi^{(s)}(\vec{p})$ in the momentum representation. They transform under Poincaré transformations with generators of the form (43), with $\alpha = 1$ in the integer-spin cases, and $\alpha = \frac{3}{2}$ in the half-odd-integer-spin cases. There is therefore no difficulty in realizing the required unitary representations of ISL(2, c) with these potentials. This is of course well known in particular for the electromagnetic potential in the radiation gauge, $A_i(\vec{p}) \approx \phi^{(1)}_i(\vec{p})$. Just as Bender has defined, from the potentials, wave functions belonging to certain infinite-dimensional representations of sl(2, c), so we shall define infinite-component wave functions in terms of the $\phi^{(s)}(\vec{p})$, for more general values of α.

Consider a particular normalized element $\phi^{(r)}(\vec{p})$ of the space $G(r, \lambda, \alpha)$, $r = |\lambda|$, for fixed λ and α, and suppose that this element lies in the common invariant dense domain of p^μ, $\vec{J}$ and $\vec{K}$. When $\phi^{(r)}$ undergoes the transformation

$$\phi^{(r)} \to (\vec{L} + \vec{S}^{(s)})\phi^{(r)} ,$$

then

$$n_i \phi^{(r)} \to (\vec{L} + \vec{S}^{(s)}) n_i \phi^{(r)} + (\vec{V})_{ij}\, n_j \phi^{(r)} ,$$

where $(V_i)_{jk} = -i\epsilon_{ijk}$. Thus $\vec{n}\phi^{(r)}$ may be considered as an object belonging to the direct product of the $(2r+1)$-dimensional and three-dimensional (vector) representations of SU(2). Since

$$(r) \otimes (1) = (r+1) \oplus (r) \oplus (r-1) ,$$

$\vec{n}\phi^{(r)}$ can be resolved into components corresponding to spins $r+1$, r, and $r-1$. It is easily seen that, if $X = \vec{V} \cdot \vec{S}^{(s)}$, then

$$(X - s)(X + 1)(X + s + 1) \equiv 0 .$$

The spin-$(r+1)$, -r, and -$(r-1)$ components of $\vec{n}\phi^{(r)}$ correspond to the eigenvalues s, -1, and $-(s+1)$ of X, respectively, and therefore are obtained by applying the projection operators $(X+1)(X+s+1)/(s+1)(2s+1)$, $-(X-s)(X+s+1)/s(s+1)$, and $(X-s)(X+1)/s(2s+1)$, respectively. In this way, one obtains the resolution

$$\begin{aligned} n_i \phi^{(r)}_{jk\cdots l} = {} & (n_i^{(+)} \phi^{(r)})_{jk\cdots l} + (n_i^{(0)} \phi^{(r)})_{jk\cdots l} \\ & + (n_i^{(-)} \phi^{(r)})_{jk\cdots l} , \end{aligned} \tag{45}$$

where

$$\begin{aligned} (r+1)(2r+1)\vec{n}^{(+)} &= (r+1)^2 \vec{n} + i(r+1)(\vec{S}^{(r)} \times \vec{n}) \\ &\quad - \vec{S}^{(r)}(\vec{S}^{(r)} \cdot \vec{n}) , \\ r(r+1)\vec{n}^{(0)} &= \vec{S}^{(r)}(\vec{S}^{(r)} \cdot \vec{n}) , \\ r(2r+1)\vec{n}^{(-)} &= r^2 \vec{n} - ir(\vec{S}^{(r)} \times \vec{n}) - \vec{S}^{(r)}(\vec{S}^{(r)} \cdot \vec{n}) . \end{aligned} \tag{46}$$

Since $\phi^{(r)}$ is in $G(r, \lambda, \alpha)$, $(\vec{S}^{(r)} \cdot \vec{n})$ may be replaced by λ in these equations.

Define

$$\phi^{(r+1)}_{ijk\cdots l} = \{ (2r+1)(r+1)/[(r+1)^2 - \lambda^2] \}^{1/2} \times (n_i^{(+)} \phi^{(r)})_{jk\cdots l} .$$

Then it can be checked that

$$(\vec{S}^{(r+1)} \cdot \vec{n})\phi^{(r+1)} = \lambda \phi^{(r+1)} ,$$

and that, corresponding to transformations of $\phi^{(r)}$ with the operators (43), one has

$$\phi^{(r+1)} \to (\vec{L} + \vec{S}^{(r+1)})\phi^{(r+1)} ,$$

$$\phi^{(r+1)} \to [\vec{N} + \vec{S}^{(r+1)} \times \vec{n} + i(\alpha - 1)\vec{n}]\, \phi^{(r+1)}$$

Furthermore,

$$\phi^{(r+1)*}_{ijk\cdots l}\, \phi^{(r+1)}_{ijk\cdots l} = \phi^{(r)*}_{jk\cdots l}\, \phi^{(r)}_{jk\cdots l} ,$$

so that $\phi^{(r+1)}$ belongs to $G(r+1, \lambda, \alpha)$, and has unit norm in that space.

Evidently this procedure can be repeated indefinitely, to yield, for each $s > r$, with $s - r$ integral, a function

$$\phi^{(s)}_{ijk\cdots l} = [s(2s-1)/(s^2 - \lambda^2)]^{1/2} (n_i^{(+)} \phi^{(s-1)})_{jk\cdots l} , \tag{47}$$

belonging to $G(s, \lambda, \alpha)$, and having unit norm in that space. (Note that in the definition of $\vec{n}^{(\pm)}$ and $\vec{n}^{(0)}$, r must be replaced by the spin value of the wave function to which these operators are applied.) Straightforward but tedious calculation reveals that

$$(n_i^{(-)}\phi^{(s)})_{jk\cdots l}$$
$$=\{(s^2-\lambda^2)^{1/2}/s(2s+1)[s(2s-1)]^{1/2}\}\eta^{(s-1)}_{ijk\cdots l}, \quad s>r$$
$$=0,\quad s=r \tag{48}$$

where $\eta^{(s-1)}$ is defined in terms of $\phi^{(s-1)}$ just as $\rho^{(s-1)}$ is defined in terms of $\chi^{(s-1)}$ in Eq. (26). One then deduces from Eq. (48) that

$$\phi^{(s-1)}_{jk\cdots l}=[s(2s-1)/(s^2-\lambda^2)]^{1/2}(n_i^{(-)}\phi^{(s)})_{ijk\cdots l},\quad s>r \tag{49}$$

so that, by virtue of Eqs. (47) and (49), any of the $\phi^{(s)}$, $s\geq r$, defines all the others, and so the infinite-component object

$$\phi(\vec{p})=(\phi^{(r)}_{ij\cdots k}(\vec{p}),\phi^{(r+1)}_{lm\cdots n}(\vec{p}),\ldots).$$

Noting that, on a spin-s function

$$\vec{S}^{(s)}\times\vec{n}^{(+)}=-is\vec{n}^{(+)},\quad \vec{S}^{(s)}\times\vec{n}^{(0)}=i\vec{n}^{(0)},$$
$$\vec{S}^{(s)}\times\vec{n}^{(-)}=i(s+1)\vec{n}^{(-)}, \tag{50}$$

and writing $\vec{T}^{(s)}=\vec{S}^{(s)}\times\vec{n}+i(\alpha-1)\vec{n}$, one deduces from Eq. (45) that

$$\vec{T}^{(s)}\phi^{(s)}=i(\alpha-1-s)\vec{n}^{(+)}\phi^{(s)}+i\alpha\vec{n}^{(0)}\phi^{(s)}$$
$$+i(\alpha+s)\vec{n}^{(-)}\phi^{(s)},\quad s\geq r. \tag{51}$$

However, despite Eqs. (47), (48), and (51), it is not appropriate to assert that $\phi(\vec{p})$ belongs to an infinite-dimensional representation of sl(2, c), because

$$\phi^\dagger(\vec{p})\phi(\vec{p})\equiv\sum_{s=r}^{\infty}\phi^{(s)*}_{ij\cdots k}(\vec{p})\phi^{(s)}_{ij\cdots k}(\vec{p})$$
$$=\phi^{(r)*}_{ij\cdots k}(\vec{p})\phi^{(r)}_{ij\cdots k}(\vec{p})\sum_{s=r}^{\infty}1$$
$$=\infty.$$

This difficulty can be surmounted in various ways, depending on the values of α and λ.

Consider first the cases where $\alpha-\lambda$ is not an integer, and define

$$\chi^{(s)}(\vec{p})=\beta(s)\phi^{(s)}(\vec{p}),\quad s\geq r, \tag{52}$$

where $\beta(s)$ is some fixed, finite, nonzero complex number for each value of s, such that

$$\sum_{s=r}^{\infty}|\beta(s)|^2=1. \tag{53}$$

Then, writing

$$\chi(\vec{p})=(\chi^{(r)}_{ij\cdots k}(\vec{p}),\chi^{(r+1)}_{lm\cdots n}(\vec{p}),\ldots), \tag{54}$$

one has

$$\chi^{(s)*}_{ij\cdots k}\chi^{(s)}_{ij\cdots k}=|\beta(s)|^2\phi^{(r)*}_{lm\cdots n}\phi^{(r)}_{lm\cdots n},\quad s\geq r,$$

so that

$$\chi^\dagger(\vec{p})\chi(\vec{p})=\phi^{(r)*}_{lm\cdots n}(\vec{p})\phi^{(r)}_{lm\cdots n}(\vec{p})<\infty$$

and (55)

$$\int d^3p\,E^{(1-\alpha-\alpha^*)}\chi^\dagger(\vec{p})\chi(\vec{p})=1.$$

Furthermore, if $\phi^{(r)}$, $\phi^{(r)\prime}$ are distinct elements of $G(r,\lambda,\alpha)$, they can be seen to define distinct χ and χ' [with the same $\beta(s)$, $s\geq r$], such that

$$\chi^{(s)*}_{ij\cdots k}\chi^{(s)\prime}_{ij\cdots k}=|\beta(s)|^2\phi^{(r)*}_{lm\cdots n}\phi^{(r)\prime}_{lm\cdots n},\quad s\geq r$$

whence

$$\chi^\dagger\chi'=\phi^{(r)*}_{lm\cdots n}\phi^{(r)\prime}_{lm\cdots n}$$

and (56)

$$(\chi,\chi')=(\phi^{(r)},\phi^{(r)\prime}),$$

where

$$(\chi,\chi')=\int d^3p\,E^{(1-\alpha-\alpha^*)}\chi^\dagger(\vec{p})\chi'(\vec{p}). \tag{57}$$

Corresponding to transformations of $\phi^{(r)}$ with the operators (43), one finds

$$\chi\to(\vec{L}+\vec{S})\chi,$$
$$\chi\to(\vec{N}+\vec{T})\chi, \tag{58}$$

where the spin-s components of $\vec{S}\chi$ and $\vec{T}\chi$ are given by

$$(\vec{S}\chi)^{(s)}_{ij\cdots k}=(\vec{S}^{(s)}\chi^{(s)})_{ij\cdots k}, \tag{59}$$

and, as a consequence of Eqs. (47), (48), (51), (52),

$$(T_i\chi)^{(s)}_{jk\cdots l}$$
$$=i(\alpha-1-s)\frac{\beta(s)}{\beta(s+1)}\left[\frac{(s+1)^2-\lambda^2}{(s+1)(2s+1)}\right]^{1/2}\chi^{(s+1)}_{ijk\cdots l}$$
$$+\frac{i\alpha\lambda}{s(s+1)}(S_i\chi)^{(s)}_{jk\cdots l}$$
$$+\frac{i(\alpha+s)}{s(2s+1)}\frac{\beta(s)}{\beta(s-1)}\left[\frac{s^2-\lambda^2}{s(2s-1)}\right]^{1/2}\rho^{(s-1)}_{ijk\cdots l}, \tag{60}$$

where $\rho^{(s-1)}$ is defined in terms of $\chi^{(s-1)}$, as in Eq. (26).

A comparison with Eqs. (25), (15)–(17) shows that χ belongs to the representation $\{|\lambda|,\alpha\,\mathrm{sgn}(\lambda),\tau\}$, where

$$\tau(s)=-\frac{\beta(s)[(2s-1)(s+\alpha)]^{1/2}}{\beta(s-1)[(2s+1)(s-\alpha)]^{1/2}}. \tag{61}$$

Now it is clear from the way χ has been constructed that

$$\vec{S}\cdot\vec{n}\chi=\lambda\chi,$$
$$\vec{T}\chi=[\vec{S}\times\vec{n}+i(\alpha-1)\vec{n}]\chi. \tag{62}$$

These equations imply that

$$\vec{T}\cdot\vec{n}\chi = i(\alpha-1)\chi\,,\qquad \vec{S}\chi = (\vec{n}\times\vec{T}+\lambda\vec{n})\chi\,, \tag{63}$$

and Eqs. (62)–(63) may be combined as

$$\tilde{S}_{\mu\nu}p^{\nu}\chi = \lambda p_{\mu}\chi\,,\qquad S_{\mu\nu}p^{\nu}\chi = i(1-\alpha)p_{\mu}\chi\,. \tag{64}$$

To this stage, then, we have shown that each normalized element $\phi^{|\lambda|}_{ij\ldots k}(\vec{p})$ of a dense subspace of $G(|\lambda|,\lambda,\alpha)$, $\alpha-\lambda$ nonintegral, defines a corresponding function $\chi(\vec{p})$ belonging to the representation $\{|\lambda|, \alpha\,\mathrm{sgn}(\lambda), \tau\}$, satisfying Eqs. (64), and having unit norm with respect to the scalar product (57). Essentially by reversing the above arguments, it may be deduced that the converse of this statement is also true. If $H(\lambda,\alpha)$ denotes the Hilbert space of solutions to Eqs. (64), with scalar product (57), then there is a bijection from $G(|\lambda|,\lambda,\alpha)$ onto $H(\lambda,\alpha)$. This mapping preserves scalar products, as Eq. (56) shows, and effects the transformations

$$p^{\mu}\to p^{\mu}\,,\quad \vec{L}+\vec{S}^{(|\lambda|)}\to\vec{L}+\vec{S}\,,\qquad \vec{N}+\vec{S}^{(|\lambda|)}\times\vec{n}+i(\alpha-1)\vec{n}\to\vec{N}+\vec{T}\,. \tag{65}$$

Since the operators on the left generate the representation O_{λ^+} of ISL(2, c) in $G(|\lambda|,\lambda,\alpha)$, it follows that those on the right generate an equivalent representation in $H(\lambda,\alpha)$.

It can be seen from Eq. (57) that $H(\lambda,\alpha)$ is a subspace of the direct product of two other Hilbert spaces. One is $K(|\lambda|)$ (c.f. Sec. II) and the other is that space of (scalar) functions $f(\vec{p})$ with scalar product

$$(f,f') = \int d^3p\, E^{(1-\alpha-\alpha^*)} f^*(\vec{p}) f'(\vec{p})\,.$$

The preceding analysis shows that $\vec{L}+\vec{S}$ and $\vec{N}+\vec{T}$ are integrable in $H(\lambda,\alpha)$. Now there is some arbitrariness in the choice of the representations of sl(2, c) to which χ belongs, for given λ and α, since $\tau(s)$ depends on the choice of the numbers $\beta(s)$ as Eq. (61) shows. However, for given λ and α, there may well be no choice of the $\beta(s)$ satisfying Eq. (53) and such that $\{|\lambda|, \alpha\,\mathrm{sgn}(\lambda), \tau\}$ is integrable in $K(|\lambda|)$. For example, this is certainly the case when $\alpha = i\rho$, ρ real, since Eqs. (63) show that $\vec{T}\cdot\vec{n}$ has a complex eigenvalue $-(\rho+i)$ in $H(\lambda,\alpha)$. This operator could not conceivably have other than real eigenvalues if $\{|\lambda|, i\rho\,\mathrm{sgn}(\lambda), \tau\}$ were integrable to the *unitary* representation $[|\lambda|, i\rho\,\mathrm{sgn}(\lambda)]$ of SL(2, c). In order to understand why the nonintegrability of $\vec{T}$ in $K(|\lambda|)$ does not prevent the integrability of $\vec{N}+\vec{T}$ in $H(\lambda,\alpha)$, one observes that Eqs. (62) imply that

$$(\vec{N}+\vec{T})\chi = (\vec{N}+\vec{S}\times\vec{n}+i(\alpha-1)\vec{n})\chi. \tag{66}$$

Evidently the integrability of $\vec{S}$ in $K(|\lambda|)$ guarantees the required result.

By setting $\lambda=\pm k_0$, $\alpha=\pm c$, one exhausts the possibilities described in *(A) (i)–(ii)* of Sec. III, as λ and α run over their allowed values. In order to treat the other possibilities *(B)–(E)*, it is necessary to consider integral values of $\alpha-\lambda$. Suppose first that $\alpha-\lambda$ is integral, $|\alpha|\leq|\lambda|$, and $\lambda\neq 0$. Then the analysis goes through just as in the case of nonintegral $\alpha-\lambda$. One finds that χ belongs to $\{|\lambda|, \alpha\,\mathrm{sgn}(\lambda), \tau\}$, with τ as in Eq. (61), and by setting $\lambda=\pm k_0$, $\alpha=\pm c$, one runs over the possibilities *(B) (i)–(ii)*. Alternatively, χ may be regarded in this case as belonging to $\{|\alpha|\to\lambda\,\mathrm{sgn}(\alpha), \tau'\}$, with

$$\tau'(s) = -\frac{\beta(s)[(s+\alpha)(s-|\lambda|)]^{1/2}}{\beta(s-1)[(s-\alpha)(s+|\lambda|)]^{1/2}}\,, \tag{67}$$

and then the possibilities *(E) (i)–(ii)* are covered.

The cases with $\lambda=\pm n$, $\alpha=1$, or $\lambda=\pm(n+\frac{1}{2})$, $\alpha=\frac{3}{2}$, where n is a positive integer, are those treated by Bender.[8] In the former case, when the representation in question is $\{n, \pm 1, \tau\}$ (or $\{1\to\pm n, \tau'\}$), he has made a quite arbitrary choice of the numbers $\beta(s)$, with

$$\beta(s)/\beta(s-1) = i[s(2s-1)/(s^2-\lambda^2)]^{1/2} \tag{68}$$

so as to give

$$\tau(s) = -i(2s-1)[s(s+1)/(s^2-\lambda^2)(s-1)(2s+1)]^{1/2}.$$

It is easy to see that there are no nonzero numbers $\beta(s)$ satisfying Eqs. (53) and (68). Furthermore, as Simon *et al.* implicitly have shown,[10] there are no $\beta(s)$ satisfying Eq. (53) and

$$\frac{\beta(s)}{\beta(s-1)} = -\left[\frac{(2s+1)(s-1)}{(2s-1)(s+1)}\right]^{1/2},$$

which is what is required, when $\alpha=1$, if $\tau(s)$ is to be equal to 1. Thus χ cannot be chosen to belong to $\{n, \pm 1, 1\}$, and we conclude[17] that the integer-spin radiation-gauge potentials considered by Bender, and in particular, the electromagnetic potential in the radiation gauge, belong to nonintegrable representations $\{n, \pm 1, \tau\}$ of sl(2, c).

Next, suppose $\alpha-\lambda$ is integral, and $\alpha>|\lambda|$. If one sets

$$\chi^{(s)} = \beta(s)\phi^{(s)}\,,\qquad s\geq r$$

the analysis goes through unaltered, except that χ is found to belong to $\{|\lambda|\to\alpha\,\mathrm{sgn}(\lambda), \tau\}$, where

$$\tau(s) = \beta(s)/\beta(s-1)\,, \tag{69}$$

and in this way the possibilities *(E) (iv)* are covered. However, if in this same case one sets

$$\chi^{(s)} = \beta(s)\phi^{(s)}\,,\qquad |\lambda|\leq s<\alpha$$

$$\chi^{(s)} = 0\,,\qquad s\geq\alpha$$

then the analysis is the same, except that now χ belongs to $\{\,|\lambda| - \alpha\,\mathrm{sgn}(\lambda), \tau\}$, with

$$\tau(s) = -\beta(s)(s-\alpha)/\beta(s-1)(s+\alpha)\,.$$

As α and λ run over their allowed values, the possibilities *(D) (i)* are covered. Alternatively, χ may in this case be regarded as belonging to the finite-dimensional representation $\{\,|\lambda|, \alpha\,\mathrm{sgn}(\lambda)\}$, so that the possibilities (10) described by Weinberg[7] are covered.

Now suppose $\alpha - \lambda$ is integral, and $\alpha < -k_0$. If we set

$$\chi^{(s)} = \beta(s)\phi^{(s)}\,, \quad s \geq r$$

then χ belongs to $\{\,|\lambda| - \alpha\,\mathrm{sgn}(\lambda), \tau\}$, where τ is as in Eq. (69), and the possibilities *(D) (ii)* are covered. However, if we set

$$\chi^{(s)} = 0\,, \quad |\lambda| \leq s < -\alpha$$

$$\chi^{(s)} = \beta(s)\phi^{(s)}\,, \quad s \geq -\alpha\,,$$

then χ is found to belong to $\{\,|\lambda| - \alpha\,\mathrm{sgn}(\lambda), \tau\}$, with

$$\tau(s) = -\beta(s)(s+\alpha)/\beta(s-1)(s-\alpha)\,,$$

and the possibilities *(E) (iii)* are covered. Alternatively, in this case χ may be regarded as belonging to $\{-\alpha, -\lambda, \tau\}$, with τ as in Eq. (61), and the possibilities *(B) (iii)* are thus also covered.

Finally, suppose $\alpha = \lambda = 0$. Setting

$$\chi^{(s)} = \beta(s)\phi^{(s)}\,, \quad s \geq 0$$

we find that χ belongs to $\{0, 0, \tau\}$, with τ as in Eq. (61). This corresponds to the situation described in *(C) (i)*.

Our analysis therefore answers the question (Q2) for each of the possibilities *(A)*–*(E)* of Sec. III, with the exception of *(C) (ii)*, and shows that in each case one can find a Hilbert space $H(\lambda, \alpha)$ of solutions to the wave equations (64), carrying the corresponding representation $O_{\lambda+}$ of ISL(2, c). This Hilbert space has the scalar product (57) in each case, with α and λ related to k_0 and c as described above. It is clear why the exceptional case *(C) (ii)* is not covered by our treatment. Our analysis depends on both of Eqs. (64) being satisfied by χ. The second of these incorporates the second of Eqs. (62), which is an inevitable consequence of defining χ in terms of an element of $G(r, \lambda, \alpha)$. However, for the situation described in *(C) (ii)* the corresponding wave equations are

$$\tilde{S}_{\mu\nu} p^{\nu}\chi = 0\,,$$

$$(S_{\mu\nu} - ig_{\mu\nu})(S^{\nu\rho} - ig^{\nu\rho})\,p_\rho \chi = 0\,,$$

and the second of Eqs. (64) is *not* satisfied. We leave question (Q2) unanswered in this case.[15]

V. EXPANSORS AND RELATIONSHIPS BETWEEN DESCRIPTIONS

In this section, we shall again be concerned with operator-irreducible, su(2)-integrable representations of sl(2, c), but now we shall not be concerned with their detailed structure, nor with their integrability. Consequently, we shall label them simply $\{k_0, c\}$, $\{k_0 - c\}$, or $\{k_0 - c\}$, with no reference to the important functions $\tau(s)$ in each case.

Consider as an infinite-component operator, the sequence

$$p(\gamma) = E^{\gamma}(\xi^{(0)}, \xi_i^{(1)}, \xi_{jk}^{(2)}, \dots)$$

where $\xi^{(0)} = \kappa(0)$, $\xi_i^{(1)} = \kappa(1) n_i$, $\xi_{jk}^{(2)} = \kappa(2)(n_j n_k - \frac{1}{3}\delta_{jk}), \dots$. Here γ is an arbitrary complex number, $\kappa(s)$, $s = 0, 1, 2, \dots$, is an arbitrary nonzero complex number, and $\vec{n} = \vec{p}/E$ as before. The component $\xi_{ij\cdots k}^{(s)}$ is the symmetric, traceless O(3) tensor of rank s, with leading term $\kappa(s) n_i n_j \cdots n_k$. By calculating the commutator of $E^{\gamma}\xi^{(s)}$ with the $\vec{L}$ and $\vec{N}$ of Eqs. (40), one deduces that $p(\gamma)$ transforms according to an operator-irreducible, su(2)-integrable representation of sl(2, c). This representation is $\{0, \gamma+1\}$ when $\gamma+1$ is zero or nonintegral, $\{0 - \gamma+1\}$ when $\gamma+1$ is a positive integer, and $\{0 - \gamma+1\}$ when $\gamma+1$ is a negative integer. However, when $\gamma+1$ is a positive integer, one may also set $\kappa(s) = 0$, $s \geq (\gamma+1)$, and the resultant operator $p(\gamma)'$ then transforms according to the finite-dimensional representation $\{0, \gamma+1\}$. Similarly, when $\gamma+1$ is a negative integer, one may set $\kappa(s) = 0$, $0 \leq s < -(\gamma+1)$, and the resultant operator $p(\gamma)'$ transforms according to the representation $\{-(\gamma+1), 0\}$.

For example, when $\gamma = 1$,

$$p(1) = E\,(\kappa(0), \kappa(1) n_i, \kappa(2)(n_j n_k - \tfrac{1}{3}\delta_{jk}), \dots)$$

transforms according to $\{0 - 2\}$. However, the components of

$$p(1)' = E\,(\kappa(0), \kappa(1) n_i)$$

are scalar multiples of those of the four-vector p^{μ}, so that $p(1)'$ transforms according to $[0, 2]$.

Objects like $p(\gamma)$ and $p(\gamma)'$ have been discussed by Dirac,[18] and, more recently, by Bender and Griffiths.[19] Following Dirac, we shall call $p(\gamma)$ and $p(\gamma)'$ homogeneous expansor operators of degree γ.

Next consider wave functions $\chi(\vec{p})$, with invariant helicity $\lambda = \epsilon k_0$, $\epsilon^2 = 1$, and belonging to the $(2k_0+1)$-dimensional representation $\{k_0, \epsilon(k_0+1)\}$. What is the effect of the expansors $p(\gamma)$ and $p(\gamma)'$ on $\chi(\vec{p})$? We know that $\chi = \chi_{ij\cdots k}^{(k_0)}$ belongs to $G(k_0, \epsilon k_0, k_0+1)$, and hence, from (44), that $E^{\gamma}\chi$ belongs to $G(k_0, \epsilon k_0, k_0+\gamma+1)$. But then $E^{\gamma}\chi$ is the spin-k_0 component of a function $\chi'(\vec{p})$, belonging to an operator-irreducible, su(2)-integrable representation of sl(2, c), and satisfying Eqs. (64) with $\lambda = \epsilon k_0$, $\alpha = k_0 + \gamma + 1$. Furthermore, the operators $\xi^{(s)}$ carry $E^\gamma \chi$ into linear combinations of the various spin components of χ', as a consequence of Eqs. (45)–(48). Thus $p(\gamma)\chi$ or $p(\gamma)'\chi$, regarded as a new wave function, belongs to another operator-irreducible, su(2)-integrable representation of sl(2, c), has invariant helicity $\lambda = \epsilon k_0$, the same as χ, and satisfies Eqs. (64) with $\alpha = (k_0 + \gamma + 1)$. The particular representation to which $p(\gamma)\chi$ or $p(\gamma)'\chi$ belongs may be determined from the values of λ and α as in Sec. IV.

It is now clear that any of the finite- or infinite-component wave functions χ', having invariant helicity λ, as described in Secs. III and IV, may be regarded as being of the form $p(\gamma)\chi$ or $p(\gamma)'\chi$, for appropriate γ, with χ belonging to the $(2k_0+1)$-dimensional representation $\{k_0, \epsilon(k_0+1)\}$, where $\lambda = \epsilon k_0$. This generalizes a result obtained by Weinberg.[7] He showed that, if $\psi(x)$ belongs to $\{k_0, \epsilon(k_0+1)\}$, and has invariant helicity $\lambda = \epsilon k_0$, then

$$\psi' = P_\mu P_\nu \cdots P_\rho \psi, \quad P_\mu = i\partial/\partial x^\mu$$

belongs to the finite-dimensional representation $\{k_0, \epsilon(k_0+n+1)\}$, and has the same helicity. (Here n is the rank of the tensor $P_\mu P_\nu \cdots P_\rho$.) In the momentum representation, this is the result that $p(n)'\chi$ belongs to $\{k_0, \epsilon(k_0+n+1)\}$. Weinberg has explained his result as follows: The function ψ' belongs to the direct product of two finite-dimensional representations. One is $\{k_0, \epsilon(k_0+1)\}$, and the other is $\{0, n+1\}$. This direct product reduces into a finite direct sum of finite-dimensional representations:

$$\{k_0, \epsilon(k_0+1)\} \otimes \{0, n+1\}$$
$$= \{k_0, \epsilon(k_0+n+1)\} \oplus \{k_0-1, \epsilon(k_0+n)\} \oplus \cdots,$$

but the Eqs. (4) satisfied by ψ ensure that ψ' only has components in $\{k_0, \epsilon(k_0+n+1)\}$.

In the more general case, one is dealing with the direct product of $\{k_0, \epsilon(k_0+1)\}$ and R, the representation according to which $p(\gamma)$ or $p(\gamma)'$ transforms. If γ is nonintegral, this direct product reduces to a finite direct sum of irreducible representations, but if γ is integral, it will not always be fully reducible. In any event, the Eqs. (64) satisfied by χ, with $\alpha = k_0 + 1$, ensure that $p(\gamma)\chi$ or $p(\gamma)\chi'$ has components in only one operator-irreducible representation. This representation must of course be present in the reduction of $\{k_0, \epsilon(k_0+1)\} \otimes R$. These ideas can be extended to treat the relationships between wave functions χ_1, χ_2, belonging to different representations, but having the same invariant helicity $\lambda = \epsilon k_0$. For if

$$\chi_1 \approx p(\alpha)\chi, \quad \chi_2 \approx p(\beta)\chi,$$

where χ belongs to $\{k_0, \epsilon(k_0+1)\}$, then

$$\chi_1 \approx p(\alpha - \beta)\chi_2.$$

We are now in a position to give a very simple interpretation of some of the results obtained by Bender and Griffiths.[8,19] The electromagnetic potential in the radiation gauge, A_i, is related to the electromagnetic field $F_{\mu\nu}$ by

$$P_0 A_i = F_{0i},$$
$$P_i A_j - P_j A_i = F_{ij}.$$

In the momentum representation then, $A_i(\vec{p}) = E^{-1}F_{0i}(\vec{p})$. Now the part of $F_{\mu\nu}$ with helicity ± 1 belongs to $\{1, \pm 2\}$, so that the corresponding part $A_i^{(\pm)}$ of A_i belongs to

$$A^{(\pm)} \approx p(-1)\chi^{(\pm)},$$

where $\chi^{(\pm)}$ belongs to $\{1, \pm 2\}$. It follows at once from the results of Sec. IV that $A^{(\pm)}$ belongs to $\{1, \pm 1\}$. More generally, one has

$$\phi^{(\pm)} \approx p(-s)\chi^{(\pm)},$$

where $\phi^{(\pm)}$ is the radiation-gauge potential with integral helicity $\pm s$, and $\chi^{(\pm)}$ belongs to $\{s, \pm(s+1)\}$. Then $\phi^{(\pm)}$ belongs to $\{s, \pm 1\}$. For half-odd-integral spins, one has

$$\phi^{(\pm)} \approx p(\tfrac{1}{2} - s)\chi^{(\pm)},$$

with $\chi^{(\pm)}$ belonging to $\{s, \pm(s+1)\}$, so that $\phi^{(\pm)}$ belongs to $\{s, \pm\frac{3}{2}\}$.

Bender and Griffiths have noted that, if $\psi^{(\pm)}(x)$ is the coordinate-space object corresponding to $\phi^{(\pm)}$, then $(P_0)^n \psi^{(\pm)}$ transforms according to the representation $\{s, \pm(n+1)\}$ [or $\{s, \pm(n+\frac{3}{2})\}$, if the spin is half-odd-integral], provided $(n+1) \leq s$ [or $(n+\frac{3}{2}) \leq s$]. For $n = s$ (or $n = s - \frac{1}{2}$), the resultant representation is $\{s \to \pm(s+1)\}$. These results also can easily be understood by considering the effect of $p(n)'$ on $\phi^{(\pm)}$.

Note that $p(\gamma)$ has length dimension $-\gamma$. Suppose $\chi' \approx p(\gamma)\chi$, where $\chi(\vec{p})$ has helicity $\lambda = \epsilon k_0$ and belongs to $\{k_0, \epsilon(k_0+1)\}$. Since the canonical length dimension of χ is $(\frac{3}{2} - k_0)$, that of χ' is $(\frac{3}{2} - k_0 - \gamma)$. The corresponding coordinate space objects ψ and ψ' have canonical dimensions $-(k_0+1)$ and $-(k_0+\gamma+1)$, respectively.

VI. CONCLUSION

The main conclusion to be drawn is that wave functions belonging to nonintegrable representations of the Lie algebra sl(2, c) can be used to realize unitary representations of the Poincaré group ISL(2, c).

A class of operator-irreducible, su(2)-integrable representations of sl(2, c) has been described. If $S_{\mu\nu}$ are the operators of a typical representation in this class, then the matrix elements of $S_{\mu\nu}$, in an su(2) basis, are the same as those of $S'_{\mu\nu}$ belonging to a corresponding integrable representation. But a specific choice of domains for the $S_{\mu\nu}$ distinguishes them from the $S'_{\mu\nu}$ in many cases, so that many of the representations in the class are nonintegrable. Nevertheless, it has been shown that wave functions belonging to such representations can be used to realize the representations O_{λ^+} of ISL(2, c), corresponding to free, massless particles with helicity λ. It is clear that corresponding free, massless, infinite-component fields can be constructed in each case, using the techniques described by Weinberg[7] and Frishman *et al.*[9]

These observations have enabled us to resolve an apparent paradox, which arises when one attempts to reconcile the results of Bender,[8] Frishman *et al.*,[9] and Simon *et al.*[10] In particular, while Bender claims to have described fields belonging to representations of SL(2, c), he has in fact dealt only with representations of sl(2, c). We are able to conclude that the free, electromagnetic potential in the radiation gauge, for example, belongs to a representation of sl(2, c) which is not integrable.

In a sense it seems absurd to introduce a field with an infinite number of components in order to describe a massless particle, whose "spin" is essentially a one-component thing. More than one field component is needed if a "manifestly covariant" description is to be given. However, as Weinberg[7] has shown, such a description of a free, massless particle with helicity $\lambda = \epsilon s$, $\epsilon^2 = 1$, can be given using fields or wave functions with $2s+1$ components. Why then consider descriptions using infinite-component objects? The point is really that two equivalent descriptions of free particles, using fields or wave functions belonging to different representations of sl(2, c), may behave very differently when one attempts to extend the description to include interactions. In particular, the canonical length dimensions of the fields will vary with the representation to which they belong. The significance of this point has been emphasized by Weinberg.[7] It is therefore important to find different possible ways of describing free particles, and more generally, to investigate new realizations of the representations of the Poincaré group.

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