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PHYSICAL REVIEW D VOLUME 10, NUMBER 4 15 AUGUST 1974

Absence of runaways and divergent self-mass in nonrelativistic quantum electrodynamics*

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(Received 14 January 1974)

We derive the quantum-mechanical operator equation of motion for a point electron and show that (i) the electrostatic self-mass of a point electron is zero in quantum mechanics; (ii) the equation of motion does not admit runaway solutions; (iii) the correspondence limit of the *solutions* of the quantum-mechanical equation likewise does not display runaways; and (iv) the solutions do not violate the principle of causality in quantum mechanics.

The Lorentz-Dirac theory of a classical nonrelativistic point electron, interacting with its self-field and subject to an external force $\vec{F}(t)$, leads to the equation of motion[1-3]

$$m_0 \frac{d^2\vec{R}(t)}{dt^2} = \vec{F}(t) - \delta m \frac{d^2\vec{R}(t)}{dt^2} + \frac{2e^2}{3c^3}\frac{d^3\vec{R}(t)}{dt^3}, \qquad (1)$$

where m_0 is the mechanical mass of the electron and δm is its electrostatic self-mass (divergent for a point electron). This equation is beset by the well-known difficulties that it admits runaway solutions (exponentially growing accelerations even in the absence of external forces) and violates causality (when the runaways are eliminated by the imposition of suitable boundary conditions, the electron accelerates before the force acts). While these results mar the internal consistency of classical electrodynamics, the view is often adopted that since preacceleration occurs on such a short time scale (10^{-23} sec for an electron) the noncausal effects belong in the domain of quantum theory, to which one must turn for a resolution of the problem.

To prove that runaways are not present in quantum theory one would have to show that no Heisenberg-picture operator in the theory has an exponentially growing time dependence. This result has not been established for the standard Hamiltonian[4] governing the interaction of a nonrelativistic point electron with a quantized electromagnetic field,[5,6] on which the conclusions of this paper are based, and it is not our purpose here to give such a proof. Instead, starting from the operator form of Maxwell's equation and the Lorentz force equation which follow from the Hamiltonian, we derive a quantum-mechanical operator equation of motion for a point electron which reduces to Eq. (1) in the correspondence limit (i.e., $\hbar \to 0$) and show that (i) the electrostatic self-mass of a point electron is zero in quantum mechanics, (ii) the equation of motion does not admit runaway solutions, (iii) the correspondence limit of the *solutions* of the quantum-mechanical equations likewise does not display runaways, and (iv) the solutions do not violate the principle of causality in quantum mechanics. These calculations thus show how nonrelativistic quantum electrodynamics is compatible with requirements (no runaways, etc.) which presumably follow from the general principles of quantum mechanics and in addition possesses a physically reasonable correspondence limit.

The results in the quantum-mechanical case are best understood by comparison to the classical results for an extended charge distribution. For a spherically symmetric static charge distribution, Eq. (1) is replaced by[7]

$$m_0\frac{d^2\vec{R}(t)}{dt^2}=\vec{F}(t)-\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}\gamma_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}}$$
$$+(\text{small nonlinear terms}). \qquad (2)$$

The coefficients γ_n are proportional to L^{n-1}, where L is the effective charge radius, and are structure-dependent. For simple charge distributions these coefficients can be explicitly evaluated and the series summed. Thus, for a spherical shell, one obtains[8,9]

$$\frac{d^2\vec{R}(t)}{dt^2}=\xi\left[\frac{d\vec{R}(t-2L/c)}{dt}-\frac{d\vec{R}(t)}{dt}\right]+\frac{\vec{F}(t)}{m(1-c\tau/L)}, \qquad (3)$$

neglecting the small nonlinear terms.[10] In Eq. (3),

$$\xi=\frac{(c/2L)(c\tau/L)}{(1-c\tau/L)},$$
$$\tau=2e^2/3mc^3,$$

and

$$m\equiv m_0+2e^2/3Lc^2.$$

Analysis of Eq. (3) shows[8,9] that, if $L>c\tau$ (i.e., $\xi>0$) there are no runaways and no preacceleration, while if $L\ll c\tau$ this equation reduces, as it must, to Eq. (1) with $m_0+\delta m=m$. Clearly the higher-order time derivatives $d^{n+2}\vec{R}(t)/dt^{n+2}$ in Eq. (2), proportional to positive powers of L, are essential in suppressing the runaways.

Turning to the quantum mechanics of a point electron, we find similar results with the electron Compton wavelength λ formally playing the role of a size parameter. First we derive the (Heisenberg-picture) operator form of the Lorentz force equation from the standard Hamiltonian[4] and then, proceeding in analogy with the classical theory, we introduce the operator form of the retarded solution to Maxwell's equations, eliminating the self-fields from the equation of motion for the electron. The resulting equation of motion is[11]

$$m_0\frac{d^2\vec{R}(t)}{dt^2}=\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-i)^{n+1}}{n!c^n}\int d\vec{x}\,d\vec{x}'\tfrac{1}{2}[\rho(\vec{x}-\vec{R}(t))|\vec{x}-\vec{x}'|^{n-1},(\mathrm{ad}^{n+1}H)\vec{j}(\vec{x}',t)]_+$$
$$+e\left\{\vec{E}_{\mathrm{in}}(\vec{R}(t),t)+\frac{1}{2c}\left[\frac{d\vec{R}(t)}{dt}\times\vec{B}_{\mathrm{in}}(\vec{R}(t),t)-\vec{B}_{\mathrm{in}}(\vec{R}(t),t)\times\frac{d\vec{R}(t)}{dt}\right]\right\}, \qquad (4)$$

where $\rho(\vec{x}-\vec{R}(t))$ is the electron's charge density normalized to unity,

$$\vec{j}(\vec{x},t)\equiv\tfrac{1}{2}\left[\rho(\vec{x}-\vec{R}(t)),\frac{d\vec{R}(t)}{dt}\right]_+,$$

and $(\mathrm{ad}A)B=[A,B]_-$, $(\mathrm{ad}^2A)B=[A,[A,B]_-]_-$, etc.

We first study the motion of an electron in the absence of external forces, i.e., for effectively negligible $\vec{E}_{\mathrm{in}}(\vec{R}(t),t)$ and $\vec{B}_{\mathrm{in}}(\vec{R}(t),t)$. Evaluation of the commutators in Eq. (4) gives, after a lengthy calculation,[9]

$$m_0\frac{d^2\vec{R}(t)}{dt^2}=-\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!c^n}A_n\frac{d^{n+2}\vec{R}(t)}{dt^{n+2}}, \qquad (5)$$

where

$$A_n=\left(1+\frac{\lambda}{3(n+2)}\frac{\partial}{\partial\lambda}\right)B_n,$$
$$B_n=\sum_{k=0}^{\infty}\frac{n!}{(n+2k)!}\binom{n+1-2k}{2k}\left(-\frac{\lambda^2}{4}\right)^k\times\int d\vec{x}\,d\vec{x}'\rho(x)|\vec{x}-\vec{x}'|^{n-1+2k}(\nabla_{\vec{x}'}{}^2)^{2k}\rho(x'),$$

and $\lambda\equiv\hbar/mc$ is the electron Compton wavelength. Note that each structure coefficient B_n is a power series in λ^2/L^2 and that if we retain only the $k=0$ term in the series for each B_n, corresponding to the $\hbar\to 0$ limit, we recover Eq. (2) with the external force term omitted. In deriving Eq. (5) nonlinear terms have been omitted as in the classical derivation.

The electron's electrostatic self-mass is given by

$$\delta m=\frac{2e^2}{3c^2}A_0=\frac{2e^2}{3c^2}\left(1+\frac{\lambda}{6}\frac{\partial}{\partial\lambda}\right)B_0,$$

where

$$B_0=\left(1+\lambda\frac{\partial}{\partial\lambda}\right)B_0', \qquad (6)$$
$$B_0'=\frac{2}{\pi}\mathcal{P}\int_0^{\infty}dk\,\frac{\rho^2(k)}{1-(\lambda k/2)^2}=\int d\vec{x}\,d\vec{x}'\frac{\rho(x)}{|\vec{x}-\vec{x}'|}(1+\tfrac{1}{4}\lambda^2\nabla_{\vec{x}'}{}^2)^{-1}\rho(x'),$$

with $\mathcal{P}$ denoting the principal value, and $\rho(k)$ is the Fourier transform of $\rho(x)$. This result was obtained by summing the entire series for B_0 as given in Eq. (5). Notice that if one sets $\lambda=0$ in Eq. (6) and *then* goes to the point limit $[\rho(k)=1]$, one obtains the classical linearly divergent result

for δm, whereas if one goes to the point limit keeping λ finite, $\delta m = 0$. For a Yukawa form factor, $\rho(k) = (1+k^2L^2)^{-1}$, the self-mass assumes its maximum value when $L \sim \lambda$, in which case $(\delta m)_{\max} \sim \alpha m_0$, where α is the fine-structure constant. Furthermore, we find that the self-mass given by Eq. (6) can even be *negative* for L small compared to the Compton wavelength.

We now turn to the remaining coefficients. *In the point limit,*

$$A_n = \begin{cases} 0, & \text{for } n \text{ even} \\ \dfrac{2n(4n+5)}{3(n+1)(n+2)}(2n-1)!!(-1)^{(n-1)/2}\lambda^{n-1}, & \text{for } n \text{ odd}. \end{cases} \tag{7}$$

Note that A_1 agrees with the classical result, as expected since it is dimensionless, and that the higher coefficients A_n are proportional to λ^{n-1} (n odd). Equations (5) and (7) show an obvious similarity to the classical equation (2).

Next we investigate the solutions of Eq. (5). Taking matrix elements of the operator $d\vec{R}(t)/dt$ between stationary states of the Hamiltonian, we have $[d\vec{R}(t)/dt]_{mn} = \exp(iE_{mn}t/\hbar)[d\vec{R}(0)/dt]_{mn}$, where $E_{mn} \equiv E_m - E_n$. If there are runaway solutions, there must be states $|m\rangle$ and $|n\rangle$ for which $[d\vec{R}(0)/dt]_{mn} \neq 0$ and for which $\beta \equiv iE_{mn}/\hbar$ has a positive real part. Supposing $[d\vec{R}(0)/dt]_{mn} \neq 0$ and taking the matrix elements of Eq. (5) between states $|m\rangle$ and $|n\rangle$, one obtains a power series in the variable $\eta \equiv \beta(\lambda/c)$. Inside its circle of convergence $|\eta| < \frac{1}{2}$, this power series may be summed to obtain the following equation for β (after factoring out the root $\beta = 0$):

$$\begin{aligned} 1 &= \tfrac{2}{3}\alpha f(\eta), \\ f(\eta) &= -\frac{4i}{3}[(1-2i\eta)^{-1/2} - (1+2i\eta)^{-1/2}] \\ &\quad - \frac{7}{3\eta}[(1-2i\eta)^{1/2} + (1+2i\eta)^{1/2} - \tfrac{2}{7}] \\ &\quad + \frac{2i}{3\eta^2}[(1-2i\eta)^{3/2} - (1+2i\eta)^{3/2}]. \end{aligned} \tag{8}$$

Detailed analysis shows that, for the physical value of the fine-structure constant α, Eq. (8) has no roots inside the circle of convergence $|\eta| < \frac{1}{2}$. The condition $|\eta| < \frac{1}{2}$ corresponds physically to the condition $E_{mn} < \frac{1}{2}mc^2$ and our nonrelativistic formalism, with its systematic neglect of terms of order $v^2/c^2, (v^2/c^2)^2, \ldots$, can be expected to be consistent only if this condition is respected. But within this domain, we see that there are no runaways. Likewise it is clear that runaways are not present in the correspondence limit of the allowed solution to Eq. (5):

$$\beta = 0,$$

$$\left[\frac{d\vec{R}(t)}{dt}\right]_{mn} = \left[\frac{d\vec{R}(0)}{dt}\right]_{mn} = \text{constant}.$$

Finally, we discuss preacceleration. Assuming a c-number time-dependent external force $\vec{F}(t)$, the acceleration is given by

$$m\vec{A}(\omega) = \frac{\vec{F}(\omega)}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)} \tag{9}$$

where $f(\eta)$ is defined in Eq. (8) and

$$\vec{A}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt \frac{d^2\vec{R}(t)}{dt^2} e^{-i\omega t},$$

$$\vec{F}(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{+\infty} dt\, \vec{F}(t) e^{-i\omega t}.$$

For consistency we must require the force to have no Fourier components for frequencies $|\omega| > c/2\lambda$. This condition corresponds closely to that discussed above (i.e., $E_{mn} < \frac{1}{2}mc^2$) and means that the applied force changes slowly compared to the time required for light to cross the electron's Compton wavelength. Equation (9) can then be rewritten as

$$m\frac{d^2\vec{R}(t)}{dt^2} = \int_{-\infty}^{+\infty} dt'\, \vec{F}(t')G(t-t'), \tag{10}$$

with

$$G(t-t') = \int_{-c/2\lambda}^{+c/2\lambda} \frac{d\omega}{2\pi} \frac{e^{i\omega(t-t')}}{1 - \frac{2}{3}\alpha f(i\omega\lambda/c)}. \tag{11}$$

Investigation of Eq. (11) shows that the response function $G(t-t')$ is spread about the origin $(t-t'=0)$ with a width given by the characteristic time $\Delta t \sim \hbar/E_{mn} \sim \lambda/c$. This time scale, set jointly by the dynamics [through Eqs. (5) and (7)] and the uncertainty principle, is very large compared to the time $\tau \sim \alpha\lambda/c$ associated with classical preacceleration, implying that there is no observable violation of causality in quantum mechanics.

The correspondence limit (i.e., the limit $\hbar \to 0$) of the solution given by Eqs. (10), (11) is the solution of the equation[12]

$$\begin{aligned} m\frac{d^2\vec{R}(t)}{dt^2} &= \vec{F}(t) + \tau\frac{d\vec{F}(t)}{dt} + \tau^2\frac{d^2\vec{F}(t)}{dt^2} + \cdots \\ &\simeq \vec{F}(t) + \tau\frac{d\vec{F}(t)}{dt}, \end{aligned} \tag{12}$$

the last approximate equality being a consequence of the fact that $\vec{F}(t)$ changes very little on the time scale of τ. In effect, then, quantum theory yields the usual radiative damping effects in the correspondence limit (e.g., the radiative linewidth of a low-frequency oscillator) while eliminating the possibility of observable noncausal behavior.

The authors wish to offer their warmest thanks to Professor Henry Primakoff for stimulating conversations and helpful advice on this problem and for his careful scrutiny of the manuscript. Hearty appreciation is also expressed to Professor S. L. Adler, Professor R. T. Powers, and Professor A. S. Wightman for their valued comments on a preliminary draft of the paper.

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*This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contracts Nos. AT(11-1)-3069 and AT(30-1)-3071, and by the National Science Foundation under Contract No. GT-33959X.

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[10]The nonlinear and magnetic terms are of order v^2/c^2 or smaller times the terms retained. The neglect of such terms in studying the motion of a nonrelativistic electron would appear to be justified providing analysis of the linear equation does not reveal runaway solutions.

[11]In obtaining the equation of motion we have introduced a static charge density into the Hamiltonian (Ref. 4). We will pass to the point limit at an appropriate stage of the calculation.

[12]One can derive an integrodifferential equation relating the acceleration and the external force which is equivalent to the Lorentz-Dirac equation together with an asymptotic condition which is suitable for the elimination of runaways. Equation (12) follows from this equation when it is assumed that the force at $t' = t + \tau s$ can be expanded in a Taylor series about $s = 0$. See the works cited in Ref. 3 as well as D. Iwanenko and A. Sokolov, *Klassiche Feldtheorie* (Akademie, Berlin, 1953); R. Haag, Z. Naturforsch. 10A, 752 (1955); G. N. Plass, Rev. Mod. Phys. 33, 37 (1961); F. Rohrlich, Ann. Phys. (N.Y.) 13, 93 (1961).