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Field theory of relativistic strings. I. Trees*

Michio Kaku and K. Kikkawa
Department of Physics, The City College of the City University of New York, New York, New York 10031
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We present an entirely new kind of field theory, a field theory quantized not at space-time points, but quantized along an extended set of multilocal points on a string. This represents a significant departure from the usual quantum field theory, whose free theory represents a definite set of elementary particles, because the field theory on relativistic strings can accommodate an infinite set of linearly rising Regge trajectories. In this paper, we (1) present canonical quantization and the Green's function of the free string, (2) introduce three-string interactions, (3) resolve the question of multiple counting, (4) complete the counting arguments for all N-point trees, and (5) introduce four-string interactions which yield a Yang-Mills structure when the zero-slope limit is taken. In future work, we will discuss loops and explicitly calculate Reggeon-Pomeron and Pomeron-Pomeron interactions.

I. INTRODUCTION

Quantum field theory provides one with a convenient formalism in which to describe the world of strong interactions. Building on the phenomenal successes of quantum electrodynamics, quantum field theory yields scattering matrices which are Lorentz-invariant, crossing-symmetric, unitary, and analytic. Quantum field theory, however, is a clumsy formalism in which to describe simple phenomenological properties of hadronic interactions, such as Regge behavior, resonances, factorization, and duality. S-matrix theory, on the other hand, is another hadronic formalism, but one which avoids the problems of finiteness and renormalizability. S-matrix theory, though it can conveniently accommodate these features of strong interactions, can, at best, only yield qualitative predictions concerning hadronic scattering amplitudes.

In this paper we present *an entirely new kind of field theory*,[1] a field theory defined on multilocal relativistic strings, which combines the attractive features of both quantum field theory and S-matrix theory, and reproduces the properties of the dual resonance model. Because the free string has well-defined resonances, this new field theory easily describes a hadronic world with infinitely rising Regge trajectories. This new field theory, though it has all the features of quantum field theory (such as canonical quantization relations, Green's functions, equations of motion, etc.), represents a significant departure from the usual theory, for it is a field theory defined not at a *point* in space-time, but a field theory which is defined on an infinite set of multilocal points on a string. It is a field theory which describes not just the interactions of a certain set of fundamental particles; it is a field theory which accommodates the interactions of an infinite number of hadrons.

In this paper we will write down a multilocal Lagrangian formalism from which the properties of the dual resonance model[2] can be rigorously derived. We will introduce equal-time commutation relations and Green's functions defined on strings. We will also introduce interactions, which essentially represent the successive splittings and reconstitutions of the free string. The interaction comes in *two* terms: The first represents the breaking of one string into two smaller ones; the second represents a *four*-string interaction. Using the usual Dyson product and the Wick expansion, we can calculate the unitary S matrix. We find that each term in the expansion has a simple intuitive picture. In Fig. 1, for example, the four-point function can be represented as two strings which emerge from the infinite past, which then momentarily merge and then split up, and then propagate to the infinite future. We will show that these "Feynman" diagrams are equivalent to the amplitudes introduced earlier in the dual resonance model.

Though theorists have suspected that the dual resonance model could be reformulated in terms of a second-quantized field theory, attempts to isolate the underlying field theory have always been frustrated for several reasons.

First, there is the problem of multiple counting. Any field theory, including this one, yields Born terms which contain *both* s- and t-channel poles. In the dual resonance model, however, *one* dual diagram can be expressed either as a sum over s- *or* t-channel poles; hence there emerges the problem of multiple counting, which becomes quite severe as we go to higher-point functions.

Second, there are formal difficulties in writing down a second-quantized dual theory. As we shall see in Sec. III, there does not exist a simple canonical momentum to the field variable if we quantize covariantly.

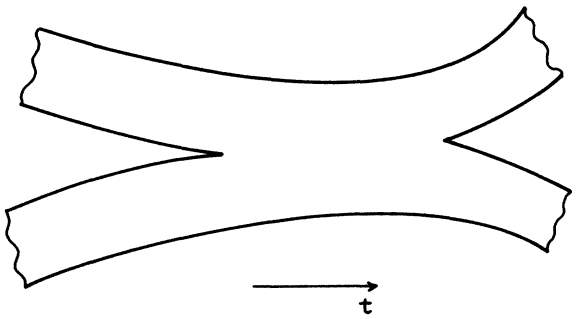

FIG. 1. Scattering of two pairs of strings into another pair of strings (4-point amplitude).

And lastly, the gauge problem persists in the second-quantized approach. The formulation of gauges, crucial to ghost elimination, is difficult to express in second-quantized form.

All these difficulties are resolved once we specify light-cone quantization. In particular, in this choice of gauge, though it yields both s- and t-channel pole diagrams as shown in Figs. 2 and 3, the *sum* of these two diagrams yields the usual 4-point beta function. The generalization to N-point functions is straightforward once we know that *certain specific sums of light-cone diagrams equal one N-point dual amplitude,* thus resolving the question of multiple counting. The other questions, as we shall see in Sec. III, are also resolved once the light-cone gauge is specified.

In addition to representing a significant departure from all previous field theories, the second-quantized formalism of the dual resonance model has several distinct advantages over the usual one.

First, the field-theory approach explicitly contains a four-string interaction in the master Lagrangian, which reduces to the Yang-Mills structure in the zero-slope limit. The usual dual resonance model (DRM) contains only three-Reggeon vertices, in which it is difficult to see how a Yang-Mills theory can be reproduced to all orders. In the usual DRM, we must tediously take the zero-slope limit for N-point amplitudes before any Yang-Mills structure is revealed.

Second, theorists have long assumed that the usual DRM can be unitarized through the introduction of multiloop diagrams,[3] though there is no formal proof outside of the tree theorem. The field-theory formalism contains an explicit Hermi-

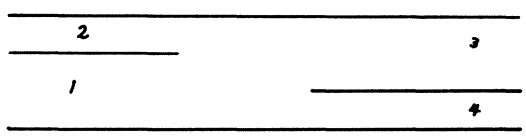

FIG. 2. A light-cone diagram of a 4-point amplitude: s-channel resonance exchange.

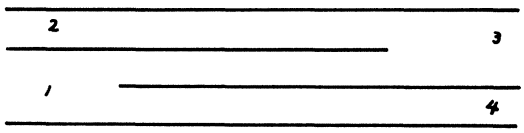

FIG. 3. Another light-cone diagram of a 4-point amplitude: t-channel resonance exchange.

tian Hamiltonian, which produces unitary S matrices (excluding the question of the tachyon).

Third, the question of the weights for the perturbation diagrams has never been resolved, except at the single-loop level.[3] The field-theory formalism automatically fixes all weights for all diagrams.

Fourth, isospin is introduced *ad hoc* via Chan-Paton factors in the usual DRM. In the field-theory approach, the Chan-Paton factors emerge naturally by assigning isospin indices to the master field variable.

Fifth, the DRM has long been suspected of being equivalent to an infinite-component field theory. In the field-theory formalism, the Lagrangian explicitly displays this character.

This paper is divided up as follows. In Sec. II we review the first-quantized formalism of the relativistic string. In Sec. III we write down the second-quantized field theory on the light cone. In Sec. IV we calculate the Green's function. In Sec. V we discuss the interaction. In Sec. VI we discuss multiple counting. In Sec. VII we discuss the four-string interaction. And in Sec. VIII we discuss the measure problem and give concluding remarks.

In summary, we wish to emphasize three important theoretical points made in this paper. First, we are presenting a new type of field theory which differs radically from all field theories previously studied, but which preserves the same canonical basis. Like the usual field theories, this field theory defined on a string possesses a renormalizable, unitary form with canonical quantization relations, Green's functions, and a Coulomb gauge condition, except that the free Lagrangian, instead of representing a few select elementary particles, now represents an infinite set of particles lying on linearly rising Regge trajectories.

Second, this field theory is the second quantization of the dual resonance model. The DRM, though it has met with considerable success in providing a theoretical framework for strong interactions, lacks a rigorous unitary formulation. There exists no proof of the conjecture that summing over all topologically distinct Riemann surfaces with equal weights generates a unitary dual amplitude. Now that the model can be expressed in terms of a field theory, the questions of uni-

tarity and weights can be studied in exactly the way usual field theories deal with the problems of unitarity.

Third, in this paper we present new results on the geometrical nature of the relativistic string. We prove that the three-Reggeon vertex function can be rigorously shown to be the amplitude for the breaking of a classical string. We also show the necessity for a hitherto unsuspected four-string interaction term.

Finally, we wish to comment on the reason why this multilocal field theory is consistent with causality. Traditionally, nonlocal field theories have been plagued with the problem of introducing interactions which do not violate relativity. In our model, however, the problem of causality does not appear for two reasons.

First, the quantization procedure is defined on relativistic strings, whose classical, free behavior is known to be consistent with causality. We can also demonstrate Poincaré invariance in both first- and second-quantized forms.

Second, the strings interact with each other only at points. For example, the 3-string interaction only takes place when the end points of two strings join to form a third. Likewise, the 4-string interaction takes place when two strings touch at an interior point.

II. FIRST QUANTIZATION

Before considering the question of second quantization, it will be instructive to reexamine the first-quantized approach to the theory of the relativistic string.

Nielsen, Nambu, and Susskind[4] formulated the DRM in terms of a first-quantized coordinate $X_\mu(\sigma, t)$, which obeys the two-dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \sigma^2}\right) X_\mu(\sigma, t) = 0 , \tag{2.1}$$

which sweeps out a two-dimensional Riemann surface, with t going from $-\infty$ to $+\infty$ and σ going from 0 to π. From the Lagrangian

$$\begin{aligned} &L = \frac{1}{4\pi}(\dot{X}_\mu{}^2 - X'^2_\mu), \\ &\dot{X}_\mu = \frac{\partial}{\partial t} X_\mu , \quad X'_\mu = \frac{\partial}{\partial \sigma} X_\mu , \end{aligned} \tag{2.2}$$

we can introduce the canonical momentum

$$P_\mu(\sigma, t) = \frac{\delta L}{\delta(\partial X_\mu / \partial t)} . \tag{2.3}$$

Variations of L near the boundary $\sigma = 0, \pi$ enable us to write $X'(0, t) = 0 = X'(\pi, t)$ and hence decompose X_μ and P_μ as follows:

$$X_\mu(\sigma, \tau) = x_\mu(\tau) + \sum_{n=1}^{\infty} (a_n^\dagger e^{-n\tau} + a_n e^{n\tau}) \left(\frac{2}{n}\right)^{1/2} \cos n\sigma , \tag{2.4}$$

$$P_\mu(\sigma, \tau) = \frac{1}{\pi}\left[p_\mu(\tau) + (-i) \sum_{n=1}^{\infty} (a_n e^{n\tau} - a_n^\dagger e^{-n\tau}) \left(\frac{n}{2}\right)^{1/2} \cos n\sigma \right], \tag{2.5}$$

such that

$$[P_\mu(\sigma), X_\nu(\sigma')] = -i g_{\mu\nu} \delta(\sigma - \sigma') . \tag{2.6}$$

The Hamiltonian in this first-quantized approach can be written as

$$H = \pi \int_0^\pi d\sigma [: P_\mu{}^2(\sigma) + (2\pi)^{-2} X'^2_\mu(\sigma) :] , \tag{2.7}$$

so that the Nambu equation of motion can be written as

$$\begin{aligned} &(L_0 + m_0^2) | X \rangle = 0 , \\ &L_0 = \sum_{n=1}^{\infty} n a_n^\dagger a_n + p_\mu{}^2 . \end{aligned} \tag{2.8}$$

The Hilbert space spanned by the harmonic oscillators $a_n^\dagger$ represents an infinite set of linearly rising Regge trajectories. Notice that, however, the Lorentz metric creates resonances with negative metric, i.e., ghosts.

A great step in reformulating the gauge problem, which sheds much light on the underlying field-theoretic nature of the DRM, was made by Nambu, Mansouri, Goto, and Chang,[5] who reexpressed the model with the Lagrangian

$$\begin{aligned} &L = -\frac{1}{2\pi} \sqrt{-g} , \\ &g = \dot{X}^2 X'^2 - (\dot{X} \cdot X')^2 . \end{aligned} \tag{2.9}$$

Since this Lagrangian is invariant under arbitrary reparameterizations

$$\begin{aligned} &\sigma \to \sigma'(\sigma, t) , \\ &t \to t'(\sigma, t) , \end{aligned} \tag{2.10}$$

we are free to choose two gauge conditions

$$\begin{aligned} &\dot{X} \cdot X' = 0 , \\ &\dot{X}^2 + X'^2 = 0 . \end{aligned} \tag{2.11}$$

We notice that the moments of these gauge conditions [which are sufficient to linearize the Lagrangian to give back Eq. (2.2)] give us the Ward identities of Virasoro[6]:

$$L_{\pm n} = \frac{\pi}{2} \int_{-\pi}^{\pi} d\sigma \, e^{in\sigma} : (\dot{X}_\mu \pm X'_\mu)^2 : , \tag{2.12}$$

$$\langle X' | L_n | X \rangle = \langle X' | (L_0 - 1) | X \rangle = 0 , \tag{2.13}$$

where $|X\rangle$ is an on-shell state of the dual Hilbert space.

The proof that the Ward identities (2.13) are sufficient to eliminate ghosts caused by the negative metric was completed by Brower[7] and also by Goddard and Thorn.[8] Because Lorentz invariance is never broken and because the Ward identities are imposed on the states themselves, we see that this approach corresponds to the Gupta-Bleuler formulation.

The Coulomb gauge formulation was finally completed by Goldstone, Goddard, Rebbi, and Thorn[9] (GGRT), who revealed the importance of the string interpretation of the DRM. Their essential observation was that a relativistic *string* $X_\mu(\sigma)$ can be represented in terms of only its transverse components, and hence the question of ghosts never appears. Their constraints

$$\begin{aligned} &\dot{X}^2 + X'^2 = 0, \\ &\dot{X}\cdot X' = 0, \\ &X_+ = \frac{1}{\sqrt{2}}(X_0 + X_{D-1}) \\ &\quad = i\tau = t \end{aligned} \tag{2.14}$$

are sufficient to determine all coordinates in terms of the physical operators $\vec{X}(\sigma)$, thereby eliminating the redundant longitudinal modes. In the GGRT gauge, the Nambu equation of motion, (2.8), is now replaced by an integral over only transverse modes. Of course, choosing a noncovariant gauge forces one to check that the Poincaré generators once again satisfy the correct commutation relations. They found that the commutator $[M_{-i}, M_{-j}]$ can only be equal to zero if the dimension of space-time is 26.

Interactions were first introduced into the string picture by Gervais and Sakita,[10] who showed that the ambiguities in the functional expression

$$\sum_{j=0}^{\infty} g^{2j+N-2} \int \frac{\mathfrak{D}X_\mu(\sigma,\tau)}{\prod_{\sigma,\tau} L(\sigma,\tau)} \times \exp\left[\int_{R_j} L(\sigma,\tau)\,d\sigma\,d\tau + \sum_{n=1}^{N} P_{\mu,n} X_\mu(z_n)\right], \tag{2.15}$$

where $L = (-2\pi)^{-1}\sqrt{-g}$, can be eliminated by choosing the gauges

$$\prod_{\sigma,\tau} \delta([\dot{X}_\mu(\sigma,\tau) \pm X'_\mu(\sigma,\tau)]^2)\,\delta(X_+ - \mathfrak{F}), \tag{2.16}$$

and then properly accounting for the Faddeev-Popov determinants. In (2.15) $P_{\mu,n}$ represents the external momentum attached at $(\sigma,\tau) = z_n$. By taking R_0 to be the upper half plane and the z_n to be real Koba-Nielsen variables, we can perform the functional integrations by completing the square, and we arrive at the usual N-point integrand. They showed that this formulation is equivalent to a Hamiltonian formulation in which the operators are all transverse, i.e., ghost free. Though their functional approach makes use of the Nambu-Goto-Chang-Mansouri (NGCM) Lagrangian (2.9) and the light-cone gauge of GGRT, their formulation lacks an intuitive string picture originally implied in the work of GGRT. (To reproduce the loop diagrams, we must sum over distinct topologies R_j, which must represent planar, nonplanar, and nonorientable topologies, and j is the loop number.)

The final step in the first-quantized string picture was completed by Mandelstam,[11] who found the conformal mapping which maps the upper half plane onto light-cone surfaces much like Figs. 2 and 3, which immediately allows a string interpretation. His mapping,

$$\begin{aligned} &\rho = \sum_{i=1}^{N} \alpha_i \ln(z - z_i), \\ &\sum_{i=1}^{N} \alpha_i = 0, \end{aligned} \tag{2.17}$$

can be easily checked to produce a mapping of narrow, parallel strips which join and split, much like relativistic strings. He was then able to isolate the three-Reggeon vertex function[12] in this formulation and prove that both Lorentz invariance and conformal invariance depended critically on the space-time dimensionality being set at 26.

The importance of Mandelstam's approach becomes evident in the second-quantized approach. We will see, in the next section, that the perturbation series produced by the field theory on strings is in one-to-one correspondence with diagrams which have the topology given by the previous mapping.

Notice that, though the string picture presented so far resembles a second-quantized theory because of the presence of an infinite number of harmonic oscillators, it is actually only a first-quantized theory, because we are only quantizing the *coordinate* $X_\mu(\sigma)$. There are an infinite number of oscillators only because they represent the normal modes of the string, i.e., because the first quantization is performed over an *extended* object. The lack of a second-quantized field theory is obvious by looking at Eq. (2.15), where we must sum over an infinite sequence of topologies. A second-quantized theory would have a master Lagrangian from which all topologies can be derived. In a first-quantized approach, like this one, we must insert the topologies in by hand.

III. SECOND QUANTIZATION ON THE LIGHT CONE[13]

In the transition from the usual first quantization of quantum field theory to second quantization, we impose canonical equal-time quantization relations not on the coordinates but on the fields themselves.

Much the same approach will be taken for the quantization on relativistic strings. We begin trying to introduce a field *functional* $\Phi[X]$ of the string variable $X_\mu(\sigma)$. Because this master field $\Phi[X]$ is a *functional* of the string variable, *it loses explicit dependence on the parameter* σ.

We divide the interval on which the string is defined ($\sigma = [0, \pi\alpha]$) into an infinite set of points (σ_1, σ_2, σ_3, . . . , σ_N). The functional Φ is then a function of *all* string variables taken at *all* points along the interval:

$$\Phi[X] = \Phi(X_\mu(\sigma_1), X_\mu(\sigma_2), X_\mu(\sigma_3), \ldots, X_\mu(\sigma_N)). \tag{3.1}$$

Because coordinates taken at *all* points along the interval appear in the functional, the functional loses explicit dependence on the interval parameter σ.

At this point, there is a technical difficulty which emerges when trying to define a unique second-quantized procedure, mentioned in the Introduction. If the Lagrangian $\mathcal{L}$ for the second-quantized string contains the field functional $\Phi[X]$, then the canonical momentum to the field functional should be given by

$$\Pi[X, \sigma] = \frac{\delta \mathcal{L}}{\delta(\partial\Phi[X]/\partial X_0(\sigma))}, \tag{3.2}$$

where the $X_0(\sigma)$'s represent an infinite of "times" defined along the string. Notice that the canonical momentum then displays a dependence on the string parameter σ, and hence there is no one-to-one correspondence between the field functional Φ and its momentum conjugate. For this reason, quantizing the relativistic string in a Lorentz-covariant fashion leads to profound problems of definition.

The lack of a well-defined canonical momentum to the field variable is immediately resolved in the light-cone formalism of GGRT. It we take the gauge conditions (2.14), then we see that the momentum

$$\Pi[X] = \frac{\delta \mathcal{L}}{\delta(\partial\Phi[X]/\partial X_+)} \tag{3.3}$$

can now be unambiguously defined, and that there is now a one-to-one correspondence between the field variable and its momentum conjugate. Previously, there was an infinite number of "times," each time corresponding to a coordinate $X_0(\sigma_i)$ defined at a point σ_i in the string interval. *Choosing the light-cone gauge fixes a common time for the entire string.*

Now, we remove all dependence of the field functional Φ on longitudinal components. Though we started originally with $\Phi[X] = \Phi[\vec{X}, X_+, X_-]$, we use conditions (2.14) to eliminate X_+ and X_-. The functional still has a dependence on the zero modes of X_-. Then we take the Fourier transform with respect to the zero mode of X_-, so the field depends on p_+. We now take the region of the parameterization of the string to be $[0, \pi\alpha]$ ($\alpha = 2p_+$) for later convenience when we have breaking strings.

Before simply writing down the Lagrangian, perhaps it is instructive to first identify the field-theoretical features which must follow from variations of the Lagrangian. This will give us an appreciation of the various components of the second-quantized Lagrangian.

First, we wish the Lagrangian to give us the Schrödinger equation on a string:

$$\pi \int_0^{\pi\alpha} d\sigma \left(-\frac{\partial^2}{\partial \vec{X}^2(\sigma)} + \frac{1}{(2\pi)^2} \vec{X}'^2(\sigma) \right) \Phi_{p_+}[X] = i \frac{\partial}{\partial X_+} \Phi_{p_+}[X] = (L_0 + m_0^2) \Phi_{p_+}[X], \tag{3.4}$$

which is the ghost-eliminated version of Nambu's Eq. (2.8). Guaranteeing condition (3.4) ensures us that the usual Hamiltonian in the first-quantized approach continues to produce translations in time for the field Φ.

The next condition that we wish the second-quantized approach to incorporate is equal-time canonical quantization relations. A desirable feature of our Lagrangian approach would be

$$\Pi[X] = i\Phi^\dagger[X], \tag{3.5}$$

which would produce

$$[\Phi_{p_+}[\vec{X}_1, \tau_1], \Phi^\dagger_{q_+}[\vec{X}_2, \tau_2]]_{\tau_1 = \tau_2} = \delta(p_+ - q_+) \prod_\sigma \delta(\vec{X}_1(\sigma) - \vec{X}_2(\sigma)). \tag{3.6}$$

We wish, therefore, to write down a master Lagrangian which will preserve the Schrödinger equation of (3.4) (which ensures that we reproduce the usual spectrum of states and the usual Hamiltonian in the first-quantized approach) and a Lagrangian which will yield (3.5), so that canonical quantization relations on the string can be written down.

We now write down the master Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_P \tag{3.7}$$

where

$$\mathfrak{L}_0=\int_0^\infty dp_+\int_0^{2\pi p_+}d\sigma\int\mathfrak{D}\vec{X}\left[\Phi^\dagger_{p_+}[X]i\frac{\partial}{\partial X_+}\Phi_{p_+}[X]-\alpha\pi^2\Phi^\dagger_{p_+}[X]\left(-\frac{\partial^2}{\partial\vec{X}^2(\sigma)}+\frac{1}{(2\pi)^2}\vec{X}'^2(\sigma)\right)\Phi_{p_+}[X]\right]. \tag{3.8}$$

($\mathfrak{L}_1$ is elaborated in Sec. V and $\mathfrak{L}_2$ in Sec. VII. $\mathfrak{L}_P$ which is related to the Pomeron contribution will be discussed in a subsequent paper.) We first notice that Π is indeed independent of σ, so that the conjugate momentum to the field functional Φ is simply its own Hermitian conjugate, i.e., Eq. (3.5). And we also notice that the Euler-Lagrange equations for the free Lagrangian yield the Schrödinger equation defined on a string, Eq. (3.4).

Much as in ordinary field theory, we can expand the field into functions of the classical solutions of the equations of motion. The Schrödinger equation (3.4), written in terms of each oscillator mode, simply reproduces the equations of the harmonic oscillator, so we can write the field function Φ in terms of Hermite polynomials. In analogy to ordinary field theory, we now write the solution to the field which satisfies both the equations of motion and the canonical commutation relations:

$$\Phi_{p_+}[X]=\int\frac{d\vec{p}}{2\pi}\sum_{\{n_l^{(i)}\}}A_{p_+,\vec{p},\{n_l^{(i)}\}}f_{p_+,\vec{p},\{n_l^{(i)}\}}(\vec{x},x_l^{(i)},X_+) \tag{3.9}$$

and

$$[A_{p_+,\vec{p},\{n_l^{(i)}\}},A^\dagger_{q_+,\vec{q},\{m_k^{(j)}\}}]=\delta(p_+-q_+)\delta(\vec{p}-\vec{q})\delta_{\{n_l^{(i)}\},\{m_k^{(j)}\}}, \tag{3.10}$$

where

$$f_{p_+,\vec{p},\{n_l^{(i)}\}}(\vec{x},x_l^{(i)},X_+)=\prod_{i=1}^{D-2}\prod_{l=1}^{\infty}H_{\{n_l^{(i)}\}}(x_l^{(i)})e^{-lx_l^{(i)2}/(4\pi)}\times\exp[i(\vec{p}\cdot\vec{x}-X_+E(p_+,\vec{p},\{n_l^{(i)}\}))], \tag{3.11}$$

$$E(p_+,\vec{p},\{n_l^{(i)}\})=\frac{1}{2p_+}\left(\sum_{l=1}^{\infty}ln_l^{(i)}-\vec{p}^{\,2}-\alpha_0\right),$$

$$X^{(i)}(\sigma)=x^{(i)}+2\sum_{l=1}^{\infty}x_l^{(i)}\cos(l\sigma/\alpha),$$

and i is the transverse Lorentz index, l is the label for the excitation level of the string, and $n_l^{(i)}$ is the number n of excitations in the ith Lorentz direction in the lth level. Notice that the Schrödinger equation, which is actually a superposition of an infinite number of harmonic-oscillator equations, yields eigenfunctions which are Hermite polynomials. In order to prove the canonical equal-time commutation relations, we need only the commutator between the operators A [Eq. (3.10)], and the completeness relation for Hermite polynomials:

$$\int\frac{d\vec{p}}{2\pi}\sum_{\{n_l^{(i)}\}}f^*_{p_+,\vec{p},\{n_l^{(i)}\}}(\vec{x},x_l^{(i)},X_+)f_{p_+,\vec{p},\{n_l^{(i)}\}}(\vec{y},y_l^{(i)},X_+)=\prod_{i=1}^{D-2}\prod_{l=1}^{\infty}\delta(x_l^{(i)}-y_l^{(i)})\delta(\vec{x}-\vec{y}). \tag{3.12}$$

At this point, we must emphasize the difference between the little $a_n^\dagger$, the harmonic-oscillator creation operator found in first-quantized dual models, and the large $A^\dagger$ that appears in the second-quantized formalism. First, the string X is no longer an operator in our formulation. $a_n^\dagger$ in the conventional formulation creates simply one oscillation mode of the string X; it does not represent the string itself. The $A^\dagger$, on the other hand, *creates an entire string,* with excitations labeled by the indices $\{n_l^{(i)}\}$. The ground state of the oscillator $A^\dagger$ is called $|0\rangle\rangle$ and should not be confused with the ground state $|0\rangle$ of the usual oscillator $a^\dagger$.

Now all that remains to be proved in the free-field theory is Lorentz invariance. Because we have explicitly taken the gauge conditions (2.14) before we quantized the field theory, we must be sure that we can define Poincaré generators such that the correct commutation relations are unchanged by the choice of gauge.

We shall explicitly construct the second-quantized Poincaré generators through a simple technique. Previously, in the work of GGRT, it was shown that the first-quantized Poincaré generators

$$M_{\mu\nu}=\tfrac{1}{2}\int_0^{\pi\alpha}d\sigma\,[X_\mu(\sigma)P_\nu(\sigma)-X_\nu(\sigma)P_\mu(\sigma)]_{\rm sym} \tag{3.13}$$

consistent with the Coulomb gauge (2.14) can be shown to satisfy the correct commutation relations if the dimension of space-time is 26. We use the fact that if $M_{\mu\nu}$ are first-quantized operators acting on fields O and $O^\dagger$, then the second-quantized generators $\mathfrak{M}_{\mu\nu}=OM_{\mu\nu}O^\dagger$ satisfy the same commutation relations as the original $M_{\mu\nu}$, if

$$[O,O^\dagger]=1. \tag{3.14}$$

It is then straightforward to show that we can define second-quantized Poincaré generators consistent with the Coulomb gauge and covariance if we define

$$\mathcal{P}_\mu = \int_0^\infty dp_+ \int \mathfrak{D}\vec{\mathbf{X}}(\sigma)\, \Phi^\dagger_{p_+}[X]\, \underline{P}_\mu \Phi_{p_+}[X],$$

$$\mathfrak{M}_{\mu\nu} = \int_0^\infty dp_+ \int \mathfrak{D}\vec{\mathbf{X}}(\sigma)\, \Phi^\dagger_{p_+}[X]\, M_{\mu\nu} \Phi_{p_+}[X],$$

where

$$\underline{P}_\mu = \int_0^{\pi\alpha} d\sigma\, P_\mu(\sigma),$$

$$P_- = \int_0^{\pi\alpha} \frac{\pi}{2p_+}\left[-\frac{\partial^2}{\partial \vec{\mathbf{X}}(\sigma)^2} + \frac{1}{(2\pi)^2}\vec{\mathbf{X}}'^2(\sigma)\right] d\sigma, \quad (3.15)$$

$$M_{\mu\nu} = \tfrac{1}{2}\int_0^{\pi\alpha} : [X_\mu(\sigma)P_\nu(\sigma) - X_\nu(\sigma)P_\mu(\sigma)]_{\text{sym}} : d\sigma.$$

We have now shown how to construct a second-quantized canonical field theory defined on free strings.

Before writing down the three-string and four-string interaction terms, it will be instructive to first construct the Green's functions of the field theory. The three- and four-string interaction terms arise naturally from a detailed discussion of the Green's functions of the free theory.

IV. GREEN'S FUNCTIONS

The construction of the Green's functions of the theory sheds much light on its relationship to path integrals defined over Riemann surfaces. We will show that the Green's functions for the freely propagating string simply reproduce the functional average over a rectangular Riemann surface with sources at either end. In this fashion, we simply reproduce the results of Gervais and Sakita,[14] except expressed in a second-quantized formalism.

If $|0\rangle\rangle$ is the ground state of the operator $A^\dagger$, then the Green's function for the field theory is

$$\langle\langle 0|\, \Phi_{p_+}[\vec{\mathbf{X}}_1, \tau_1]\Phi^\dagger_{q_+}[\vec{\mathbf{X}}_2, \tau_2]\,|\, 0\rangle\rangle$$

$$= \langle\langle 0|\, \Phi_{p_+}[\vec{\mathbf{X}}_1] e^{-(\tau_1-\tau_2)H}\Phi^\dagger_{q_+}[\vec{\mathbf{X}}_2]\,|\, 0\rangle\rangle \quad (4.1)$$

(where H is the second-quantized generator appearing in the Poincaré group generators, not the first-quantized operator of the Schrödinger equation). Expanding this, we now have

$$\int \frac{d\vec{\mathbf{p}}}{2\pi}\frac{d\vec{\mathbf{q}}}{2\pi} \sum_{\{n_l^{(i)}\}} \sum_{\{m_k^{(j)}\}} \langle\langle 0|A_{p_+,\vec{\mathbf{p}},\{n_l^{(i)}\}} A^\dagger_{q_+,\vec{\mathbf{q}},\{m_k^{(j)}\}}\,|\, 0\rangle\rangle$$

$$\times \prod_{i,j=1}^{D-2}\prod_{l,k=1}^{\infty} H_{\{n_l^{(i)}\}}(x_{1l}^{(i)}) e^{-lx_{1l}^{(i)2}/2} \exp[i(\vec{\mathbf{p}}\cdot\vec{\mathbf{x}}_1 - X_{+1}E(p_+, \vec{\mathbf{p}}, \{n_l^{(i)}\}))]$$

$$\times H_{\{m_k^{(j)}\}}(x_{2k}^{(j)}) e^{-kx_{2k}^{(j)2}/2} \exp[-i(\vec{\mathbf{q}}\cdot\vec{\mathbf{x}}_2 - X_{+2}E(q_+, \vec{\mathbf{q}}, \{m_k^{(j)}\}))]$$

$$= \prod_{l=1}^{\infty} \exp\left\{\frac{-l}{\sinh(lT/\alpha)}\left[\cosh\frac{lT}{\alpha}(\vec{\mathbf{x}}_{1l}^{\,2} + \vec{\mathbf{x}}_{2l}^{\,2}) - 2\vec{\mathbf{x}}_{1l}\cdot\vec{\mathbf{x}}_{2l}\right]\right\}$$

$$\times\left[\frac{l}{\pi\sinh(lT/\alpha)}\right]^{(D-2)/2}\left\{\exp\left[-\frac{\alpha}{4T}(\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2)^2\right]\right\}\left(\frac{\alpha}{4\pi T}\right)^{(D-2)/2}\delta(p_+ - q_+),$$

$$T = |\tau_1 - \tau_2|. \quad (4.2)$$

The above expression, representing the amplitude for propagating a string prepared in state $x_{1l}^{(i)}$ over to a string in state $x_{2k}^{(j)}$, is simply the solution to the path-integral problem, the functional average over a rectangular Riemann surface (of length $\tau_2 - \tau_1$ and height $2\pi p_+$) with sources $x_{1l}^{(i)}$ and $x_{2k}^{(j)}$ at either end. To clearly reveal this structure, it will be instructive to review the highlights of the work of Gervais and Sakita. In their work, they showed that the propagator of the dual model can always be written as functional averages over Reimann surfaces, which in turn can be reduced to the problem of finding the Neumann function defined over that surface.

Let us begin with a Lagrangian

$$L = \frac{1}{4\pi}\left[\left(\frac{\partial\vec{\mathbf{X}}}{\partial t}\right)^2 - \vec{\mathbf{X}}'^2\right]$$

found in the first-quantized formalism. Now take the functional average over a surface, so that we are now calculating the amplitude for propagating a string $x_{1l}^{(i)}$ at time τ_1 over into a string $x_{2k}^{(j)}$ at time τ_2. The prescription for the amplitude is

$$\int \mathfrak{D}\vec{\mathbf{X}}(\sigma,\tau)\exp\left(\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi\alpha} d\sigma\, L\right)$$

$$\times\prod_{\sigma,\sigma'}\delta(\vec{\mathbf{X}}(\sigma,\tau_1) - \vec{\mathbf{X}}_1(\sigma))\delta(\vec{\mathbf{X}}(\sigma',\tau_2) - \vec{\mathbf{X}}_2(\sigma')), \quad (4.3)$$

where σ and σ' both assume all values from 0 to

$\pi\alpha$. The functional integral is evaluated by the usual means, by completing the square:

$$\vec{X}(\sigma,\tau)\to\vec{X}(\sigma,\tau)+2\int N(\sigma,\tau;\sigma',\tau')\vec{J}(\sigma',\tau')d\sigma'd\tau',$$
$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\sigma^2}\right)N(\sigma,\tau;\sigma',\tau')=-\pi\delta(\sigma-\sigma')\delta(\tau-\tau'), \tag{4.4}$$

$$\int \mathfrak{D}\vec{X}(\sigma,\tau)\exp\left[\int L d\sigma\,d\tau+\int\vec{J}(\sigma,\tau)\cdot\vec{X}(\sigma,\tau)d\sigma\,d\tau\right]$$
$$=\left[\int\mathfrak{D}\vec{X}(\sigma,\tau)\exp\left(\int L d\sigma\,d\tau\right)\right]\exp\left[\int\vec{J}(\sigma,\tau)\cdot N(\sigma,\tau;\sigma',\tau')\vec{J}(\sigma',\tau')d\sigma d\sigma' d\tau d\tau'\right]. \tag{4.5}$$

We now can quote our desired result: The Fourier transform of (4.3) yields terms like

$$\exp\left[\int_0^{\pi\alpha}\vec{P}_1(\sigma)\cdot N(\sigma,\tau_1;\sigma',\tau_2)\vec{P}_2(\sigma')d\sigma\,d\sigma'\right],\quad \vec{J}(\sigma,\tau)=[\vec{P}_1(\sigma)\delta(\tau-\tau_1)+\vec{P}_2(\sigma)\delta(\tau-\tau_2)]/\sqrt{2}. \tag{4.6}$$

The exact expression for the Neumann function is not hard to find. We simply expand the Neumann function in terms of the free-string eigenfunctions and then match boundary conditions. We easily find the unique solution

$$N(\sigma,\tau;\sigma',\tau')=-\tfrac{1}{2}\max(\tau,\tau')$$
$$+\sum_{n=1}^{\infty}\frac{\cos(n\sigma/\alpha)\cos(n\sigma'/\alpha)}{n}\{e^{-n|\tau-\tau'|/\alpha}-\tfrac{1}{2}\operatorname{csch}[(n/\alpha)(\tau_1-\tau_2)][e^{(-n\tau-n\tau'+n\tau_2-n\tau_1)/\alpha}$$
$$+e^{+n(\tau_1-\tau_2)/\alpha}(e^{-n(\tau-\tau')/\alpha}+e^{-n(\tau'-\tau)/\alpha})$$
$$+e^{[n(\tau+\tau')-n\tau_2-n\tau_1]/\alpha}]\}. \tag{4.7}$$

Putting the expression for the Neumann function back into the integral, we arrive back at our original solution. In other words, we have now shown that the single-string Green's function for the second-quantized theory is simply the first-quantized functional average over a Riemann surface with sources.

In summary, our main result for this section is

$$\langle\langle 0|\Phi_{p_+}[\vec{X}_1,\tau_1]\Phi^{\dagger}_{q_+}[\vec{X}_2,\tau_2]|0\rangle\rangle$$
$$=\delta(p_+-q_+)\int\mathfrak{D}\vec{X}(\sigma,\tau)\exp\left(\int L d\sigma\,d\tau\right)\prod_{\sigma}\delta(\vec{X}(\sigma,\tau_1)-\vec{X}_1(\sigma))\prod_{\sigma'}\delta(\vec{X}(\sigma',\tau_2)-\vec{X}_2(\sigma')). \tag{4.8}$$

V. INTERACTIONS

In the previous section we have shown that the Green's function is simply the functional average over a rectangular Riemann surface with sources at either end. Given the one-to-one correspondence between Green's functions and Riemann surfaces, it is then tempting to conjecture that the interaction term, responsible for splitting a string into two smaller strings, is simply in a one-to-one correspondence with the infinitesimal Riemann surface which represents the splitting of a propagator into two smaller propagators, as in Fig. 4.

We are led, therefore, to construct an interaction term which is the limit of the Neumann function defined over the wedge-shaped region of Fig. 4 as the width goes to zero:

$$\lim_{\tau_i\to 0}\int\mathfrak{D}\vec{X}(\sigma,\tau)\exp\left(\int_R L d\sigma\,d\tau\right)\prod_{i=1}^{3}\prod_{\sigma_i}\delta(\vec{X}(\sigma_i,\tau_i)-\vec{X}_i(\sigma_i)). \tag{5.1}$$

In this integral, a string starts at "time" τ_3 and propagates until it splits into two smaller strings at $\tau=0$, which then propagate further on to "times" τ_1 and τ_2. We take each string i to be of length $\pi\alpha_i$. We will take the parameterization shown in Fig. 5:

$$0<\sigma_i<|\pi\alpha_i|,$$
$$\sigma_3=\sigma_1\quad(0<\sigma_1<\pi|\alpha_1|),$$
$$\sigma_3=\sigma_2+\pi|\alpha_1|\quad(0<\sigma_2<\pi|\alpha_2|), \tag{5.2}$$
$$\sum_{i=1}^{3}\alpha_i=0,$$
$$\alpha_3>0,\quad\alpha_1,\alpha_2<0.$$

The product over σ_i is taken to mean a product over all points defined in the region $0<\sigma_i<\pi|\alpha_i|$. The functional integral can be explicitly taken and yields terms like

$$\exp\Big[\int_0^{\pi|\alpha_3|} d\sigma_3 \times \int_0^{\pi|\alpha_i|} \prod_{i=1}^{2} d\sigma_i \vec{P}(\sigma_3)\cdot N(\sigma_3,\tau_3;\sigma_k,\tau_k)\vec{P}(\sigma_k)\Big] \quad (k=1,2), \quad (5.3)$$

where N is the Neumann function defined over region R. On the other hand, our intuition tells us that, as a string breaks, it must do so such that there is a smooth transition between the original string X_3 and its broken pieces X_1, X_2. The string picture motivates us to believe that a string, with boundary conditions $X' = 0$ at the edges, propagates until there exists a point in its interior such that $X' = 0$, where upon it may break into two pieces at that interior point. Continuity leads us to suspect that the limit of the integral in (5.1) should somehow reproduce the condition

$$\vec{X}_3(\sigma_3) = \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) + \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|). \quad (5.4)$$

We are led, therefore, to consider the following action, which will generate a Riemann sheet corresponding to interacting strings:

$$\mathcal{L}_1 = \tfrac{1}{2}g \int \prod_{i=1}^{3} \frac{dp_{+i}}{(2p_{+i})^{1/2}} \mathfrak{D}\vec{X}_i(\sigma_i)\delta(p_{+3}-p_{+1}-p_{+2})\Phi^{\dagger}_{p_{+1}}[X_1]\Phi^{\dagger}_{p_{+2}}[X_2]\Phi_{p_{+3}}[X_3] \times \prod_{\sigma_3} \delta(\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) - \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|)) + \text{H.c.}, \quad 0 < \sigma_3 < \pi|\alpha_3|. \quad (5.5)$$

Though the functional integral given in (5.1) will generate the Riemann surface which can break and join strings, our string picture motivates the action in (5.5). In order to reconcile these two different viewpoints, we must show that

$$\prod_{\sigma_3} \delta(\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) - \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|)) = \lim_{\tau_{1,2,3}\to 0} \int \mathfrak{D}\vec{X}(\sigma,\tau) \exp\Big(\int_R L d\sigma\, d\tau\Big) \times \prod_{i=1}^{3}\prod_{\sigma_i} \delta(\vec{X}(\sigma_i,\tau_i) - \vec{X}_i(\sigma_i)). \quad (5.6)$$

The proof that the Neumann function defined over the Riemann surface corresponding to a breaking string is identical to a δ function defined over the breaking string (when the interaction times approach each other) requires rather involved calculations, so the proof will be presented in Appendix A.[15] The plan for the proof is as follows: As in the case of freely propagating strings, we must construct the Neumann function over the corresponding finite Riemann surface. By expanding the Neumann function over a complete set of eigenfunctions of the free string, and then by matching boundary conditions over the various regions of the Riemann surface, we are able to construct the unique Neumann function defined over that region. When we compare this result with the calculation performed by attaching three different propagators onto the three strings appearing in the δ function and then integrating over the set of intermediate string eigenfunctions, we find that the two results are exactly identical. Thus by letting the interaction times in the Neumann function gradually approach each other, we recover the original δ-function interaction. (The procedure of explicitly constructing the Neumann function by expanding over complete eigenfunctions of the free string is quite a general program, and can be applied to all light-cone topologies, N-point functions, and multiloop diagrams.)

Also, in Appendix B we will outline yet a third method of defining the interaction between three strings. It is not hard to show that the following

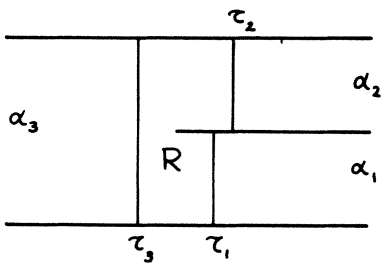

FIG. 4. A vertex diagram; splitting of a string in two. The functional integration is performed over the region R, then the limit $\tau_i \to 0$ is taken.

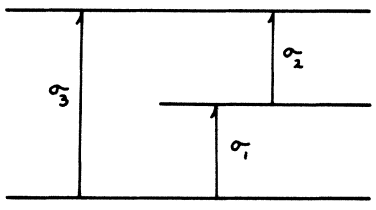

FIG. 5. The parameterization of strings.

equations, which represent the mapping of one string onto two smaller strings, can be solved exactly:

$$[\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) - \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|)]|V\rangle = 0, \tag{5.7}$$

$$[\vec{P}_3(\sigma_3) + \vec{P}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) + \vec{P}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|)]|V\rangle = 0.$$

The advantage of this proof is that we obtain the interaction vertex directly in terms of the familiar harmonic oscillators. We will show that the results are identical.

The calculation outlined in the first part of this section was performed in the eigenstates of the relativistic string, not the usual eigenstates of the first-quantized Hamiltonian. Therefore, to compare our results with the results of Mandelstam, who first constructed the vertex at infinite separations of interaction times and then reduced the vertex to finite times by adding multiplicative factors such as $e^{-p_-(\tau_i-\tau_j)}$, we must be careful not to confuse the difference in our basis states. By letting the interaction times in our finite-time vertex go to positive and negative infinity, we can show that we recover the vertex of Mandelstam.

(For our present discussion, we will temporarily omit discussion of the four-string interaction, since this vertex does not affect critically our discussion of the question of multiple counting and the formal perturbation series.) Once we know the structure of the three-string interaction term, we can insert this term into the standard Feynman-Dyson expansion of the S matrix:

$$S = 1 + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{n-1}} d\tau_n \prod_{i=1}^{n} V(\tau_i),$$

where

$$V(\tau) = e^{\tau H} \mathcal{L}_1 e^{-\tau H}. \tag{5.8}$$

We leave it as an exercise to show that we reproduce the usual Wick expansion when the Lagrangian $\mathcal{L}_1$ is inserted into the S-matrix expansion. The net effect of inserting the vertex (5.5) into the perturbation expansion is to have a set of light-cone Feynman diagrams composed out of two-dimensional Riemann surfaces. The propagators are represented by finite rectangular regions, while the vertices are simply the infinitesimal Riemann surface corresponding to the breaking of a string. By integrating over a complete set of eigenstates of the intermediate string, it is possible to smoothly construct the functional integral over the surface formed by joining together smaller surfaces. The Wick expansion creates surfaces which can, piece by piece, be joined together by summing over a complete set of string eigenstates to produce the functional integral over Riemann surfaces corresponding to multiloop amplitudes and higher-point functions. In other words, there is a one-to-one correspondence between the Riemann surface representing N-point functions or multiloop functions and each term in the perturbation series.

As in ordinary field theory, the Wick expansion gives us the standard decomposition of each term in the perturbation expansion into vertices and propagators. In the usual field theory, the vertices are points which connect propagators defined between space-time points. In this field theory, the vertices are wedge-shaped regions which join propagators defined between two extended strings. The fundamental difference between this field-theory perturbation and the usual one is that now the topology of each Feynman diagram is the topology of extended strings which propagate and split into smaller strings, while the usual field theory reproduces the topology of points which can propagate and split into other points. In both cases, we functionally integrate over all Feynman paths to calculate the propagator; only the topology is different.

The final result of contracting over all states is an amplitude which agrees with Mandelstam's original light-cone formula (3.1). The only difference is that he explicitly accounts for the "self-energy" term which arises in this dual field theory because the string has a finite extension. In our formalism we can eliminate this infinite contribution by dividing our field functional by an infinite renormalization constant. In addition we must note that our formalism gives a relativistic result, even though we have separated the transverse and longitudinal components from each other. As in Mandelstam's original paper, the longitudinal contribution to the amplitude arises when we correctly interpret the term $\exp(p_-\tau)$.

VI. MULTIPLE COUNTING

Though we have now shown that the interaction in Eq. (5.5), when inserted into the S-matrix expansion, yields a functional average over an entire surface like Fig. 6, we still must show that the counting of this field theory reproduces the counting found in the usual dual theory, and that it is consistent with the counting found in the zero-slope limit[16] of the model, i.e., the φ^3 theory (or the Yang-Mills theory,[17] if we include isospin and the four-string interaction).

First, let us discuss the four-point function and

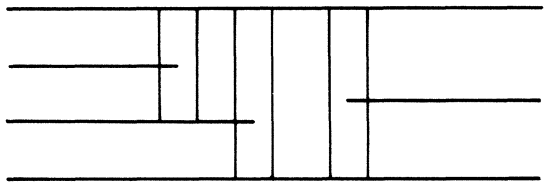

FIG. 6. Construction of scattering amplitude. Green's functions and vertices are joined together to form a light-cone diagram.

the problem of multiple counting. When the previous interaction given in Eq. (5.5) is expanded in the S matrix to second order in the coupling constant, we find 12 diagrams in the series, corresponding to four sets of s-, t-, and u-channel poles. Let us first study the behavior of the s and t graphs, as shown in Figs. 2 and 3.

It is possible to functionally integrate over all intermediate states in the perturbation expansion to get an exact expression for the Neumann function defined over a Riemann surface given in Fig. 6, but the calculation is tedious and yields little insight into the counting. A much simpler, but entirely equivalent approach, is to use the Neumann function defined over the upper half complex plane, which is known, and then conformally map the upper half plane onto the surfaces found in four-point functions. In other words, our perturbation expansion gives us expressions for the Neumann functions defined over surfaces appearing in Figs. 2 and 3, but it is much easier to simply transform the upper half plane into the corresponding surface. Mandelstam has given us the conformal mapping of the upper half plane onto stringlike surfaces, which are all that we need to discuss the counting problem:

$$\rho = \alpha_1 \ln(z-1) + \alpha_2 \ln(z-x) + \alpha_3 \ln(z),$$
$$\alpha_1, \alpha_2 > 0, \quad \alpha_3 < 0, \quad \alpha_1 + \alpha_2 > |\alpha_3|. \tag{6.1}$$

We know from the usual Koba-Nielsen formalism that the region of integration $0 \leq x \leq 1$ is sufficient to produce the four-point beta function. The question we will now investigate is whether the above mapping reproduces the s-channel diagram or the t-channel diagram.

To gain insight into the topological structure of the mapping, we first must find the points u_1 and u_2 in the z plane which correspond to the turning points in the ρ plane. To find these points, we take the derivative of ρ with respect to z and set the result to zero:

$$\frac{d\rho}{dz} = 0 \Rightarrow u_{1,2} = -\frac{1}{2\alpha_4}\{\alpha_1 x + \alpha_2 + \alpha_3 + \alpha_3 x \pm [(\alpha_1 x + \alpha_2 + \alpha_3 + \alpha_3 x)^2 + 4\alpha_3\alpha_4 x]^{1/2}\}, \qquad \sum_{i=1}^{4} \alpha_i = 0. \tag{6.2}$$

When x is close to 1, then the turning point situated between strings 1 and 2 necessarily must go to negative infinity, because then the images of $z=1$ and $z=x$ become arbitrarily close to each other in the ρ plane. Therefore, when x is close to 1, the turning points in the ρ plane are infinitely far apart from each other, so we are in the region corresponding to the s-channel diagram, $s=(P_1+P_2)^2$ (Fig. 2).

When x is close to 0, however, the limit becomes very subtle. Let x be an arbitrarily small quantity ϵ, and let us investigate the image of z in the ρ plane when z is also small, but larger than x (i.e., in the region corresponding to the separation line between strings 1 and 2):

$$\begin{aligned} &x = \epsilon, \\ &z = a\epsilon, \quad a > 1 \\ &\rho \sim i\pi(\alpha_1) + \alpha_2 \ln(a\epsilon - \epsilon) + \alpha_3 \ln\epsilon, \\ &\rho \sim i\pi\alpha_1 + (\alpha_2 + \alpha_3)\ln\epsilon. \end{aligned} \tag{6.3}$$

When $\alpha_2 + \alpha_3 < 0$ (i.e., when the "length" of string 3 is greater than the "length" of string 2), then the region corresponding to the separation between strings 1 and 2 goes to positive infinity, *so that the turning points pass each other in the complex ρ plane, corresponding to a t-channel diagram (Fig. 3)*. We see, therefore, that the mapping (6.1) is sufficient to generate *both* s- and t-channel pole diagrams.

When $\alpha_2 + \alpha_3 > 0$ (i.e., when string 3 is smaller than string 2) then we are interested in the region when x is small and positive and z is also small, but more negative than $-x$ (this corresponds, in the ρ plane, to the region of separation between strings 3 and 4):

$$\begin{aligned} &x = \epsilon, \\ &z = -a\epsilon, \quad a > 1 \\ &\rho \sim i\pi(\alpha_1 + \alpha_2 + \alpha_3) + \alpha_2 \ln(a\epsilon + \epsilon) + \alpha_3 \ln\epsilon, \\ &\rho \sim -i\pi\alpha_4 + (\alpha_2 + \alpha_3)\ln\epsilon. \end{aligned} \tag{6.4}$$

For arbitrarily small x, we see that the separation region between strings 3 and 4 goes to negative infinity, so the turning points again go past each other, corresponding to the t-channel pole diagram.

When $\alpha_2 + \alpha_3 = 0$, then the limit when x goes to

zero corresponds in the ρ plane to the situation when the turning points simply collide and then come to a halt. Therefore, in this situation, there is no t-channel diagram. As Mandelstam has shown in his previous paper, in this situation the s-channel diagram alone is sufficient to reproduce the entire dual beta function.

To summarize our result, we find that a careful study of the integration region given by the Koba-Nielsen variables in the mapping given by Mandelstam yields both the s- and t-channel pole diagrams which are produced in the second-quantized field-theoretical approach. We suspect, therefore, that the multiple counting difficulties of the dual model are resolved once we know that *certain sums of light-cone diagrams, given by the field-theoretical approach, yield one dual diagram.*

The generalization of this result to higher-point functions is not difficult. Let us assume that we have the mapping (Fig. 7) for an $(N+1)$-point function,

$$\rho = \sum_{i=1}^{M} \alpha_i \ln(z - x_i) + \sum_{j=M+1}^{N+1} \alpha_j \ln(z - x_j),$$

such that

$$\left.\begin{matrix}\alpha_i > 0 \\ \alpha_j < 0\end{matrix}\right\} x_1 = 1 \geq x_2 \geq x_3 \geq \cdots \geq x_N = 0 > x_{N+1} = -\infty,$$

$$\sum_{k=1}^{N+1} \alpha_k = 0.$$

The region (x_k, x_{k+1}) in the z plane has an image in the ρ plane corresponding to the separation region between strings k and $k+1$. The limit $x_k \to x_{k+1}$, therefore, is responsible for sending the turning point located between strings k and $k+1$ to negative infinity if $k < M$ and positive infinity if $k > M$. In other words, by allowing the Koba-Nielsen variables to come arbitrarily close to each other in successive pairings, then it is possible to send the turning points, in any order we please, to negative (positive) infinity for turning points located between the kth and $(k+1)$st strings for $k < M$ $(k > M)$.

For convenience, we will label the turning point in the ρ plane which separates the ith and $(i+1)$st strings by t_{i+1} and the jth and $(j+1)$st strings by t_j $(i < M, j \geq M+1)$. In the above paragraph, we showed that it is possible to send turning points

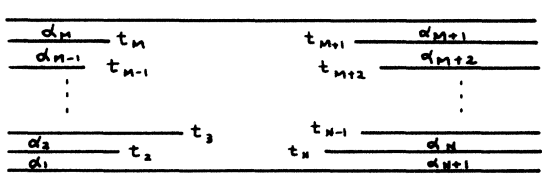

FIG. 7. A many-point tree diagram.

t_{i+1} (t_j) to negative (positive) infinity in any order we please, by simply allowing successive pairings of Koba-Nielsen variables to approach each other in the z plane. We still must show how it is possible to send the turning point t_{i+1} (t_j) to positive (negative) infinity in any order we choose.

Let us examine the region in the z plane defined by a point x as x approaches z_{i+1} $(x = z_{i+1} + \epsilon)$. This point maps onto the region corresponding to the separation of the ith and $(i+1)$st strings in the ρ plane. If all the other z's are kept fixed, then the point x is mapped into the point

$$\alpha_{i+1} \ln\epsilon + i\pi \sum_{l=1}^{i} \alpha_l, \quad \alpha_{i+1} > 0,$$

so that the turning point t_{j+1} most likely lies close to negative infinity. If we can find a configuration of Koba-Nielsen orderings which will map the point x to positive infinity, then we will have proved that the turning point t_{i+1} approaches positive infinity.

Now, let m be an integer such that t_m has an imaginary part slightly less than the imaginary part of t_{i+1}.

Then

$$\sum_{l=i+1}^{m} \alpha_l < 0, \quad m > M.$$

Now take the Koba-Nielsen ordering such that all variables z_l $(l = i+1, m)$ all lie within order ϵ of x, i.e., $x \cong z_{i+1} + \epsilon \cong z_{i+2} + \epsilon \cong \cdots x \cong z_m + \epsilon$ such that the original ordering is not destroyed. Then the map of the point x in the ρ plane is

$$\sim \sum_{l=i+1}^{m} \alpha_l \ln(x - z_l) + i\pi \sum_{l=1}^{i} \alpha_l$$

$$\sim \sum_{l=i+1}^{m} \alpha_l \ln\epsilon + i\pi \sum_{l=1}^{i} \alpha_l.$$

The mapping of the point x now approaches positive infinity, and hence turning points t_i for $i < M$ can now be sent to both positive and negative infinity independent of the ordering of the other points. [In this proof, we obviously do not want to change the ordering of the turning points, except to send t_{i+1} to positive infinity. As all the variables from z_{i+1} to z_m are collapsed to within order ϵ of each other, we must first make sure that this limit does not alter the orderings of the turning points t_n $(i+2 < n < m-1)$, which can always be done by taking first the limits on the various z_n such that the orderings in the ρ plane are unaffected.]

We have now shown that, for an arbitrary N-point function, we can always change the relative orderings of the turning points in any way we wish,

so that we include all possible orderings of N-point topology.

Perhaps an easier way to demonstrate this is to take the decay amplitude, so that we have one string on one side decaying into many strings located on the other side of the Riemann surface. In this decay amplitude where one string decays into $N-1$ strings, there are $(N-2)!$ ways in which to pair off the Koba-Nielsen variables in successive fashion, and hence $(N-2)!$ ways in which to draw planar light-cone diagrams. But the topology of each light-cone diagram defines a definite pole structure, and hence can be put into one-to-one correspondence with the pole diagrams which make up a dual diagram.

To be more precise in our language, let us define a "dual diagram" as one diagram of the original dual model with one definite ordering of external lines which is factorizable in all dual channels. Let us define a "light-cone Feynman diagram" as the various pieces which make up one dual diagram, such that each light-cone Feynman diagram corresponds to a unique pole decomposition of the original dual diagram. The number of light-cone Feynman diagrams which add up to make up one dual diagram is equal to the number of independent, noncrossing triangulations of an N-point polygon,[18] i.e.,

$$P_N = \frac{(2N-4)!}{(N-2)!\,(N-1)!}.$$

Let us define a light-cone diagram as the $(N-2)!$ different diagrams we find when we analyze a planar decay diagram in the light-cone formalism. Let us define a Feynman diagram as the usual function obtained from a ϕ^3 theory.

Since there are $(N-1)!/2$ possible cyclic orderings of N points, then $(N-1)!/2$ dual diagrams add up to form an S matrix. Each dual diagram, in addition, can be decomposed into P_N Feynman or light-cone Feynman diagrams. In the zero-slope limit, each dual diagram reduces to P_N light-cone Feynman diagrams and each Feynman diagram in the S matrix receives contributions from 2^{N-3} light-cone Feynman diagrams.

We showed that each cyclic ordering of N points contains $(N-2)!$ light-cone diagrams. Since we would like each light-cone (l.c.) diagram to correspond to a l.c. Feynman diagram, because each diagram represents a unique pole structure, we must be able to reduce the $(N-2)!$ light-cone diagrams into a smaller set of P_N light-cone Feynman diagrams.

The discrepancy between $(N-2)!$ different light-cone diagrams for one particular ordering of external lines and P_N different l.c. Feynman diagrams is resolved once we notice that not all of the $(N-2)!$ orderings correspond to diagrams with distinct pole structure. For example, in Figs. 8(a) and 8(b) we have two different light-cone diagrams, but they do not exhibit distinct pole structure.

There is a simple rule for determining if two light-cone diagrams fall within the same equivalence class. If we have two light-cone decay diagrams which are identical in topology except for the relative orderings of two turning points t_s and t_u, then the two diagrams belong to the same equivalence class if there is a v such that

$$\mathrm{Re}t_v > \max(\mathrm{Re}t_s, \mathrm{Re}t_u), \quad s < v < u$$

where the real part designates the interaction time of the turning point, and all turning points come from negative infinity.

It is not hard to see that any l.c. Feynman diagram can be uniquely represented in terms of one of the equivalence classes of light-cone diagrams, and that each equivalence class of light-cone diagrams can uniquely be represented in terms of one l.c. Feynman diagram. Therefore, since both sets include each other, then they must be identical sets.

It will be instructive to give examples of how the complete counting problem can be solved for the dual diagrams as well as for the zero-slope limit.

For example, in the case of the four-point function, the light-cone expansion yields 12 light-cone diagrams, each representing a light-cone Feynman diagram. Because two l.c. Feynman diagrams can be added to form one dual diagram, we wind up with six dual diagrams, corresponding to two (st), (su), and (tu) diagrams. Now take the zero-slope limit. Each dual diagram breaks up into two Feynman diagrams, so we eventually get 12 Feynman diagrams. But because of the factor $g/2$ in (5.5), we now recover exactly the usual ϕ^3 counting for the four-point functions, i.e., three diagrams.

In the case of the five- (six-) point function, the light-cone expansion yields 5 (14) pairs of light-cone Feynman diagrams for each ordering of the external lines, or 10 (28) total light-cone Feynman diagrams. Since there are 12 (60) cyclic orderings, the light-cone expansion yields 60 (840) pairs of l.c. Feynman diagrams, and since it takes 5 (14) l.c. Feynman diagrams to make one dual five-

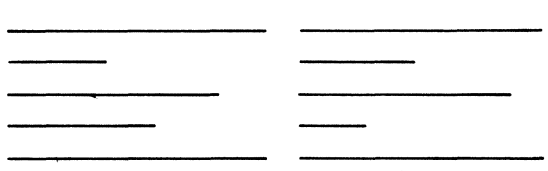

FIG. 8. The two distinct light-cone diagrams Figs. 8(a) and 8(b) form a single light-cone Feynman diagram.

(six-) point function, we now have 12 (60) pairs of dual diagrams, twice the usual case. Now, if we take the zero-slope limit, we obtain 120 (1680) Feynman diagrams, which is too many. The factor $g/2$ brings down the number of Feynman diagrams to the correct value of 15 (105).

In the N-point case, there are $2[(N-1)!/2]P_N$ light-cone Feynman diagrams and $(N-1)!$ dual diagrams. The factor $g/2$ brings down the number of Feynman diagrams to the correct $[(N-1)!\,P_N]/2^{N-2}$.

In summary, iteration of the vertex (5.5) gives twice the usual number of dual diagrams as in the usual counting except the factor $g/2$ gives each diagram weight 2^{-N+2}. When we take the zero-slope limit, this factor 2^{-N+2} is precisely the factor needed to reproduce the counting found in a ϕ^3 theory.

Notice that our weights for trees are different from the usual counting, so we find that our weights for loops are also different, viz., we find no nonorientable diagrams present in our counting. This result is still compatible with the usual unitarity arguments, because we are not taking intermediate states to be of positive charge conjugation. The factor $(1+\Omega)/2$, where Ω is the twist operator, can be dropped in our formalism. The zero-slope limit is also preserved in our formalism without nonorientable diagrams, because the orientable and nonorientable dual diagrams contribute Feynman diagrams in exactly equal numbers. By taking the weight of orientable diagrams to be twice the usual one, we achieve exactly the usual ϕ^3 counting up to the first loop. More on the counting problem for loops will be postponed until a later work. (For an explanation of the effects of the twist operator Ω, see the latter part of Appendix A.)

VII. FOUR-STRING INTERACTION

In the previous sections we have seen how to second quantize the free theory of relativistic strings and to introduce vertex functions which permit one string to split into two smaller strings at some interior point. This three-string interaction is sufficient to give the correct counting of diagrams and the correct Neumann functions for the Riemann surfaces corresponding to N-point functions, but it is not sufficient to give the entire region of the Koba-Nielsen variables.

We are forced to admit a four-string interaction[19] into our field theory for several compelling reasons. First, if we closely analyze the region of integration specified by the Koba-Nielsen variables, as specified in the mapping given by Mandelstam, we find that there is a region of integration which corresponds to a Riemann surface which cannot be decomposed into a simple propagator with two interaction vertices. We will find that this region of integration maps onto surfaces which have a continuously deforming topology. Second, if we are to successfully reproduce an infinite component field theory from our original Lagrangian, and if we are to recover a Yang-Mills interaction when isospin indices are added to our field functionals, then we are forced to consider the addition of a four-point interaction which will satisfy the properties of second-kind gauge-invariant theories. A four-string interaction would incorporate the necessary four-point interactions needed to give local gauge invariance.

The intuitive reason for the four-string interaction lies in the fact that the string may interact with another string at points which lie in the interior region of both strings. In the usual picture, one string propagates in time, until it breaks at some interior point, whereupon two smaller strings continue to propagate. In this new set of diagrams, two strings propagate freely, until they come in contact with each other at some point interior to both strings, whereupon they merge and change topology (see Fig. 9). Because the interaction point at which the two strings merge can vary along the length of the interiors of the strings, we conclude that we must integrate along the interaction point (this does not correspond to an integration over "time," and hence the four-string interaction is still fixed in τ). The fact that strings can merge at points interior to both strings introduces a new feature to the dual model. While the old first-quantization methods only explicitly revealed a three-Reggeon structure, which implicitly contained four-point structure when analyzing four-point dual functions in the zero-slope limit, the second quantization of the relativistic string contains this interaction in the Lagrangian itself. Furthermore, if spontaneous symmetry breaking is to be incorporated into the string model, then it is preferable to have a Φ^4 interaction rather than a Φ^3 interaction.

The mathematical reason for requiring a four-string interaction lies in the fact that the previous Lagrangian studied up to now, with only three-Reggeon couplings, does not produce the entire region of integration of the (tu) graphs. Using

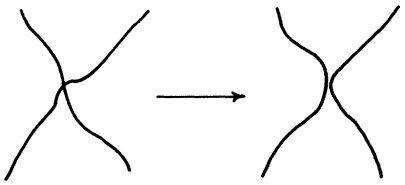

FIG. 9. Mechanism of four-string interaction.

perturbation theory to second order in the coupling constant, we showed that it was possible to produce s-, t-, and u-channel diagrams. We demonstrated that the s-channel diagram could be added to the t-channel diagram, to produce one complete dual diagram. By carefully analyzing the Riemann surface generated by the mapping of Eq. (6.1), we demonstrated that this map is sufficient to generate both the s- and t- channel graphs, and that there is no topological ambiguity in defining the transition between the s- and t-channel graphs. Likewise, s- and u-channel graphs can be added in much the same way to reproduce the Riemann surface of the (su) dual four-point function.

For the (tu) channel graphs, however, there are serious problems in resolving the ambiguities arising out of defining the transition between the t and u graphs. In Fig. 10, for example, we show the surface corresponding to the t-channel light-cone diagram. Notice that the turning points, when they approach each other, eventually collapse into a single point and cannot go past each other, as was the case when deforming the s-channel graph into the t graph. On the other hand, Fig. 11 describes the u-channel light-cone diagram, which also has the same ambiguities when the turning points gradually collapse with each other into a point.

When the turning points of both graphs eventually collapse into the same point, we arrive at the configuration given in Fig. 12, which seems to be the same for both the t- and u-channel diagrams. It is tempting to speculate that the t-channel graph collapses into a configuration like Fig. 12 only momentarily, and immediately transforms into the corresponding u-channel graph. Unfortunately, this conjecture is incorrect, because the topological structure of the t- and u-channel graphs differs radically, and so there must be a continual deformation of one graph into the other. *In other words, the four-string interaction corresponds to the continuous deformation of the t-channel topology into the u channel through the configuration shown in Fig. 12.* The continuous deformation of one topology into another can only be described by a Riemann surface in which four strings are intimately intertwined.

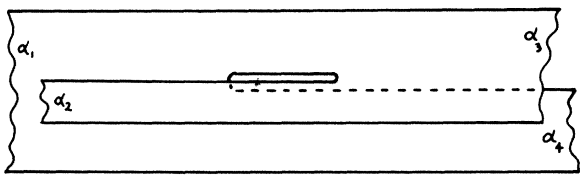

FIG. 10. A light-cone diagram with a t-channel pole. The diagram represents the Reimann sheet structure of the mapping (7.4) for $0 < x < x_1$.

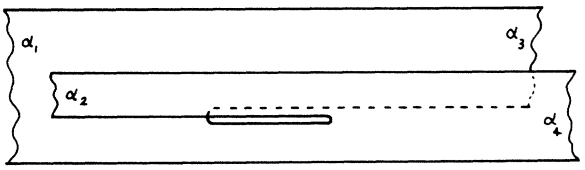

FIG. 11. A light-cone diagram with a u-channel pole. The diagram represents the Reimann sheet structure of the mapping (7.4) for $x_2 < x < 1$.

The change of one topology into another can be described as follows. In the t-channel graph of Fig. 10, the turning points gradually approach one another, until they collapse into a point A (see Fig. 12). Then the two turning points merge into one point, which then journeys from A and moves down the Riemann surface, until it eventually arrives at point B, whereupon the two turning points once again separate and reproduce the topology of the u-channel diagram. The intermediate region where the turning points are momentarily merged into one point (called C) corresponds to the four-string interaction. As C moves continuously from point A (t-channel diagram) to the point B (u-channel diagram), this point represents the point where strings are rearranged, so that it acts like a "zipper" which interlaces a new sheet structure as it moves down the path. The path of this point, from point A to C, represents one particular connection of sheets, while the path from C to B represents another.

It will be instructive to trace the Riemann sheet structure of the vertex. Let us begin at a point which lies on sheet 1 in Fig. 13 and journey down this strip until we intersect the path (ACB). If we intersect the path (ACB) at a point which lies above C, then we enter sheet 3. If we intersect the path (ACB) at a point below point C, then we will enter into sheet 4. Likewise, if we begin our journey at a point on sheet 2, then if our journey intersects the line (ACB) at a point above (below) point C, then we will enter string 4 (3). Thus, the point C acts like a "zipper" which continually realigns the sheet structure of the Riemann surfaces as it moves from point A to B.

Now that we have carefully mapped out the

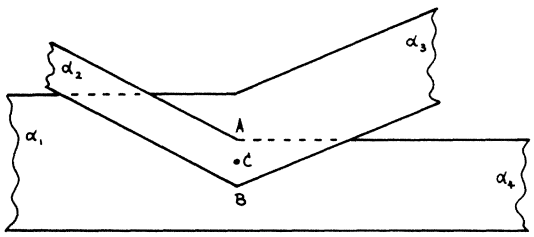

FIG. 12. The Riemann sheet structure which shows the transition from Fig. 10 to Fig. 11, for $x_1 < x < x_2$.

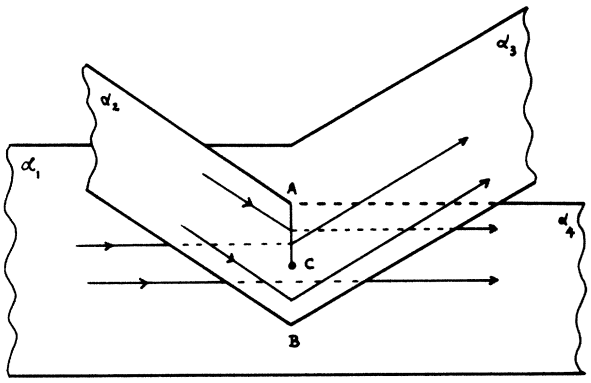

FIG. 13. The Riemann sheet structure of a 4-string interaction term.

Riemann surface structure of the four-string interaction, we can write down the correspondence between strings. As before, we can decompose strings 1 and 2 in terms of the strings 3 and 4 as follows:

$$
\begin{aligned}
&(0<\sigma_i<\pi|\alpha_i|),\\
&\sigma_1=\sigma_4 \quad (0<\sigma_4<\pi|\alpha_4|),\\
&\sigma_1=\sigma_2+\pi(\alpha_1-|\alpha_3|) \quad (0<\sigma_2<\pi\alpha_2),\\
&\sigma_1=\sigma_3+\pi(\alpha_1-|\alpha_3|) \quad (0<\sigma_3<\pi|\alpha_3|),\\
&\alpha_1,\ \alpha_2>0, \quad \alpha_3,\ \alpha_4<0, \qquad (7.1)\\
&\sum_{i=1}^{4}\alpha_i=0,\\
&\alpha_1>|\alpha_3|>\alpha_2, \quad \alpha_1>|\alpha_4|>\alpha_2;
\end{aligned}
$$

$$
\begin{aligned}
&\vec{X}_1(\sigma_1)=\vec{X}_3(\sigma_3)\theta(\sigma_1-\sigma_0)+\vec{X}_4(\sigma_4)\theta(\sigma_0-\sigma_1),\\
&\vec{X}_2(\sigma_2)=\vec{X}_3(\sigma_3)\theta(\sigma_0-\sigma_1)+\vec{X}_4(\sigma_4)\theta(\sigma_1-\sigma_0), \qquad (7.2)\\
&\pi(\alpha_1-|\alpha_3|)<\sigma_0<\pi|\alpha_4|.
\end{aligned}
$$

The parameterization is defined in Fig. 14. Notice that eventually we must integrate over the point σ_0. It is not hard to incorporate the topological structure of the above equations into a four-string interaction:

$$
\begin{aligned}
\mathcal{L}_2=\tfrac{1}{4}g^2\int\prod_{i=1}^{4}\mathfrak{D}\vec{X}_i(\sigma_i)\int_0^\infty\frac{dp_{+i}}{(2p_{+i})^{1/2}}\,\delta(p_{+1}+p_{+2}-p_{+3}-p_{+4})\Phi^\dagger_{p_{+1}}[X_1]\Phi^\dagger_{p_{+2}}[X_2]\Phi_{p_{+3}}[X_3]\Phi_{p_{+4}}[X_4]\\
\times\int_{\pi(\alpha_1-|\alpha_3|)}^{\pi|\alpha_4|}d\sigma_0\prod_{\sigma_1}\delta(\vec{X}_1(\sigma_1)-\vec{X}_3(\sigma_3)\theta(\sigma_1-\sigma_0)-\vec{X}_4(\sigma_4)\theta(\sigma_0-\sigma_1))\\
\times\prod_{\sigma_2}\delta(\vec{X}_2(\sigma_2)-\vec{X}_3(\sigma_3)\theta(\sigma_0-\sigma_1)-\vec{X}_4(\sigma_4)\theta(\sigma_1-\sigma_0))+\text{H.c.} \qquad (7.3)
\end{aligned}
$$

Now that we have shown the origin of the four-string interaction and have presented the interaction in second-quantized form, we will now carefully discuss the light-cone mappings which produce these interactions.

As before, we start with the Mandelstam mapping, only now we will take

$$
\begin{aligned}
&\rho=\alpha_1\ln(z-1)+\alpha_3\ln(z-x)+\alpha_2\ln(z),\\
&\alpha_1,\ \alpha_2>0,\ \alpha_3<0,\\
&\alpha_1>|\alpha_3|>\alpha_2, \qquad (7.4)\\
&\alpha_1>|\alpha_4|>\alpha_2,\\
&\sum_{i=1}^{4}\alpha_i=0.
\end{aligned}
$$

With these conditions, we can produce either the t- or u-channel graphs, depending on the value of x. As before, the turning points are given by Eq. (6.2), but now we will be interested in the region where the term under the square root vanishes, because the mapping differs tremendously from the conditions given in (6.1). With the new restrictions on the α's it is not hard to show that the two turning points in the z plane, u_1 and u_2, now *both* lie in the region between 0 and x (for one particular value of x) and *both* in the region between $-\infty$ and 0 (for another value of x). Before, the (st) configuration placed one turning point between 1 and x and the other between $-\infty$ and 0. In the (tu) configuration, both turning points u_1 and u_2 are located in the same region, and hence can collapse into the same point to produce four-string interactions.

Let x_1 and x_2 be the values for x for which the discriminant vanishes ($x_2>x_1$). When $0<x<x_1$, then the two turning points u_1 and u_2 are located as follows:

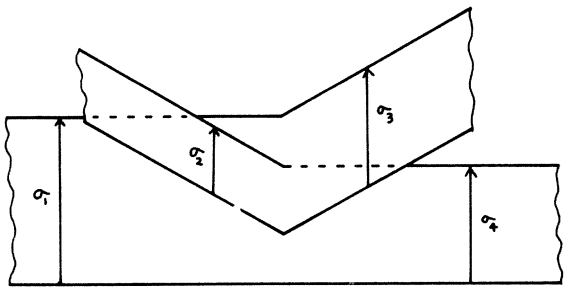

FIG. 14. The parameterization of the 4-string interaction term.

$$-\infty < u_1 < u_2 < 0 . \tag{7.5}$$

When $x_2 < x < 1$, then the turning points are located as follows:

$$0 < u_1 < u_2 < x ,$$

$$\text{if } x = x_1 \text{ then } -\infty < u_1 = u_2 < 0 , \tag{7.6}$$

$$\text{if } x = x_2 \text{ then } 0 < u_1 = u_2 < x . \tag{7.7}$$

The region $x_1 < x < x_2$, then, corresponds to the Riemann surface interlocking four Riemann strips. In this region, u_1 and u_2 become complex ($u_1 = u_2^*$). One of them descends into the lower half plane, which is not included in the z plane that we are studying. The other complex point migrates in the upper half plane in a path which links the regions $(-\infty, 0)$ and $(0, x)$. The image of this point in the ρ plane is the point C we referred to earlier in our discussion. The image of the path taken by the point u which migrates in the upper half z plane is the path taken by the "zipper" (ACB).

We, of course, wish to have a vertex which is local in "time," so that the four-string Lagrangian is not a function of τ. When the path (ACB) is actually calculated by using Eq. (4.5), we find that the real part of the complex point C changes as C moves from A to B, i.e., the four-string interaction given by the previous mapping is not local in τ. For the four-point function, however, this need not concern us, because we can always translate the Riemann surface an arbitrary distance in τ, and hence force the path ACB to be a straight vertical line. Because the four-point Riemann surface is infinite in extent, and because we are only interested in *differences* in the interaction times, we are free to translate the infinite strip any finite distance. For the four-point function, therefore, we encounter no problems in using a four-string vertex local in τ.

For the higher-point functions, however, there remains the delicate point of whether the line ACB can be arbitrarily fixed with respect to the other turning points. The field-theory perturbation with a four-string vertex local in τ tells us that we must be able to integrate over *all* relative orientations of the real parts of the turning points and also over *all* relative orientations of the turning points with respect to the vertical line ACB. It is not obvious that the mapping given in Eq. (2.17) yields Riemann surfaces which are in one-to-one correspondence with the set of diagrams which include all the various relative orientations of the real parts of turning points with the line ACB.

In the case of the four-point function, there was a finite line segment over which x could range which would map onto four-string interactions. In the five-point function (one planar vertex sewn onto a four-string interaction), there are two integration variables x and y. The integration region which corresponds to the four-string interaction is a two-dimensional region in the xy plane. In general, for an N-point function, there are $N-3$ Koba-Nielsen variables which describe an $(N-3)$-dimensional space. The region corresponding to one arrangement of four strings can be represented by a closed $(N-3)$-dimensional surface in this Koba-Nielsen space.

We must prove that the orientations defined by the field theory are sufficient to parameterize the entire $(N-3)$-dimensional closed surface and not only parts of it. In other words, the line ACB may be a *curved* line in ρ space when the relative positions of the turning points are kept fixed, but this does not mean we must abandon our local four-string vertex given in Eq. (7.3). The solution to the problem is to take a different reparameterization of our variables such that if we fix the position of C, then the other planar turning points assume all possible relative orientations. Another way to state this is the following: If we fix the relative orientations of the planar turning points, then there must exist a parameterization of ACB such that ACB goes in a vertical line.

In conclusion, if we were to take the mapping in Eq. (7.4) and try to solve for the path ACB, we will find that, in general, this path is curved, while our four-string vertex given in Eq. (7.3) yields only vertical lines, i.e., lines local in τ. This is not a problem, because there exists a parameterization of the region of the four-string vertex such that ACB is a straight line with respect to *all* possible orientations of the other turning points.

To prove that we can always take a parameterization such that the line ACB is a vertical line, i.e., with fixed τ, is equivalent to showing that the real part of point C can be made to vary independent of the orientations of the other turning points, so that all possible orientations are obtained. A simple way of seeing this result is to reexamine the mapping in Eq. (2.17). While we originally started with the choice z_i as independent variables, we can always take new variables as $\tau_1, \tau_2, \ldots, \tau_n$, i.e., the relative distances between the real parts of the interaction times. The Jacobian transformation from the set (z_i) to the set (τ_i) is a smooth one, so all the τ's can be made to vary independent of each other. In addition, we know that the limits placed on the integration region for the (z_i's) are mapped into regions in the τ space which correspond to taking each interaction time from $-\infty$ to $+\infty$. Because the path taken by the "zipper" point is the path taken by one of the turning points (after they have col-

lapsed), we can be sure that the point C can be made to vary independently from all other τ_i, so that we have shown that all possible orientations are possible. The orientations we prefer to choose, of course, are those in which the line ACB is fixed in τ with respect to all orientations of the other turning points.

The next question that we must answer is: Are there five-, six-, and higher-string interactions? Have we really exhausted the set of all Riemann surfaces which cannot be generated by three- and four-string interactions? The answer is that three- and four-string interactions are sufficient to generate all trees.

Because, in order to have a multistring interaction, several pairs of turning points must collapse into each other and become complex variables in the upper half plane, we discard the lower half plane solution. Then these complex turning points must stick together in the upper half plane. But since we know that the Jacobian for the transformation produces a one-to-one mapping, we are not allowed to have these turning points coincide for a region of the integration range, or else the τ variables would not be really independent of each other.

In closing this section, we wish to comment on the Yang-Mills structure of our four-string interactions. Since the light-cone gauge has been chosen in our quantization formulation, the interactions of the first excited states (spin 1, massless) have to be compared with the Yang-Mills theory quantized in the same gauge. It is not difficult to convince ourselves that both the isospin and the helicity structure of four-interaction coincide with those of the Yang-Mills theory investigated by Tomboulis.[20] In doing this however, the following points must be taken into account:

(1) In defining (7.3), the correct symmetrization $(1 \pm \Omega)/2$ combined with the Chan-Paton factor must be done.

(2) In the light-cone gauge, the polarization factor is independent of momentum[20]:

$$e^{\mu}(p, i)e_{\mu}(q, j) = \delta_{ij} .$$

(3) A certain part of isospin factors in the Yang-Mills theory, which does not appear in (7.3), can be eliminated by using the relation for the structure constants

$$\epsilon_{acd}\epsilon_{bde} - \epsilon_{bcd}\epsilon_{ade} = -\epsilon_{abd}\epsilon_{dce} ,$$

which is true for any simple group. The detailed discussion on this problem, however, will be shown in a later article.

VIII. MEASURE AND CONCLUDING REMARKS

It is not hard to use the perturbation machinery defined in this paper to produce explicit expressions for multitree diagrams expressed in terms of Neumann functions over the corresponding Riemann surface. In effect, we have reproduced Mandelstam's Eq. (3.1).

But there are a few yet unresolved points that have to be made. The first unresolved question is the Lorentz invariance of our three-Reggeon vertex function. Mandelstam has recently shown, by dividing up the string into a series of points, that the application of a Lorentz transformation on the three-string vertex has the correct properties only if the dimension of space-time is 26. (Previously, he demonstrated the Lorentz invariance of the finite-time vertex function; this new calculation is for infinitesimal times.) The relativistic invariance of the four-string interaction remains unresolved.

A more pressing question is the problem of measure. Unless we can calculate the measure explicitly, we cannot claim to have fully derived the N-point functions of the original dual model. Unfortunately, when calculating N-point functions in our formalism, there remains a determinant of the Neumann function defined over the N-point surface. Since our field-theory prescription began with an integration over the "time" variables, and since we would ultimately like to obtain the usual Koba-Nielsen variables, then we suspect that the determinant of the Neumann function is, indeed, the Jacobian of the transformation from one set of variables to another.

There are several reasons for suspecting the Jacobian of the transformation to be equal to the determinant over the Neumann function. First, we know that if the α_i's are set either equal to zero or to the length of the original strip, then the determinant reduces to 1, no matter in what order the limits are taken (assuming, of course, that the limits do not change the topology of the diagram). There is also another function of the parameters that reduces to a trivial result when the α's are taken to be either zero or the length of the strip, in any order consistent with the topology, and this is the Jacobian. Second, another reason for believing that the determinant reproduces the Jacobian is that Mandelstam evaluated the four-point interaction explicitly in his formalism with operators. The same calculation, of course, could have been performed by functional techniques, in which case we would have found that the determinant arising from the functional calculation corresponds to the Jacobian coming out of the operator approach. But because he

took a very symmetric configuration, we certainly cannot conclude that this is a general proof.

One last unresolved question is the problem of constructing a Lorentz-invariant second-quantized Lagrangian. Simply taking a trivial covariant generalization of (3.8) *will certainly not produce* the correct Lagrangian, because this new Lagrangian does not have the gauge invariance which allows us to take the gauge (2.14) in the first place. Construction of a gauge-invariant Lorentz-invariant Lagrangian is a highly nontrivial problem. And because the canonical momentum conjugate to Φ is a function of σ, there are also profound questions of interpretation.

ACKNOWLEDGMENTS

We are happy to acknowledge many fruitful and valuable conversations with Professor S. Mandelstam and especially Professor B. Sakita.

APPENDIX A

In this section our goal is to prove Eq. (5.6), i.e., the statement that the δ functions which define the breaking of a relativistic string are equal to the functional integral over the Riemann surface corresponding to the two-dimensional surface of interaction as the width of this surface goes to zero. Once we have established this result, then the cumbersome procedure of solving directly for Neumann functions can be dropped in favor of the δ functional technique, which very quickly gives us the solution to the N-point functions and the multiloop amplitudes. In fact, the δ functional technique allows us the chance to formulate general rules for writing down the general solution to any light-cone topology by inspection. It is possible to state, but without proof, the general structure of all the matrices that occur when calculating Neumann functions over quite general light-cone surfaces.

The plan of the proof is quite simple. First, we will actually perform the integrations over the δ functions by attaching three distinct Green's function propagators onto the three various strings. By functionally integrating over the intermediate-string states, we will obtain the amplitude for splitting relativistic strings taken at arbitrary, finite interaction times. Second, we will actually construct the Neumann function over a wedge-shaped Riemann surface which defines the interaction taken at finite times. The construction of the Neumann function is achieved by expanding the Neumann function into normal-mode eigenstates of the free string and then by matching boundary conditions. Because we know that the boundary conditions are sufficient to determine the Neumann function, and because we know that the Neumann function is unique, we have now calculated the unique result, which agrees exactly with the calculation performed by integrating over the δ functions.

Later, in Appendix B, we will calculate the interaction vertex by yet another procedure, by explicitly solving Eq. (5.7). This technique of constructing the vertex is yet another method which yields the same result as the previous two methods. The advantage of this technique is that the equations are well posed and lead to solutions in terms of the usual harmonic oscillators.

We begin the calculation by attaching three Green's function propagators onto the three strings defined in the δ function. We will represent the string at the instant of interaction by $\vec{X}$, while the string taken at finite times will be represented by $\vec{X}'$: Let

$$G(\vec{X}_i', \tau_i; \vec{X}_i, 0) = \langle\langle 0 | \Phi_{p_-}[\vec{X}_i', \tau_i]\Phi_{p_+}^{\dagger}[\vec{X}_i, 0] | 0\rangle\rangle$$

$$= \left(\frac{\alpha}{4\pi\tau_i}\right)^{(D-2)/2} \exp[-p_+(\vec{x}_i' - \vec{x}_i)^2/2\tau_i]$$

$$\times \prod_{n=1}^{\infty}\left[\frac{n}{\pi \sinh(n\tau_i/\alpha)}\right]^{(D-2)/2} \exp\left\{-\frac{n}{\sinh(n\tau_i/\alpha)}[(\vec{x}_{i,n}'^{\,2} + \vec{x}_{i,n}^{\,2})\cosh(n\tau_i/\alpha) - 2\vec{x}_{i,n}' \cdot \vec{x}_{i,n}]\right\} . \quad \text{(A1)}$$

We now construct

$$V(\vec{X}_{1,2,3}'; \tau_{1,2,3}) = \prod_{i=1}^{3}\prod_{n=0}^{\infty}\int \mathfrak{D}\vec{X}_{i,n} G(\vec{X}_i', \tau_i; \vec{X}_i, 0) \prod_{\sigma_3} \delta(\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) - \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|)) . \quad \text{(A2)}$$

The product over δ functions can be expressed as

$$\int \mathfrak{D}\vec{P}(\sigma_3) \exp\left[i\int d\sigma_3 \vec{P}(\sigma_3)(\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta(\pi|\alpha_1| - \sigma_3) - \vec{X}_2(\sigma_2)\theta(\sigma_3 - \pi|\alpha_1|))\right]. \quad \text{(A3)}$$

Let

$$\vec{X}_i(\sigma_i)=\left[\vec{x}_i+\sum_{n=1}^{\infty}2\vec{x}_{i,n}\cos(n\sigma_i/\alpha_i)\right]$$
$$\vec{P}_i(\sigma_i)=\left(\frac{1}{\pi\alpha_i}\right)\left[\vec{p}_i+\sum_{n=1}^{\infty}\vec{p}_{i,n}\cos(n\sigma_i/\alpha_i)\right], \tag{A4}$$

and define the functional Fourier transform of $F[\vec{X}_1,\vec{X}_2,\vec{X}_3]$ by

$$\tilde{F}[\vec{P}_1,\vec{P}_2,\vec{P}_3]=\int\prod_{i=1}^{3}\mathfrak{D}\vec{X}_i F[\vec{X}_1,\vec{X}_2,\vec{X}_3]\exp\left[i\int_0^{\pi\alpha_3}d\sigma_3\sum_{i=1}^{3}\vec{P}_i(\sigma_i)\cdot\vec{X}_i(\sigma_i)\theta_i(\sigma)\right],$$
$$\sigma_3=\sigma_1\quad(0<\sigma_1<\pi\alpha_1),\qquad \alpha_1+\alpha_2=\alpha_3,$$
$$\sigma_3=\sigma_2+\pi\alpha_1\quad(0<\sigma_2<\pi\alpha_2),\qquad \alpha_i>0; \tag{A5}$$
$$-\theta_1(\sigma)\equiv\theta(\pi\alpha_1-\sigma_3),\quad -\theta_2(\sigma)\equiv\theta(\sigma_3-\pi\alpha_1),\quad \theta_3\equiv 1.$$

Then, the Fourier transform of (A3) reduces to

$$\prod_n\delta\left(\vec{p}_{1,n}-\sum_{m=0}^{\infty}A^{13}_{n,m}\vec{p}_{3,m}\right)\delta\left(\vec{p}_{2,n}-\sum_{m=0}^{\infty}A^{23}_{n,m}\vec{p}_{3,m}\right), \tag{A6}$$

where the overlapping matrix A^{i3} is defined by

$$A^{i3}_{n,m}=\frac{2}{\pi\alpha_3}\int_0^{\pi\alpha_i}\cos\left(\frac{m\sigma_3}{\alpha_3}\right)\cos\left(\frac{n\sigma_i}{\alpha_i}\right)d\sigma_i\ ,\quad n,m\neq 0$$
$$A^{i3}_{0,n}=\frac{1}{\pi\alpha_3}\int_0^{\pi\alpha_i}\cos\left(\frac{n\sigma}{\alpha_i}\right)d\sigma=0,$$
$$A^{i3}_{n,0}=\frac{2}{\pi\alpha_3}\int_0^{\pi\alpha_i}\cos\left(\frac{n\sigma}{\alpha_3}\right)d\sigma=\frac{2}{\pi n}\sin\left(\frac{\pi n\alpha_i}{\alpha_3}\right), \tag{A7}$$
$$A^{i3}_{0,0}=\alpha_i/\alpha_3,\quad \alpha_i>0,\quad \alpha_1+\alpha_2=\alpha_3.$$

Using (A6), one can show that the Fourier transform of (A2) is

$$\tilde{V}(\vec{P}'_{1,2,3};\tau_{1,2,3})$$
$$=\int\prod_n d\vec{P}_{3,n}\exp\left[-\sum_{i=1}^{3}\left\langle\vec{P}'_i\middle|C_i\frac{1}{2N}S_i^{-1}\middle|\vec{P}'_i\right\rangle-\left\langle\vec{P}_3\middle|\sum_{i=1}^{3}A_{3,i}C_i\frac{1}{2N}S_i^{-1}A_{i,3}\middle|\vec{P}_3\right\rangle+2\sum_{i=1}^{3}\left\langle\vec{P}_3\middle|A_{3,i}\frac{1}{2N}S_i^{-1}\middle|\vec{P}'_i\right\rangle\right]$$
$$=(\det M)^{(D-2)/2}\exp\left[-\sum_{i,j}^{3}\langle\vec{P}'_i|N_{ij}|\vec{P}'_j\rangle\right], \tag{A8}$$

where

$$|\vec{P}_i\rangle=(\vec{p}_i,\vec{p}_{i,1},\vec{p}_{i,2},\dots),$$
$$(N)_{n,m}=n\delta_{n,m},$$
$$(C_i)_{n,m}=\delta_{n,m}\cosh(n\tau_i/\alpha_i),$$
$$(S_i)_{n,m}=\delta_{n,m}\sinh(n\tau_i/\alpha_i).$$

The overlapping matrix A_{ij} is defined by (A7) $[(A_{33})_{nm}=\delta_{n,m}]$. In the last expression of (A8),

$$N_{ii}=S_i^{-1}\frac{1}{2N}A_{i3}M^{-1}A_{3i}S_i^{-1}-C_i\frac{1}{2N}S_i^{-1}, \tag{A9}$$

$$N_{ij}=S_i^{-1}\frac{1}{2N}A_{i3}M^{-1}A_{3j}\frac{1}{2N}S_j^{-1}, \tag{A10}$$

with

$$M=\sum_{i=1}^{3}A_{3i}C_i\frac{1}{2N}S_i^{-1}A_{i3}. \tag{A11}$$

In (A9)–(A11), the zero-frequency part of $(1/2N)S_i^{-1}$ is defined to be 1. Now that we have derived an explicit form for the finite-time interaction vertex, we still must show that this result is identical to the calculation performed through defining Neumann functions in the finite-time Riemann surface.

Our notation will be defined through Fig. 5. We will divide the Riemann surface into three areas, each corresponding to one of the three strings.

Because the Neumann function depends on the values taken at two independent points, we will split the Neumann function into nine separate parts, depending on which of the three regions the two points are located. In addition, there will be a tenth term, corresponding to the singularity which corresponds to the logarithmic diverging term.

Our plan is simple. We first expand the Neumann function into eigenfunctions of the free string which satisfy the conditions

$$\left.\frac{\partial}{\partial\sigma}N(\sigma,\tau;\sigma',\tau')\right|_{\substack{\sigma_3=0,\pi\alpha_3;\\ \sigma_3=\pi\alpha_1 \text{ if } \tau>0}}=0,$$

$$\left|\frac{\partial}{\partial\tau}N(\sigma,\tau;\sigma',\tau')\right|_{\tau=\tau_1,\tau_2,\tau_3}=\tfrac{1}{2} \tag{A12}$$

for all σ', τ' within the Riemann wedge. Then we will simply match boundary conditions as σ', τ' roams throughout the Riemann surface, keeping σ, τ fixed. And finally, we will match the first derivative of the Neumann function throughout the Reimann surface.

We start with the following expansion for the Neumann function:

$$N(\sigma'\xi';\sigma\xi)=N_0(\sigma',\xi';\sigma,\xi)+\sum_{i\geq j}^{3}\langle\vec{\mathrm{P}}_i(\sigma_i',\xi_i')|N_{ij}|\vec{\mathrm{P}}_j(\sigma_j,\xi_j)\rangle, \tag{A13}$$

where

$$|\vec{\mathrm{P}}_i(\sigma_i,\xi_i)\rangle=\cos n\sigma_i\cosh n(\xi_i-\tau_i/\alpha_i), \tag{A14}$$

with $\sigma_i=\sigma/\alpha_i$ and $\xi_i=\tau/\alpha_i$, and

$$N_0(\rho',\rho)=\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\sigma_i')\cos(n\sigma_j)\delta_{ij}[e^{-n|\xi_i'-\xi_j|}+e^{-n\delta(i)(\xi_i'+\xi_j)}e^{2n\delta(i)\xi_i}]-\tfrac{1}{2}\max(\tau_i,\tau_i') \tag{A15}$$

with $\delta(3)=\delta(1)=-\delta(2)=1$. The expansion of $N(\rho',\rho)$ with respect to (A14) guarantees the boundary condition (A12) except along the lines $\tau_i=0$, where the three Neumann functions are to be glued (see the region R in Fig. 4).

Requiring $N(\sigma_i',\xi_i';\sigma_i,-\epsilon)=N(\sigma_i',\xi_i';\sigma_i,+\epsilon)$, for example, one obtains constraints over (A13) ($\epsilon\to 0$):

$$\left(K_3\frac{2}{N}+N_{33}C_3\right)=N_{31}C_1A_{13}+N_{32}C_2A_{23}, \tag{A16}$$

$$\begin{aligned}A_{13}C_3N_{31}&=\left(C_1N_{11}+\frac{2}{N}K_1^{-1}\right)\left(\frac{\alpha_1}{\alpha_3}\right),\\ A_{23}C_3N_{32}&=\left(C_2N_{22}+\frac{2}{N}K_2^{-1}\right)\left(\frac{\alpha_2}{\alpha_3}\right),\end{aligned} \tag{A17}$$

$$\begin{aligned}A_{23}C_3N_{31}&=C_2N_{21}\left(\frac{\alpha_2}{\alpha_3}\right),\\ A_{13}C_3N_{32}&=C_1N_{12}\left(\frac{\alpha_1}{\alpha_3}\right)\end{aligned} \tag{A18}$$

[one finds the same overlapping matrix A_{ij} as defined in (A7)]. The new matrix K_i is given by

$$(K_i)_{n,m}=\delta_{n,m}e^{n\xi_i}. \tag{A19}$$

Next, requiring the continuity of

$$(\partial/\partial\xi_j)N(\sigma_i',\xi_i';\sigma_j,\xi_j)|_{\xi_j=\pm\epsilon}$$

one obtains

$$\begin{aligned}(2K_3+N_{33}NS_3)A_{31}&=N_{31}NS_1,\\ (2K_3+N_{33}NS_3)A_{32}&=N_{32}NS_2,\end{aligned} \tag{A20}$$

$$\begin{aligned}-A_{13}S_3NN_{31}&=2K_1^{-1}-NS_1N_{11},\\ -A_{23}S_3NN_{32}&=2K_2^{-1}-NS_2N_{22},\end{aligned} \tag{A21}$$

$$\begin{aligned}A_{23}S_3NN_{31}&=S_2NN_{21},\\ A_{13}S_3NN_{31}&=S_1NN_{12}.\end{aligned} \tag{A22}$$

Because of the identity

$$1=A_{31}\frac{\alpha_3}{\alpha_1}A_{13}+A_{32}\frac{\alpha_3}{\alpha_2}A_{23}$$

only six are independent among Eqs. (A16)–(A22). One can, therefore, solve them with respect to N_{ij}. Using (A16) and (A20), for example, one obtains

$$N_{33}=2K_3\left[\left(\frac{1}{2N}\right)(-1+S_3^{-1}C_3)M^{-1}-1\right]\left(\frac{1}{N}\right)S_3^{-1},$$

where M is defined in (A11). Then, taking N_0 in (A13) into account, one can show that, if $\xi_3'=\tau_3$,

$$N(\sigma_3',\xi_3';\sigma_3\xi_3)=S_3\left(\frac{1}{2N}\right)M^{-1}\left(\frac{1}{2N}\right)S_3^{-1}-C_3\left(\frac{1}{2N}\right)S_3^{-1}, \tag{A23}$$

which agrees with (A9). Similarly, all other relations in (A9) and (A10) are obtained.

As a consequence of the above arguments, the equivalence of the dual model vertex expressed by the Neumann function to the overlapping condition has been proved. Q.E.D.

In the above discussion we considered only an unsymmetrized vertex function. The symmetrization of the vertex, however, can be easily accomplished by twisting some of the strings which

appear in the vertex, i.e., if the ith string is to be twisted, the parameter σ_i in (A4) and also in the step functions should be replaced by $\pi\alpha_i - \sigma_i$. This procedure simply replaces the overlapping matrix $(A^{ji})_{n,m}$ to $(A^{ji})_{n,m}(-1)^m$. The rest of the discussion is valid without any change. We stress that we have the option of adopting either the untwisted or the twisted (with definite charge conjugation) vertex, but the only model which is physically relevant is the model with symmetrized vertices and isospin.

APPENDIX B

In this appendix we would like to show another simple derivation of vertex functions. The method we use is an extended version of the one developed by Goto and Naka[21] in their bilocal field theory. The advantage of this method is that the continuity conditions (or overlapping conditions) enable us to obtain the representation of the vertex function in which the energy matrix is diagonal.

Let us assume that a string $X_3(\sigma)$ breaks into $X_1(\sigma)$ and $X_2(\sigma)$ at $\tau=0$, say, then the vertex function (at $\tau=0$) should satisfy the following conditions:

$$[\vec{X}_3(\sigma_3) - \vec{X}_1(\sigma_1)\theta_1(\sigma) - \vec{X}_2(\sigma_2)\theta_2(\sigma)]|\,V\rangle = 0\,,$$
$$\theta_1(\sigma) \equiv \theta(\pi|\,\alpha_1\,| - \sigma_3)\,,$$
$$\theta_2(\sigma) \equiv \theta(\sigma_3 - \pi|\,\alpha_1\,|)\,; \quad \text{(B1)}$$

$$[\vec{P}_3(\sigma_3) + \vec{P}_1(\sigma_1)\theta_1(\sigma) + \vec{P}_2(\sigma_2)\theta_2(\sigma)]|\,V\rangle = 0\,, \quad \text{(B2)}$$

where

$$[\vec{P}_i(\sigma), \vec{X}_j(\sigma')] = -i\delta(\sigma - \sigma')\delta_{ij}\,. \quad \text{(B3)}$$

Let us introduce a new state $|\,V_0\rangle$ by

$$|V\rangle = \exp\left[+i\int \vec{P}_3(\sigma)(\vec{X}_1(\sigma_1)\theta_1(\sigma) + \vec{X}_2(\sigma_2)\theta_2(\sigma))\,d\sigma\right]|V_0\rangle\,. \quad \text{(B4)}$$

Then, one can confirm that $|\,V_0\rangle$ should obey

$$\vec{X}_3(\sigma_3)|\,V_0\rangle = 0 \quad \text{(B5)}$$

because of (B1). Furthermore, from (B2) it is easy to show that

$$\vec{P}_1(\sigma_1)\theta_1(\sigma)|\,V_0\rangle = \vec{P}_2(\sigma_2)\theta_2(\sigma)|\,V_0\rangle = 0\,. \quad \text{(B6)}$$

The expansion formulas

$$\vec{X}_i(\sigma) = \left[\vec{x}_i + \sum_{n=1}^{\infty} 2\vec{x}_{i,n}\cos\left(\frac{n\sigma}{\alpha_i}\right)\right],$$
$$\vec{P}_i(\sigma) = \left(\frac{1}{\pi\alpha_i}\right)\left[\vec{p}_i + \sum_{n=1}^{\infty}\vec{p}_{i,n}\cos\left(\frac{n\sigma}{\alpha_i}\right)\right] \quad \text{(B7)}$$

show that (B5) and (B6) are equivalent to

$$(\vec{a}_{3,n} + \vec{a}^{\dagger}_{3,n})|\,V_0\rangle = 0\,,$$
$$(\vec{a}_{2,n} - \vec{a}^{\dagger}_{2,n})|\,V_0\rangle = 0\,, \quad \text{(B8)}$$
$$(\vec{a}_{1,n} - \vec{a}^{\dagger}_{1,n})|\,V_0\rangle = 0\,,$$

where

$$\vec{x}_{i,n} = \frac{1}{(2n)^{1/2}}(\vec{a}_{i,n} + \vec{a}^{\dagger}_{i,n})\,,$$
$$\vec{p}_{i,n} = (\tfrac{1}{2}n)^{1/2}(\vec{a}_{i,n} - \vec{a}^{\dagger}_{i,n})/i\,. \quad \text{(B9)}$$

The solution to (B8) is now given by

$$|\,V_0\rangle = \exp\{\tfrac{1}{2}[(a_1^{\dagger}|\,a_1^{\dagger}) + (a_2^{\dagger}|\,a_2^{\dagger}) - (a_3^{\dagger}|\,a_3^{\dagger})]\}|0\rangle\,. \quad \text{(B10)}$$

Substituting (B10) into (B4) and eliminating all the annihilation operators, one obtains

$$|\,V\rangle = \exp\Bigg\{\tfrac{1}{2}\sum_{i=1}^{2}(a_i^{\dagger}|\,a_i^{\dagger}) - \tfrac{1}{2}(a_3^{\dagger}|\,a_3^{\dagger}) - \sum_{j=1,2}\Bigg[\left\langle a_3^{\dagger}\right|\frac{1}{1+\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}}\bar{A}_{3,j}\left|a_j^{\dagger}\right\rangle + \left\langle a_3^{\dagger}\right|\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}\frac{1}{1+\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}}\bar{A}_{3,j}\left|a_j^{\dagger}\right\rangle\Bigg]$$
$$+ \left\langle a_3^{\dagger}\right|\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}\frac{1}{1+\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}}\left|a_3^{\dagger}\right\rangle\Bigg\}|0\rangle\exp\left\{-\frac{D-2}{2}\mathrm{Tr}\left[\ln\left(1+\sum_{i=1}^{2}\bar{A}_{3,i}\bar{A}_{i,3}\right)\right]\right\}, \quad \text{(B11)}$$

where

$$\bar{A}_{3,i} = \tfrac{1}{4}\sqrt{N}A_{3,i}\frac{1}{\sqrt{N}}, \quad \text{(B12)}$$

with $A_{3,i}$ being the overlapping matrix. The four-string interaction vertex can also be constructed by the same method.

APPENDIX C

The explicit expression for the four-string interaction has been obtained by attaching four Green's functions to the δ function vertex and then by functionally integrating over the intermediate-string states. Here, however, we will calculate the four-string interaction by using the methods of Goto and Naka.

For the sake of brevity, we will only present the highlights of the calculation. We will omit

the integration over the "zipper" variable, and we will drop some contributions due to zero modes:

$$\alpha_i > 0, \quad \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4, \quad \alpha_1 > \alpha_3 > \alpha_2,$$
$$\alpha_1 > \alpha_4 > \alpha_2, \quad 0 < \sigma_i < \pi\alpha_i,$$
$$\theta_a(\sigma) \equiv \theta(\sigma_1 - \sigma_0), \quad \theta_b(\sigma) \equiv \theta(\sigma_0 - \sigma_1)$$
$$\pi(\alpha_1 - \alpha_3) < \sigma_0 < \pi\alpha_4,$$

$$[\vec{\mathbf{X}}_1(\sigma_1) - \vec{\mathbf{X}}_3(\sigma_3)\theta_a(\sigma) - \vec{\mathbf{X}}_4(\sigma_4)\theta_b(\sigma)]|\, V\rangle = 0,$$
$$[\vec{\mathbf{X}}_2(\sigma_2) - \vec{\mathbf{X}}_3(\sigma_3)\theta_b(\sigma) - \vec{\mathbf{X}}_4(\sigma_4)\theta_a(\sigma)]|\, V\rangle = 0,$$
$$[\vec{\mathbf{P}}_1(\sigma_1) + \vec{\mathbf{P}}_3(\sigma_3)\theta_a(\sigma) + \vec{\mathbf{P}}_4(\sigma_4)\theta_b(\sigma)]|\, V\rangle = 0,$$
$$[\vec{\mathbf{P}}_2(\sigma_2) + \vec{\mathbf{P}}_3(\sigma_3)\theta_b(\sigma) + \vec{\mathbf{P}}_4(\sigma_4)\theta_a(\sigma)]|\, V\rangle = 0,$$

$$|\, V\rangle \equiv \exp\left(-i\int_0^{\pi\alpha_1} d\sigma_1\{\vec{\mathbf{P}}_1(\sigma_1)[\vec{\mathbf{X}}_3(\sigma_3)\theta_a(\sigma) + \vec{\mathbf{X}}_4(\sigma_4)\theta_b(\sigma)] + \vec{\mathbf{P}}_2(\sigma_2)[\vec{\mathbf{X}}_4(\sigma_4)\theta_a(\sigma) + \vec{\mathbf{X}}_3(\sigma_3)\theta_b(\sigma)]\}\right)|\, V_0\rangle,$$

$$\vec{\mathbf{X}}_1(\sigma_1)|\, V\rangle = i\frac{\delta}{\delta\vec{\mathbf{P}}_1(\sigma_1)}\Big|\, V\Big\rangle \Rightarrow \vec{\mathbf{X}}_1(\sigma_1)|\, V_0\rangle = 0,$$
$$\vec{\mathbf{X}}_2(\sigma_2)|\, V\rangle = i\frac{\delta}{\delta\vec{\mathbf{P}}_2(\sigma_2)}\Big|\, V\Big\rangle \Rightarrow \vec{\mathbf{X}}_2(\sigma_2)|\, V_0\rangle = 0,$$
$$\vec{\mathbf{P}}_3(\sigma_3)|\, V\rangle = -i\frac{\delta}{\delta\vec{\mathbf{X}}_3(\sigma_3)}\Big|\, V\Big\rangle \Rightarrow \vec{\mathbf{P}}_3(\sigma_3)|\, V_0\rangle = 0,$$
$$\vec{\mathbf{P}}_4(\sigma_4)|\, V\rangle = -i\frac{\delta}{\delta\vec{\mathbf{X}}_4(\sigma_4)}\Big|\, V\Big\rangle \Rightarrow \vec{\mathbf{P}}_4(\sigma_4)|\, V_0\rangle = 0,$$
$$\Rightarrow |\, V_0\rangle = \exp\left[-\frac{1}{2}\sum_{i=1}^{2}(a_i^\dagger|\, a_i^\dagger) + \frac{1}{2}\sum_{j=3}^{4}(a_j^\dagger|\, a_j^\dagger)\right]|\, 0\rangle,$$
$$(M^{ij})_{n,m} \equiv \frac{n}{m\pi\alpha_1}\int_{a_{ij}}^{b_{ij}} d\sigma_1 \cos\left(\frac{n\sigma_i}{\alpha_i}\right)\cos\left(\frac{m\sigma_j}{\alpha_j}\right), \qquad i = 1, 2;\ j = 3, 4,$$

$$\begin{cases} a_{14} = 0, & b_{14} = \sigma_0, \\ a_{13} = \sigma_0, & b_{13} = \pi\alpha_1, \\ a_{23} = \pi(\alpha_1 - \alpha_3), & b_{23} = \sigma_0, \\ a_{24} = \sigma_0, & b_{24} = \pi\alpha_4, \end{cases}$$

$$\Rightarrow |\, V\rangle = \exp\left[-\sum_{i=1}^{2}\sum_{j=3}^{4}(a_i - a_i^\dagger|\, M_{ij}|\, a_j + a_j^\dagger)\right]\Big|\, V_0\Big\rangle,$$
$$\alpha \equiv \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \quad M_{\alpha\beta} \equiv \begin{pmatrix} M_{13} & M_{14} \\ M_{23} & M_{24} \end{pmatrix},$$
$$\Rightarrow |\, V\rangle = \exp\Big[4\Big\langle\alpha^\dagger\Big|M_{\alpha\beta}\frac{1}{1+4M_{\beta\alpha}M_{\alpha\beta}}M_{\beta\alpha}\Big|\,\alpha^\dagger\Big\rangle - 4\Big\langle\beta^\dagger\Big|M_{\beta\alpha}\frac{1}{1+4M_{\alpha\beta}M_{\beta\alpha}}M_{\alpha\beta}\Big|\,\beta^\dagger\Big\rangle + 4\Big\langle\alpha^\dagger\Big|\frac{1}{1+4M_{\alpha\beta}M_{\beta\alpha}}M_{\alpha\beta}\Big|\,\beta^\dagger\Big\rangle - \frac{1}{2}\sum_{i=1}^{2}(a_i^\dagger|\, a_i^\dagger) + \frac{1}{2}\sum_{j=3}^{4}(a_j^\dagger|\, a_j^\dagger)\Big]|\, 0\rangle$$
$$\times \exp[-\tfrac{1}{2}(D-2)\,\mathrm{Tr}\ln(1+4M_{\alpha\beta}M_{\beta\alpha})].$$

We adopt an unconventional, but obvious, summation convention over α and β.

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Absence of runaways and divergent self-mass in nonrelativistic quantum electrodynamics*

E. J. Moniz
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139† and Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

D. H. Sharp
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
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We derive the quantum-mechanical operator equation of motion for a point electron and show that (i) the electrostatic self-mass of a point electron is zero in quantum mechanics; (ii) the equation of motion does not admit runaway solutions; (iii) the correspondence limit of the *solutions* of the quantum-mechanical equation likewise does not display runaways; and (iv) the solutions do not violate the principle of causality in quantum mechanics.

The Lorentz-Dirac theory of a classical nonrelativistic point electron, interacting with its self-field and subject to an external force $\vec{F}(t)$, leads to the equation of motion[1-3]

$$m_0 \frac{d^2\vec{R}(t)}{dt^2} = \vec{F}(t) - \delta m \frac{d^2\vec{R}(t)}{dt^2} + \frac{2e^2}{3c^3}\frac{d^3\vec{R}(t)}{dt^3}, \qquad (1)$$

where m_0 is the mechanical mass of the electron and δm is its electrostatic self-mass (divergent for a point electron). This equation is beset by the well-known difficulties that it admits runaway solutions (exponentially growing accelerations even in the absence of external forces) and violates causality (when the runaways are eliminated by the imposition of suitable boundary conditions, the electron accelerates before the force acts). While these results mar the internal consistency of classical electrodynamics, the view is often adopted that since preacceleration occurs on such a short time scale (10^{-23} sec for an electron) the noncausal effects belong in the domain of quantum theory, to which one must turn for a resolution of the problem.

To prove that runaways are not present in quantum theory one would have to show that no Heisenberg-picture operator in the theory has an exponentially growing time dependence. This result has not been established for the standard Hamiltonian[4] governing the interaction of a nonrelativistic point electron with a quantized electromagnetic field,[5,6] on which the conclusions of this paper are based, and it is not our purpose here to give such a proof. Instead, starting from the operator form of Maxwell's equation and the Lorentz force equation which follow from the Hamiltonian, we derive a quantum-mechanical operator equation of motion for a point electron which reduces to Eq. (1) in the correspondence limit (i.e., $\hbar \to 0$) and show that (i) the electrostatic self-mass of a point electron is zero in quantum mechanics, (ii) the equation of motion does not admit runaway solutions, (iii) the correspondence limit of the *solutions* of the quantum-mechanical equations likewise does not display runaways, and (iv) the solutions do not violate the principle of causality in quantum mechanics. These calculations thus show how nonrelativistic quantum electrodynamics is compatible with requirements (no runaways, etc.) which presumably follow from the general principles of quantum mechanics and in addition possesses a physically reasonable correspondence limit.

The results in the quantum-mechanical case are best understood by comparison to the classical results for an extended charge distribution. For a spherically symmetric static charge distribution, Eq. (1) is replaced by[7]