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Electromagnetic fields in curved spaces: A constructive procedure*

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A new procedure for computing electromagnetic fields in curved space-times is presented. With this procedure, the problem is reduced to solving one complex linear scalar wave equation. Many space-times of astrophysical importance may be treated in this manner, including black-hole and neutron-star spaces and cosmological models.

I. INTRODUCTION

A major source of difficulty in integrating the electromagnetic field equations in a given curved space-time is the coupled structure of the Maxwell system, which consists of eight partial differential equations in six unknowns. Standard devices for the flat-space treatment which successfully decouple the equations fail in general relativity, since space-time curvature leads to a more strongly coupled system. One such device, however, has not been fully exploited in the context of curved spaces: the Debye or two-component Hertz potential formalism. It is the purpose of this paper to show that the Hertz formalism can be extended to all curved space-times, and that the Debye formalism can be extended to a wide and astrophysically interesting class of spaces, in each of which the potential obeys one (decoupled) linear scalar wave equation.[1] Included are, for example, the Friedmann cosmological models, the Kerr and Schwarzschild solutions of black holes and neutron stars, the Gödel universe, Taub-NUT (Newman-Unti-Tamburino) space, the Bondi and Kantowski-Sachs universes, and other universes of various Bianchi types. In fact, the results of Ellis[2] and Wainwright[3] show that this method applies to every perfect-fluid model with local rotational symmetry. Mathematically, the class of space-times to which the scalar Debye formalism has been extended is the generalized Goldberg-Sachs[4-6] class: every algebraically special geometry, in the sense of Petrov,[7] which admits a shear-free congruence of null geodesics along the repeated principal direction of the Weyl tensor. (One must omit from among these the spaces with strong background electromagnetic fields, as required in the test-field approximation.)

In Sec. II we summarize the Hertz and Debye potential theory in flat space and formulate the problem of generalizing to curved spaces. Section III couches the theory in the covariant language of differential forms; the notation and powerful theorems of this formalism provide the desired generalization of the Hertzian scheme. Section IV translates these results into the standard tensor notation, both to assist the reader unfamiliar with forms, and also to facilitate the eventual use of the fully explicit Newman-Penrose[5] (NP) formalism. In Sec. V, translation of the above results into the concise and explicit NP formalism enables us to construct a decoupled linear wave equation for the scalar Debye potential in the generalized Goldberg-Sachs class of space-times. Examples for important spaces are given in Sec. VI. The aim of Appendix A is to illustrate the procedure of explicitly writing differential form equations in a definite Cartan frame, which is of use at several points in the text. Appendix B is intended to enable the reader unfamiliar with the spinor or the NP spin-coefficient formalisms to understand the latter from a purely tetrad- or Cartan-frame viewpoint, and to calculate spin coefficients, necessary in the applications, by standard Cartan methods. These methods provide a straightforward procedure for computing spin coefficients with a minimum of calculation.

II. HERTZ AND DEBYE POTENTIALS IN FLAT SPACE

This section is a brief summary of the flat-space theory and is largely based on the paper of Nisbet[8]; the reader is referred there for a more detailed discussion. We mention only those results necessary for the subsequent generalization.

Hertz[9] introduced a potential for the Maxwell field while investigating electric dipole fields; the true covariant bivector nature of this potential was noted considerably later by Laporte and Uhlenbeck.[10] This type of bivector (antisymmetric second-rank tensor) potential is related by second derivatives to the physical field, hence by first derivatives to the familiar four-vector potential. In fact,

$$\varphi = -\vec{\nabla}\cdot\vec{P}_E, \quad \vec{A} = \frac{\partial \vec{P}_E}{\partial t} + \vec{\nabla}\times\vec{P}_M, \tag{2.1}$$

and

$$\vec{E} = \vec{\nabla}\vec{\nabla}\cdot\vec{P}_E - \frac{\partial^2\vec{P}_E}{\partial t^2} - \vec{\nabla}\times\frac{\partial\vec{P}_M}{\partial t}$$

$$= -\vec{\nabla}\times\frac{\partial\vec{P}_M}{\partial t} + \vec{\nabla}\times(\vec{\nabla}\times\vec{P}_E),$$

$$\vec{B} = \vec{\nabla}\times\frac{\partial\vec{P}_E}{\partial t} + \vec{\nabla}\times(\vec{\nabla}\times\vec{P}_M) \tag{2.2}$$

$$= \vec{\nabla}\vec{\nabla}\cdot\vec{P}_M - \frac{\partial^2\vec{P}_M}{\partial t^2} + \vec{\nabla}\times\frac{\partial\vec{P}_E}{\partial t}$$

are the relations in question. The notation is standard; we choose $c = 1$ and denote the Hertz bivector by $\vec{P}_E$ and $\vec{P}_M$ according to the natural electric and magnetic labeling of components. The conditions imposed upon the Hertz potential by Eq. (2.2) and the Maxwell equations

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 0, \quad \vec{\nabla}\cdot\vec{E} = 0 \tag{2.3}$$

are just

$$\Box\vec{P}_E = 0, \quad \Box\vec{P}_M = 0 \tag{2.4}$$

in the source-free vacuum case (where $\Box = \partial^2/\partial t^2 - \nabla^2$ is the d'Alembertian operator).

A new type of gauge freedom, termed by Nisbet "gauge transformations of the third kind," is associated with the Hertzian potentials. Here we consider gauge transformations of the sources, that is, those gauge terms which may appear as sources in Eq. (2.4) while preserving the source-free property of the Maxwell field itself. These turn out to be bivectors of the form

$$\vec{Q}_E = \vec{\nabla}\times\vec{G}, \quad \vec{Q}_M = -\frac{\partial\vec{G}}{\partial t} - \vec{\nabla}g$$

and

$$\vec{R}_E = -\frac{\partial\vec{W}}{\partial t} - \vec{\nabla}w, \quad \vec{R}_M = -\vec{\nabla}\times\vec{W}, \tag{2.5}$$

where $(\vec{G}, g)$ and $(\vec{W}, w)$ are arbitrary four-vectors. In this scheme the wave equations (2.4) for the potentials are modified to become

$$\Box\vec{P}_E = \vec{Q}_E + \vec{R}_E, \quad \Box\vec{P}_M = \vec{Q}_M + \vec{R}_M; \tag{2.6}$$

the new fields given by

$$\vec{E} = \vec{R}_E + \vec{\nabla}\vec{\nabla}\cdot\vec{P}_E - \frac{\partial^2\vec{P}_E}{\partial t^2} - \vec{\nabla}\times\frac{\partial\vec{P}_M}{\partial t}$$

$$= -\vec{Q}_E - \vec{\nabla}\times\frac{\partial\vec{P}_M}{\partial t} + \vec{\nabla}\times(\vec{\nabla}\times\vec{P}_E),$$

$$\vec{B} = -\vec{R}_M + \vec{\nabla}\times\frac{\partial\vec{P}_E}{\partial t} + \vec{\nabla}\times(\vec{\nabla}\times\vec{P}_M) \tag{2.7}$$

$$= \vec{Q}_M + \vec{\nabla}\vec{\nabla}\cdot\vec{P}_M - \frac{\partial^2\vec{P}_M}{\partial t^2} + \vec{\nabla}\times\frac{\partial\vec{P}_E}{\partial t}$$

may be verified to remain source-free by substitution into the Maxwell equations (2.3).

Nisbet's[8] reduction of the Hertz bivector to two purely radial vectors (Debye[11] potentials) utilizes the gauge transformations just discussed. In this representation, the potential is given by $\vec{P}_E = \hat{r}P_E$, $\vec{P}_M = \hat{r}P_M$ (with $\hat{r}$ the unit radial vector), and the gauge bivectors are obtained from Eq. (2.5) with $\vec{G} = \vec{W} = 0$ and $g = 2P_E/r$, $w = 2P_M/r$. The functions P_E and P_M are the Debye potentials and Eq. (2.6) implies that they each obey the wave equation (which differs in the radial operator from the scalar d'Alembertian operator)

$$-\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right) = 0. \tag{2.8}$$

The solutions, which are of the form $\psi = e^{-ikt}rz_l(kr)Y_l^m(\theta,\varphi)$ with $z_l(kr)$ a spherical Bessel function and $Y_l^m(\theta,\varphi)$ a spherical harmonic, give rise to the static ($k = 0$) and dynamic electric ($P_M = 0$) and magnetic ($P_E = 0$) multipoles of order l when inserted into the prescription (2.7) for the electromagnetic field.[12] Only the monopole field is missing in this scheme, since the differential operations (2.7) annihilate the $l = 0$ solution to Eq. (2.8).

We emphasize the essential role played in the treatment sketched above by the gauge terms g and w. For, if the d'Alembertian operators of Eq. (2.6) are computed explicitly upon the Debye choice $\vec{P}_E = \hat{r}P_E$, $\vec{P}_M = \hat{r}P_M$ of Hertzian vectors, the resultant expressions each contain three components; only by adding the specified gauge terms to the right-hand side does one reduce two components of each equation to identities and the third component to Eq. (2.8).

A remarkable economy is achieved by the Debye potentials: The arbitrary source-free Maxwell field is specified by two scalar functions which obey a single separable second-order wave equation. Roughly speaking, one might expect that since a zero-rest-mass field possesses two degrees of freedom, no more economical representation of the Maxwell field is possible.

With the intention of formulating a covariant generalization of the Debye potentials, one may consider the two-potential representation from the following viewpoint. By a suitable choice of bivector direction in space-time—the rt direction (and its dual $\theta\varphi$ direction)—one has succeeded in "diagonalizing" the Hertz potential, in the sense of Synge.[13] That is, one has found the principal directions (and values) of the Hertz bivector. Of course, one may in this sense "diagonalize" any bivector, with the resultant principal directions algebraically dependent upon the bivector itself. Remarkably, the Debye scheme shows that all source-free Maxwell fields may be represented by Hertzian bivector potentials with the *same* principal directions, independent of the Maxwell field, and determined *a priori*.

The problem of covariant generalization of the Debye scheme may thus be viewed as the search for a special bivector direction in space-time, determined *a priori* and presumably geometrically, independently of the details of any particular Maxwell field.

We shall see how the generalized Goldberg-Sachs theorem[4-6] provides special directions of the required sort in a wide class of space-times.

III. DIFFERENTIAL FORMS

The reader totally unfamiliar with the language of differential forms may omit this section; the chief loss will be a certain lack of motivation for some of the formulas of the next section, which is provided here.

Once the results of Sec. II are translated from the three-space vector notation into the notation of differential forms,[14-19] the framework will be provided for investigation of the curved-space problem, since the latter is a covariant notation. Comparison of this section with Sec. IV may convince the reader of the superior adaptation of the present notation over standard tensor analysis for problems of this sort (where antisymmetric tensors play a central role).

We make use of the operators $*$, the Hodge dual; d, the exterior derivative; $\delta \equiv *d*$, the co-derivative; and $\Delta \equiv d\delta + \delta d = d*d* + *d*d$, the harmonic operator. The operator Δ has the property of reducing in Minkowski space (or flat three-space) to the d'Alembertian (or Laplacian) operator.

The flat-space equations of Sec. II are now presented in the differential-forms notation; once an equation is written in this formalism, it is fully covariant. That the translations are correct may be verified by explicitly writing out the equations below in some Cartan frame. This procedure is illustrated in Appendix A for selected equations.

In terms of the Maxwell 2-form $f = \frac{1}{2} f_{\mu\nu}\omega^\mu \wedge \omega^\nu$, where $f_{\mu\nu}$ is the Maxwell tensor and ω^α, $\alpha = 0, 1, 2, 3$ are the basis forms, the Maxwell equations (2.3) become

$$df = 0, \quad \delta f = 0. \tag{3.1}$$

The equations (2.1) relating the Hertz bivector (2-form) P to the four-vector (1-form) potential A become

$$A = \delta P. \tag{3.2}$$

The analog of Eq. (2.2) giving f in terms of P is

$$f = d\delta P = -\delta d P, \tag{3.3}$$

the equality of the last two expressions requiring

$$\Delta P = 0, \tag{3.4}$$

the analog of Eq. (2.4). Now the fact that an f given by Eq. (3.3) is a Maxwell field [i.e., satisfies Eq. (3.1)] is a trivial consequence of the identity $d^2 \equiv 0$—the exterior derivative applied twice annihilates any form. That is, we have

$$df = d(d\delta P) \equiv 0, \quad \delta f = \delta(-\delta d P) \equiv 0, \tag{3.5}$$

where we have used the corollary $\delta^2 \equiv 0$ [a consequence of $*^2 = \pm$ the identity, the sign depending on the dimension of the form, so that $\delta^2 = (*d*)(*d*) = \pm *d^2* \equiv 0$].

The 2-form gauge terms (2.5) are

$$Q = dG, \quad R = *dW, \tag{3.6}$$

where G and W are arbitrary 1-forms. The wave equation (2.6) with gauge terms is therefore

$$\Delta P = dG + *dW, \tag{3.7}$$

so that the gauge-transformed field (2.7) becomes

$$f = d\delta P - dG = *dW - \delta d P. \tag{3.8}$$

That the transformed fields (3.8) still obey the Maxwell equations (3.1) is again a trivial application of $d^2 \equiv \delta^2 \equiv 0$.

Equations (3.7) and (3.8) represent a fully covariant generalization of the Hertz potential scheme to all curved space-times.

The Debye two-component reduction of this formalism in flat space may now be summarized as follows. We use the spherical orthonormal Cartan frame $\omega^0 = dt$, $\omega^1 = dr$, $\omega^2 = r\,d\theta$, $\omega^3 = r\sin\theta\, d\varphi$; we choose the Hertz 2-form to be of the form $P = P_E\omega^0 \wedge \omega^1 + P_M\omega^2 \wedge \omega^3$, and select gauge 1-forms $G = (2P_E/r)\omega^0$, $W = (2P_M/r)\omega^0$. Then, just as in Sec. II, Eq. (3.7) results in the wave equation (2.8) for each of P_E and P_M (see Appendix A), and Eq. (3.8) yields the standard electromagnetic multipoles.

It should be remarked that the entire Hertzian scheme as generalized above to curved spaces would fail if the world were Riemannian (as opposed to pseudo-Riemannian), for, if space-time were a compact Riemann space, a result in Hodge theory[15,16] would say that $\Delta P = 0$ if and only if dP and δP vanish. But then the prescriptions (3.3) for the Maxwell field would be identically zero.

IV. TENSOR NOTATION

There is no unique translation of the formulas of Sec. II into tensor notation; in particular, many second-order tensor operators reduce in flat space to the wave operator of Eq. (2.4) (two examples are the contracted second covariant derivative operator, and the operator which we in fact adopt below). There is, however, a unique translation of the formulas of Sec. III into tensor notation, since the forms language is covariant; we choose this unique prescription. In effect, we are allowing the operators defined on forms to make the choice for us. The reader who has noted the elegance of the formulas of the previous section (based on powerful properties of the operators in the theory of differential forms) will see the motivation for this choice.

For relations between the operators defined on forms and the covariant derivative operator of conventional tensor analysis, Ref. 16 is especially recommended.

In the present notation, the Maxwell equations (2.3) or (3.1) become

$$\nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu} = 0\,, \qquad \nabla^\mu f_{\mu\nu} = 0\,, \tag{4.1}$$

where ∇_μ denotes the covariant derivative. The Hertzian bivector potential $P_{\mu\nu}$ is related to the four-vector potential A_μ [see Eqs. (2.1) or (3.2)] by

$$A_\mu = -\nabla^\lambda P_{\lambda\mu}\,. \tag{4.2}$$

Equations (2.2) or (3.3) giving $f_{\mu\nu}$ in terms of $P_{\mu\nu}$ become

$$\begin{aligned} f_{\mu\nu} &= -\nabla_\mu \nabla^\lambda P_{\lambda\nu} + \nabla_\nu \nabla^\lambda P_{\lambda\mu} \\ &= \nabla_\lambda \nabla^\lambda P_{\mu\nu} - \nabla^\lambda \nabla_\mu P_{\lambda\nu} + \nabla^\lambda \nabla_\nu P_{\lambda\mu}\,. \end{aligned} \tag{4.3}$$

The last equality specifies the wave equation analogous to Eqs. (2.4) or (3.4) to be

$$-\nabla_\lambda \nabla^\lambda P_{\mu\nu} + (\nabla^\lambda \nabla_\mu - \nabla_\mu \nabla^\lambda) P_{\lambda\nu} + (\nabla_\nu \nabla^\lambda - \nabla^\lambda \nabla_\nu) P_{\lambda\mu} = 0\,. \tag{4.4}$$

It is readily seen via the Ricci identities that the last two terms are proportional to the Riemann tensor, so that the operator of Eq. (4.4) could not be preferred over the contracted second covariant derivative (the first term alone) purely from the standpoint of a generalization from flat space. Thus an investigator working in the tensor formalism might not have succeeded in finding the operator of Eq. (4.4).

The fact that the field tensor (4.3) obeys Eq. (4.1) is no longer proved by inspection as in the notation of forms. We present the proof.

For the first set of Maxwell equations [the first of Eq. (4.1)] we choose the first of Eq. (4.3) for $f_{\mu\nu}$ and express it in terms of A_μ [Eq. (4.2)], so that $f_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Then

$$\begin{aligned} \nabla_\mu f_{\nu\lambda} + \nabla_\nu f_{\lambda\mu} + \nabla_\lambda f_{\mu\nu} &= \nabla_\mu \nabla_\nu A_\lambda - \nabla_\mu \nabla_\lambda A_\nu + \nabla_\nu \nabla_\lambda A_\mu - \nabla_\nu \nabla_\mu A_\lambda + \nabla_\lambda \nabla_\mu A_\nu - \nabla_\lambda \nabla_\nu A_\mu \\ &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A_\lambda + (\nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda) A_\nu + (\nabla_\nu \nabla_\lambda - \nabla_\lambda \nabla_\nu) A_\mu \\ &= R^{\ \sigma}_{\lambda\ \nu\mu} A_\sigma + R^{\ \sigma}_{\nu\ \mu\lambda} A_\sigma + R^{\ \sigma}_{\mu\ \lambda\nu} A_\sigma \text{ (by the Ricci identities)} \\ &= 3R^{\sigma}{}_{[\lambda\nu\mu]} A_\sigma = 0 \text{ (by the cyclic symmetry of the Riemann tensor).} \end{aligned}$$

Similarly, for the second of Eq. (4.1) we express $f_{\mu\nu}$ by the second of Eq. (4.3) and introduce a potential $B_{\lambda\mu\nu}$ whose four-divergence (as opposed to A_μ whose curl) gives $f_{\mu\nu}$. Thus

$$B_{\lambda\mu\nu} = \nabla_\lambda P_{\mu\nu} - \nabla_\mu P_{\lambda\nu} + \nabla_\nu P_{\lambda\mu}$$

and $f_{\mu\nu} = \nabla^\lambda B_{\lambda\mu\nu}$ together yield the second of Eq. (4.3) (note the total antisymmetry of $B_{\lambda\mu\nu}$). Now the second Maxwell equation gives

$$\begin{aligned} \nabla^\mu f_{\mu\nu} &= \nabla^\mu \nabla^\lambda B_{\lambda\mu\nu} \\ &= \tfrac{1}{2}(\nabla^\mu \nabla^\lambda - \nabla^\lambda \nabla^\mu) B_{\lambda\mu\nu} \text{ (by the antisymmetry of } B_{\lambda\mu\nu}) \\ &= \tfrac{1}{2}(R^{\sigma}{}_{\lambda}{}^{\lambda\mu} B_{\sigma\mu\nu} + R^{\sigma}{}_{\mu}{}^{\lambda\mu} B_{\lambda\sigma\nu} + R^{\sigma}{}_{\nu}{}^{\lambda\mu} B_{\lambda\mu\sigma}) \text{ (by the Ricci identities).} \end{aligned}$$

The first two terms each vanish since the first factor is symmetric and the second antisymmetric in σ, μ (or σ, λ for the second term). The third term is

$$-\tfrac{1}{2} R_{\nu\sigma\lambda\mu} B^{\sigma\lambda\mu} = -\tfrac{1}{2} R_{\nu[\sigma\lambda\mu]} B^{\sigma\lambda\mu}$$

by the antisymmetry of $B^{\sigma\lambda\mu}$, and also vanishes by the Riemann tensor symmetry $R_{\nu[\sigma\lambda\mu]} = 0$.

The gauge terms of Eqs. (2.5) and (3.6) in tensor notation are

$$Q_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu \,, \qquad R_{\mu\nu} = -\nabla^\lambda W_{\lambda\mu\nu} \,, \tag{4.5}$$

where G_μ and $W_{\lambda\mu\nu}$ are arbitrary tensors, except that $W_{\lambda\mu\nu}$ is totally antisymmetric (W is essentially the dual of the arbitrary 4-vector or 1-form with the same kernel letter of the previous sections).

With gauge terms included, the wave equation for the potential becomes [Eqs. (2.6) and (3.7)]

$$-\nabla_\lambda\nabla^\lambda P_{\mu\nu} + (\nabla^\lambda\nabla_\mu - \nabla_\mu\nabla^\lambda)P_{\lambda\nu} + (\nabla_\nu\nabla^\lambda - \nabla^\lambda\nabla_\nu)P_{\lambda\mu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu - \nabla^\lambda W_{\lambda\mu\nu} \,, \tag{4.6}$$

and the gauge-transformed field tensor [Eqs. (2.7) and (3.8)] is given by

$$f_{\mu\nu} = -\nabla_\mu\nabla^\lambda P_{\lambda\nu} + \nabla_\nu\nabla^\lambda P_{\lambda\mu} - \nabla_\mu G_\nu + \nabla_\nu G_\mu = \nabla_\lambda\nabla^\lambda P_{\mu\nu} - \nabla^\lambda\nabla_\mu P_{\lambda\nu} + \nabla^\lambda\nabla_\nu P_{\lambda\mu} - \nabla^\lambda W_{\lambda\mu\nu} \,. \tag{4.7}$$

Using a proof similar to that given above for the field (4.3), one can show that the transformed field still obeys the source-free Maxwell equations.

Equations (4.6) and (4.7) comprise a covariant generalization of the Hertz potential formalism to all space-times.

The flat-space Debye two-component reduction of the potential is now given in tensor notation. In the natural spherical coordinate basis for tensors, we choose $P_{tr} = -P_{rt} = P_E$; $P_{\theta\varphi} = -P_{\varphi\theta} = r^2 \sin\theta\, P_M$; all other components vanish. For gauge terms we take $G_t = 2P_E/r$, other components zero; and $W_{r\theta\varphi} = 2r^2\sin\theta\, P_M/r$, other components given antisymmetrically by permuting indices, or else vanish. Again, the statement is that with these choices, Eq. (4.6) yields the wave equation (2.8) for each of P_E, P_M and that Eq. (4.7) gives the standard electromagnetic multipole fields.

V. DEBYE POTENTIALS IN CURVED SPACES

With the covariant machinery for the Hertzian potentials having been set up in the last two sections, we are in a position to formulate a two-component (or one complex component) reduction of the potential analogous to the Debye scheme in flat space. The problem consists of finding special bivector directions in space-time so that Eqs. (4.6) or (3.7) yield decoupled wave equations for the corresponding components of the potential for some choice of gauge terms (4.5) or (3.6). In this section we show that in a class of space-times, the Weyl tensor provides such special bivectors through its principal directions. These are, as required, defined geometrically by the space-time itself and independently of the Maxwell fields to be computed.

The considerations which lead to the principal null directions of the Weyl tensor for specification of the preferred bivectors are as follows. In flat space, the asymptotic form of a spherical radiation multipole is a null bivector field, with principal null direction just the radial propagation 4-vector.[20] This is associated in a simple way with the special bivector of the Debye potential [namely, $\omega^0 \wedge \omega^1 = \frac{1}{2}(\omega^0 - \omega^1)\wedge(\omega^0 + \omega^1)$, where $\omega^0 \pm \omega^1$ are the null propagation vectors]. We conjecture, then, that in curved space, we seek those bivector directions associated in this way with propagation vectors of (source-free) null Maxwell fields. But the Mariot[21] -Robinson[22] theorem states that these propagation vectors are just the tangents of shear-free null geodesics; in fact, Robinson's theorem[22] characterizes them as such. Finally, the Goldberg-Sachs[4,5] theorem makes the connection between shear-free congruences of null geodesics and repeated principal null directions of the Weyl tensor (for vacuum space-times): The former are the integral curves of the latter. Thus one is guided in the search for a special bivector in curved space to choose a null tetrad (Cartan frame) with one element aligned along the repeated principal null direction of the Weyl tensor of an algebraically special space-time. The preferred bivector will by these considerations presumably be the exterior product of two elements of such an aligned null tetrad.

In order to check whether one has obtained decoupled wave equations as individual components of Eqs. (4.6) or (3.7), a fully explicit formalism is necessary. For this reason and because of the key role played in the formulation by null vectors, we adopt the NP[5] formalism for the following explicit computations and for a concise statement of the final results.

In this treatment, the approach to the NP formalism will be from a purely tetrad- or Cartan-frame point of view, so that the reader with no knowledge of spinors or NP formalism but familiar with the tetrad or frame formalism will have no difficulty proceeding.

It should first be remarked that all formulas of Sec. IV may be reinterpreted from a frame viewpoint with only the conceptual change that each index be understood as a tetrad index. With that in mind, the translation of the equations of Sec. IV into the NP formalism becomes purely mechanical, with the dictionary of changes in notation provided in Appendix B. The covariant derivatives of the last section are given by the standard frame or tetrad formulas, for arbitrary tensors T of indicated rank,

$$\begin{aligned}\nabla_\beta T_\mu &= \omega_\beta(T_\mu) - \eta^{\sigma\rho}\gamma_{\rho\mu\beta}T_\sigma\,,\\ \nabla_\beta T_{\mu\nu} &= \omega_\beta(T_{\mu\nu}) - \eta^{\sigma\rho}\gamma_{\rho\mu\beta}T_{\sigma\nu} - \eta^{\sigma\rho}\gamma_{\rho\nu\beta}T_{\mu\sigma}\,,\\ \nabla_\beta T_{\mu\nu\lambda} &= \omega_\beta(T_{\mu\nu\lambda}) - \eta^{\sigma\rho}\gamma_{\rho\mu\beta}T_{\sigma\nu\lambda}\\ &\quad - \eta^{\sigma\rho}\gamma_{\rho\nu\beta}T_{\mu\sigma\lambda} - \eta^{\sigma\rho}\gamma_{\rho\lambda\beta}T_{\mu\nu\sigma}\,,\end{aligned} \tag{5.1}$$

where the matrix $\eta^{\mu\nu}$ for raising frame indices in an NP null tetrad is

$$\eta^{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix} \quad (\mu, \nu = 1, 2, 3, 4)\,. \tag{5.2}$$

The Ricci rotation coefficients $\gamma_{\rho\mu\beta}$, or linear combinations of them, are relabeled by individual letters in the NP formalism (these are the spin coefficients); this scheme is given in Appendix B, as are the individual labels for the frame derivatives ω_μ (the intrinsic tetrad derivatives). The reader is warned about a redundance of notation in the literature: Δ represents both the harmonic operator and one of the NP frame derivatives; δ represents both the co-derivative on forms and one of the NP frame derivatives; context, however, will make it clear which meaning is intended.

In an NP null tetrad aligned so that the vector l is oriented along the repeated principal direction of the Weyl tensor, we choose a Hertzian bivector with one independent complex scalar component given by $P_{24} \equiv P_{n\bar{m}} = -P_{\bar{m}n} = \psi$ (and $P_{23} \equiv P_{nm} = -P_{mn} = \bar{\psi}$ as required for real components in real frames), all other components zero. A direct computation shows that for this choice, Eq. (4.6) with suitable gauge terms yields a decoupled equation for the scalar potential ψ. To see this, we compute the left-hand side of Eq. (4.6), writing all sums out explicitly, making use of Eq. (5.1) and the NP notational scheme of Appendix B. The three independent components (all others are related to these by antisymmetry or complex conjugation) are found to be

$$\begin{aligned}& n\bar{m}\text{: } -2[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})(D + 2\epsilon - \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta - \tau)]\psi\,,\\ & ln \text{ or } \bar{m}m\text{: } -[(D + \epsilon + \bar{\epsilon} + \rho - \bar{\rho})(\delta + 2\beta - \tau) + (-\delta + \bar{\alpha} - \beta - \bar{\pi} - \tau)(D + 2\epsilon - \rho)]\psi = 2[(D + \epsilon + \bar{\epsilon} - \bar{\rho})\tau - (\delta + \bar{\pi} - \bar{\alpha} + \beta)\rho]\psi\\ & lm\text{: } 0\,,\end{aligned} \tag{5.3}$$

where terms proportional to κ and σ have been omitted ($\kappa = \sigma = 0$ is the shear-free null geodesic condition on l); the second expression for the ln component is derived from the first by the use of several of the NP equations.[5] The choice of gauge terms $G_n = W_{n\bar{m}m} = 2\tau\psi + 2\bar{\tau}\bar{\psi}$, $G_{\bar{m}} = W_{ln\bar{m}} = 2\rho\psi$, $G_m = W_{lnm} = 2\bar{\rho}\bar{\psi}$ (other components obtained antisymmetrically by permuting indices, or else zero) is seen to decouple the equations, for, with these gauge terms, the three components of the right-hand side of Eq. (4.6) are

$$\begin{aligned}& n\bar{m}\text{: } 4[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})\rho - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})\tau]\psi,\\ & ln \text{ or } \bar{m}m\text{: } 2[(D + \epsilon + \bar{\epsilon} - \bar{\rho})\tau - (\delta + \bar{\pi} - \bar{\alpha} + \beta)\rho]\psi,\\ & lm\text{: } 0\,.\end{aligned} \tag{5.4}$$

Comparison with Eq. (5.3) shows that the ln and lm components are identically satisfied, and that the $n\bar{m}$ component yields a single decoupled wave equation:

$$[(\Delta - \bar{\gamma} + \gamma + \bar{\mu})(D + 2\epsilon + \rho) - (\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\delta + 2\beta + \tau)]\psi = 0\,. \tag{5.5}$$

This linear equation may be treated by separation of variables in important space-times.

The complex scalar ψ contains all of the information of the Maxwell field. Indeed, the Maxwell tensor is given explicitly in terms of the solutions for ψ by Eq. (4.7) (with $P_{\mu\nu}$, G_μ, and $W_{\lambda\mu\nu}$ as specified above in terms of ψ):

$$\begin{aligned}\varphi_0 &\equiv f_{lm} = [-(D - \epsilon + \bar{\epsilon} - \bar{\rho})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}\,,\\ \varphi_1 &\equiv \tfrac{1}{2}(f_{ln} + f_{\bar{m}m}) = [-(D + \bar{\epsilon} + \epsilon)(\bar{\delta} + 2\bar{\beta} + \bar{\tau})\\ &\qquad + (\pi + \bar{\tau})(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}\,,\\ \varphi_2 &\equiv f_{\bar{m}n} = [-(\bar{\delta} + \alpha + \bar{\beta} - \bar{\tau})(\bar{\delta} + 2\bar{\beta} + \bar{\tau}) + \lambda(D + 2\bar{\epsilon} + \bar{\rho})]\bar{\psi}\,,\end{aligned} \tag{5.6}$$

where the φ notation is the NP labeling of the tetrad components of the field, as specified. Thus, the *Maxwell field is computed by straightforward differentiation* of the solution of a linear scalar wave equation.

Just as in flat space, the monopole field is not included in the solutions (5.6), although it may easily be computed in any particular geometry. That one complex potential (rather than the two real potentials of the flat-space formulation) is sufficient to yield the arbitrary field tensor is made plausible by the following consideration. In the two-real-component scheme, the second component yields Maxwell fields dual to those yielded by the first ("magnetic" vs "electric" multipoles). In the null tetrad treatment, however, the duality operation is accomplished merely by the multiplication of a given scalar potential ψ by i (this follows from the facts that the dual of the bivector $l \wedge m$ is just $il \wedge m$, and that the dual of the potential yields the dual of the field). Hence if a given Maxwell field is obtained from a scalar ψ satisfying Eq. (5.5), then its dual is obtained from another solution of Eq. (5.5), namely $i\psi$, by linearity of the equation. In short, a scalar potential along an independent bivector direction is not necessary to specify the dual of a field, unlike the flat-space case.

The coordinate components of the Maxwell tensor are given in terms of the NP components of Eq. (5.6) by standard basis transformation procedures:

$$F_{\mu\nu} = 2(\varphi_1 + \overline{\varphi}_1) n_{[\mu} l_{\nu]} + 2\varphi_2 l_{[\mu} m_{\nu]} + 2\overline{\varphi}_2 l_{[\mu} \overline{m}_{\nu]} + 2\varphi_0 \overline{m}_{[\mu} n_{\nu]} + 2\overline{\varphi}_0 m_{[\mu} n_{\nu]} + 2(\varphi_1 - \overline{\varphi}_1) m_{[\mu} \overline{m}_{\nu]} , \tag{5.7}$$

where antisymmetrization is denoted by the square brackets.

Equation (5.7) gives Maxwell fields in regions free of electromagnetic sources; since they are exact solutions, including near-zone fields, they may be matched to bounded sources by standard methods. Extended sources may be treated by Green's-function techniques as in flat space,[23] and by a curved-space extension of the method of stream potentials (see Ref. 8).

Finally, it should be remarked that in Type D space-times admitting a Debye potential scheme as in Eqs. (5.5) and (5.6), an additional formulation is possible. The basic framework is still that of the previous sections; the difference is that the potential is of the form $P_{13} = P_{lm} = -P_{ml} = \psi$ (with $P_{14} = P_{l\overline{m}} = -P_{\overline{m}l} = \overline{\psi}$ required by reality of physical components). Or, the new scheme may be derived from Eqs. (5.5) and (5.6) by application of the transformation $l \leftrightarrow n$, $m \leftrightarrow \overline{m}$ of the NP formalism. The justification is that in a Type D space-time admitting shear-free congruences of null geodesics along each of the two repeated principal directions of the Weyl tensor, the vectors l and n, if oriented along these special directions, are equivalent. The alternate equations are

$$[(D + \overline{\epsilon} - \epsilon - \overline{\rho})(\Delta - 2\gamma - \mu) - (\delta + \overline{\pi} - \overline{\alpha} - \beta)(\overline{\delta} - 2\alpha - \pi)]\psi = 0 , \tag{5.8}$$

$$\varphi_0 = [-(\delta - \beta - \overline{\alpha} + \overline{\pi})(\delta - 2\overline{\alpha} - \overline{\pi})]\overline{\psi} ,$$

$$\varphi_1 = [-(\Delta - \overline{\gamma} - \gamma)(\delta - 2\overline{\alpha} - \overline{\pi}) - (\tau + \overline{\pi})(\Delta - 2\overline{\gamma} - \overline{\mu})]\overline{\psi} , \tag{5.9}$$

$$\varphi_2 = [-(\Delta + \gamma - \overline{\gamma} + \overline{\mu})(\Delta - 2\overline{\gamma} - \overline{\mu})]\overline{\psi} .$$

We emphasize that in this scheme, all of κ, σ, λ, and ν are assumed to vanish, so that each of l and n is required to be a shear-free null geodesic direction. In these space-times, either the scheme (5.8), (5.9) or the scheme (5.5), (5.6) is sufficient for computations; it is not necessary to use the two formulations together. The choice will presumably be made on the basis of which of Eqs. (5.5) or (5.8) is the easier differential equation to solve in any particular application.

For completeness we present a third scheme which, like the second, is valid in Type D space-times with null geodesic congruences along each of the repeated principal directions of the Weyl tensor. Unlike the wave equations (5.5) and (5.8), the corresponding equation in the third scheme, Eq. (5.10), is not separable in important spaces, and is thus less likely to be useful in applications than the above two treatments. In the third formulation the wave equation and Maxwell field components are

$$[-(\Delta - \gamma - \overline{\gamma} + \overline{\mu} - \mu)D + (\overline{\delta} - \alpha + \overline{\beta} - \pi - \overline{\tau})\delta]\psi = 0 , \tag{5.10}$$

$$\varphi_0 = [(\delta - \overline{\alpha} - \beta + \overline{\pi})D + (D - \epsilon + \overline{\epsilon} - \overline{\rho})\delta]\overline{\psi} ,$$

$$\varphi_1 = [(\Delta - \gamma - \overline{\gamma} + \overline{\mu} - \mu)D + (\delta - \overline{\alpha} + \beta + \overline{\pi} + \tau)\overline{\delta}]\overline{\psi} , \tag{5.11}$$

$$\varphi_2 = [(\Delta + \gamma - \overline{\gamma} + \overline{\mu})\overline{\delta} + (\overline{\delta} + \alpha + \overline{\beta} - \overline{\tau})\Delta]\overline{\psi} ,$$

where the Hertzian bivector potential is given in terms of the scalar ψ by $\frac{1}{2}(P_{ln} + P_{\overline{m}m}) = \psi$.

The space-times covered by the alternate formulations, Eqs. (5.8) and (5.9) and Eqs. (5.10) and (5.11), in practice include the Kerr and Schwarzschild geometries, as well as the matter-filled cosmologies mentioned above, which Wainwright[3] has shown to have both the required Petrov Type (D) and the required congruences of null geodesics. The vacuum Type D spaces are automatically included by the Goldberg-Sachs theorem. All of these are of course included in the first formulation [Eqs. (5.5) and (5.6)].

VI. ILLUSTRATIONS OF THE METHOD

Although the physical-frame Debye potential formulation of flat space (Sec. II) may not always have as elegant a generalization to curved space as does the null-frame scheme (Sec. V), it may nevertheless be extended to important space-times. In particular, we show below that its extension to the spherically symmetric Schwarzschild and Friedmann space-times yields the so-called vector (more correctly, bivector) spherical harmonics representation of the field. Where the symmetry of the space-time allows such a treatment, it is preferred over the null-frame scheme since the wave equation is real, and the Maxwell field is given directly in physical components.

We illustrate the Debye potential method in Schwarzschild[24] space (black holes, neutron stars), the Friedmann[25] universe models, and the Kerr[26] solution (rotating black holes, neutron stars).

A. Schwarzschild space (Petrov Type D)

Israel[27] and Anderson and Cohen[28] have found the static multipoles; Mo and Papas[29] have studied the dynamical case. In fact, Mo and Papas have introduced Debye potentials for spherical space-times, though from a three-vector analysis viewpoint. Below we relate their scheme to ours, and compare our results with those of Refs. 27 and 28.

In the orthonormal frame

$$\omega^0 = A\,dt,\ \omega^1 = B\,dr,\ \omega^2 = r\,d\theta,\ \omega^3 = r\sin\theta\,d\varphi, \tag{6.1}$$

where $A^2(r) = B^{-2}(r) = 1 - 2M/r$, with M the gravitational mass in geometrized units, we choose $P = P_E\omega^0\wedge\omega^1 + P_M\omega^2\wedge\omega^3$ and gauge terms $G = (2P_E/Br)\omega^0$, $W = (2P_M/Br)\omega^0$. Then Eq. (3.7) yields

$$-\frac{B}{A}\frac{\partial^2\psi}{\partial t^2} + \frac{\partial}{\partial r}\left(\frac{A}{B}\frac{\partial}{\partial r}\right)\psi + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right) = 0, \tag{6.2}$$

just the vacuum case of Eq. (6) of Ref. 29, for each of P_E and P_M. Separation of variables leads to $\psi = e^{-ikt}R_{lk}(r)Y_l^m(\theta,\varphi)$, where $R_{lk}(r)$ must in the dynamical case be found by numerical integration of

$$k^2R + \frac{A}{B}\frac{d}{dr}\left(\frac{A}{B}\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}\frac{A}{B}R = 0. \tag{6.3}$$

In the static ($k=0$) case of Eq. (6.3), the functions $R_{l0}(r)$ may be expressed in terms of Legendre functions.[27,28] In fact,

$$R_{l0}(r) = r^2\left\{\left(\frac{r}{2M} - 1\right)\left[P_l\left(1 - \frac{r}{M}\right)\right]_{,r}\right\}_{,r},$$

where replacement of the Legendre function P_l of the first kind by Q_l of the second kind yields the linearly independent solution. The solutions containing Q_l are well behaved at infinity; those containing P_l are well behaved at the horizon; linear combinations of the two kinds can be matched to physical source distributions. It should also be remarked that approximation techniques have proved useful in solving radial equations like Eq. (6.3) in the Schwarzschild geometry.[30,29] [In the flat-space limit $M\to 0$ ($A\to 1, B\to 1$), Eq. (6.3) of course becomes the wave equation (2.8) and $R_{lk}(r)\to rz_l(kr)$, where $z_l(kr)$ is a spherical Bessel, Neumann, or Hankel function.[12]] In terms of these functions the fields are, in physical components in the frame (6.1), as given by Eq. (3.8),

$$\begin{aligned}
E_1 &= \frac{l(l+1)}{r^2}e^{-ikt}R_{lk}(r)Y_l^m(\theta,\varphi), & B_1 &= 0,\\
E_2 &= \frac{e^{-ikt}}{Br}\frac{d}{dr}[R_{lk}(r)]\frac{\partial}{\partial\theta}[Y_l^m(\theta,\varphi)], & B_2 &= \frac{kme^{-ikt}}{Ar\sin\theta}R_{lk}(r)Y_l^m(\theta,\varphi),\\
E_3 &= \frac{ime^{-ikt}}{Br\sin\theta}\frac{d}{dr}[R_{lk}(r)]Y_l^m(\theta,\varphi), & B_3 &= \frac{ike^{-ikt}}{Ar}R_{lk}(r)\frac{\partial}{\partial\theta}[Y_l^m(\theta,\varphi)].
\end{aligned} \tag{6.4}$$

These are the electric multipoles (except for $l=0$, which is $E_1 = 1/r^2$), both static and dynamic; the magnetic multipoles are obtained similarly from Eq. (3.8) and are related to Eq. (6.4) by inserting an independent solution ψ to Eq. (6.2) for P_M, and performing the duality operation $E_i\to B_i$, $B_i\to -E_i$.

B. The Friedmann universe models (Petrov Type O or conformally flat)

In this section we treat the three types of Friedmann or Robertson-Walker universes of positive, zero, and negative spatial curvature. This problem has been considered by other authors. Schrödinger[31] has studied the spatially continuous electromagnetic modes, which might be termed the bivector hyperspherical harmonics, of the closed Friedmann model. Infeld and Schild[32] have analyzed all electromagnetic multipole fields, including those with singularities corresponding to point sources at the origin, in the closed universe model. Lifshitz[33] has considered the vector harmonics, which differ slightly from the source-free Maxwell fields, and has obtained solutions for the $l=1$, $m=0$ (aligned dipole) case (his "most symmetric vector" harmonics) in the closed and open models. Here we sketch the treatment of this problem by the Debye potential method, and we present *all multipole fields in the* *three curvature cases* in terms of elementary functions.

Again, the calculation is performed in the orthonormal frame obtained by normalizing the spherical coordinate basis:

$$\omega^0 = a(\eta)d\eta,\ \ \omega^1 = a(\eta)du,$$
$$\omega^2 = a(\eta)f(u)d\theta,\ \ \omega^3 = a(\eta)f(u)\sin\theta\, d\varphi, \tag{6.5}$$

where the three curvature cases are covered by the choices

$$f(u) = \begin{cases} \sin u, & 0 \le u \le \pi \\ u, & 0 \le u \le \infty \\ \sinh u, & 0 \le u \le \infty \end{cases} \quad \text{for spatial curvature} \begin{cases} \text{positive} \\ \text{zero} \\ \text{negative} \end{cases}$$

(these models are closed, open, and open, respectively).[34] The radius function $a(\eta)$ also depends upon the curvature case, and can be found in Refs. 34. We choose the Hertzian 2-form potential $P = P_E\,\omega^0 \wedge \omega^1 + P_M\,\omega^2 \wedge \omega^3$ and gauge 1-forms

$$G = (2f_{,u}/af)P_E\,\omega^0 + [(a^2)_{,\eta}/a^3]\,P_E\,\omega^1,$$
$$W = (2f_{,u}/af)P_M\,\omega^0 + [(a^2)_{,\eta}/a^3]\,P_M\,\omega^1 .$$

Then Eq. (3.7) yields the wave equation, for each of P_E and P_M,

$$\frac{\partial^2\psi}{\partial\eta^2} - \frac{\partial^2\psi}{\partial u^2} - \frac{1}{f^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right) = 0, \tag{6.6}$$

which is separable and can be solved analytically in terms of elementary functions. Solutions are of the form $\psi = e^{-ik\eta}f(u)Z_{lk}(u)Y_l^m(\theta,\varphi)$, where the $f(u)$ is factored out for notational convenience and the $Z_{lk}(u)$ obey

$$\frac{1}{f^2(u)}\frac{d}{du}f^2(u)\frac{dZ_{lk}}{du} + \left[G_k - \frac{l(l+1)}{f^2(u)}\right]Z_{lk}(u) = 0, \tag{6.7}$$

where $G_k = k^2 - 1$, k^2, or k^2+1 for the positive-, zero-, or negative-curvature cases, respectively. In the flat case where $f(u) = u$, Eq. (6.7) is just the spherical Bessel equation, as expected from conformal properties of the Maxwell field.

A solution $Z_{0k}(u)$ to Eq. (6.7) with $l = 0$ yields a solution with $l \ne 0$ by the operation[35]

$$Z_{lk}(u) = [-f(u)]^l \frac{d^l}{[f(u)du]^l} Z_{0k}(u); \tag{6.8}$$

we therefore turn attention to the $Z_{0k}(u)$.

For the positive-curvature case the independent solutions for $Z_{0k}(u)$ are $e^{\pm iku}/\sin u$ or $\sin ku/\sin u$ and $\cos ku/\sin u$, with static ($k=0$) limits $u/\sin u$ and $1/\sin u$. Suitably normalized linear combinations of these will yield the unit multipoles with sources at $u=0$ and continuous at $u=\pi$; those with sources at $u=\pi$ and continuous at $u=0$; and the traveling waves. The harmonics, or globally continuous modes, follow from $Z_{0n}(u) = \sin nu/\sin u$, with $n = 2, 3, 4, \ldots$, as seen from the Sturm-Liouville problem associated with Eq. (6.7).

For the negative-curvature case the $Z_{0k}(u)$ are $e^{\pm iku}/\sinh u$ or $\sin ku/\sinh u$ and $\cos ku/\sinh u$, $k \ne 0$, with static limits $u/\sinh u$ and $1/\sinh u$.

The fields themselves as given by Eq. (3.8) are, for electric type,

$$\begin{aligned} E_1 &= \frac{l(l+1)e^{-ik\eta}}{a^2(\eta)f(u)} Z_{lk}(u)Y_l^m(\theta,\varphi), & B_1 &= 0, \\ E_2 &= \frac{e^{-ik\eta}}{a^2(\eta)f(u)}\frac{d}{du}[f(u)Z_{lk}(u)]\frac{\partial}{\partial\theta}[Y_l^m(\theta,\varphi)], & B_2 &= \frac{kme^{-ik\eta}}{a^2(\eta)\sin\theta}Z_{lk}(u)Y_l^m(\theta,\varphi), \\ E_3 &= \frac{ime^{-ik\eta}}{a^2(\eta)f(u)\sin\theta}\frac{d}{du}[f(u)Z_{lk}(u)]Y_l^m(\theta,\varphi), & B_3 &= \frac{ike^{-ik\eta}}{a^2(\eta)}Z_{lk}(u)\frac{\partial}{\partial\theta}[Y_l^m(\theta,\varphi)] . \end{aligned} \tag{6.9}$$

The magnetic multipoles follow likewise from Eq. (3.8) or may be obtained from the dual ($E_i \to B_i$, $B_i \to -E_i$) of the above fields, with an independent solution ψ of Eq. (6.6) for P_M. The "monopole" electric field is $E_1 = 1/a^2(\eta)f^2(u)$, other components zero.

As an illustration of the above fields we consider the $l=1$, $m=0$ case in the closed model [$f(u) = \sin u$] with $k = 2, 3, 4, \ldots$ (the aligned dipole harmonics). For this case $Z_{0n} = \sin nu/\sin u$, $n = 2, 3, 4, \ldots$, so that $Z_{1n} = -\sin u(d/\sin u\, du) \times \sin nu/\sin u$, from Eq. (6.8). Then Eq. (6.9) gives

$$E_1 = \frac{2e^{-in\eta}}{a^2(\eta)\sin u}\frac{d}{du}\left(\frac{\sin nu}{\sin u}\right)\cos\theta,$$

$$E_2 = -\frac{e^{-in\eta}}{a^2(\eta)\sin u}\frac{d}{du}\sin u\frac{d}{du}\left(\frac{\sin nu}{\sin u}\right)\sin\theta,$$

$$B_3 = -\frac{ine^{-in\eta}}{a^2(\eta)} \frac{d}{du}\left(\frac{\sin nu}{\sin u}\right)\sin\theta\,,$$

$$n=2,3,4,\ldots \text{ (other components vanish)}.$$

These are to be compared with the most symmetric vector harmonics of Lifshitz (which lack the B_3 component since they are vectors rather than bivectors); all other multipoles, continuous or with singularities for point sources, follow equally readily from Eqs. (6.8) and (6.9).

It should be remarked that in their investigation of the above fields in the closed universe, Infeld and Schild noticed that their solutions for the four-vector potential, derived by conformal transformation methods, could be obtained by differentiating a single complex scalar. Their scalar ψ, whose significance has not been entirely clear, is related to our potentials by $\psi = P_E + iP_M$.

C. The Kerr solution (Petrov Type D)

A new and complete treatment of Maxwell fields in the Kerr geometry is provided by the Debye potential method.

In vacuum Type D spaces, previous authors[36–38] have obtained decoupled wave equations for each of the three complex field components by working directly with the field tensor itself in null frames. It has been shown that two of these three equations are separable,[37,38] and that by integrating (nonseparable) Pfaffian differential equations, one may obtain the remaining two components in terms of a given one.[36]

The present scheme yields all three field components by straightforward differentiation of the solution of one separable wave equation, which we now derive. No simple scheme for decoupling in orthonormal frames seems to be possible here, so the general results of Sec. V will be used. In order to write Eqs. (5.5) and (5.6) explicitly in coordinates, we shall need a choice of coordinates and of null frame (which is arbitrary except for the direction of l); we follow Teukolsky[37] in his choice of tetrad[39] and of Boyer-Lindquist[40] coordinates. In these coordinates the Kerr metric is

$$ds^2 = -(1-2Mr/\Sigma)dt^2 - [(4Mar\sin^2\theta)/\Sigma]\,dt\,d\varphi + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 + (\sin^2\theta)[r^2+a^2+(2Ma^2r\sin^2\theta)/\Sigma]\,d\varphi^2\,, \quad (6.10)$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$ (not to be confused with the NP operator Δ) and M and a are the mass and rotation parameters of the black hole or neutron star. The nonvanishing spin coefficients are[37]

$$\rho = -1/(r - ia\cos\theta), \quad \beta = -2^{-3/2}\bar\rho\cot\theta, \quad \pi = 2^{-1/2}ia\rho^2\sin\theta\,,$$
$$\tau = -2^{-1/2}ia\rho\bar\rho\sin\theta, \quad \mu = \rho^2\bar\rho\Delta/2, \quad \gamma = \mu + \rho\bar\rho(r-M)/2, \quad \alpha = \pi - \bar\beta; \quad (6.11)$$

the intrinsic derivatives are $D = l^\mu\partial/\partial x^\mu$, $\Delta = n^\mu\partial/\partial x^\mu$, and $\delta = m^\mu\partial/\partial x^\mu$, with μ summing over coordinate indices, and with Boyer-Lindquist $[t, r, \theta, \varphi]$ components of the tetrad vectors given by

$$l^\mu = [(r^2+a^2)/\Delta,\ 1,\ 0,\ a/\Delta]\,, \quad n^\mu = [r^2+a^2,\ -\Delta,\ 0,\ a]/2\Sigma,$$
$$m^\mu = 2^{-1/2}[ia\sin\theta,\ 0,\ 1,\ i/\sin\theta]/(r+ia\cos\theta). \quad (6.12)$$

With the substitutions (6.11) and (6.12), the wave equation (5.5) becomes

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\,\partial\varphi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2\psi}{\partial\varphi^2}$$
$$-\Delta\frac{\partial^2\psi}{\partial r^2} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + 2\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi} + 2\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + \frac{1}{\sin^2\theta}\psi = 0\,. \quad (6.13)$$

It is useful to note that Eq. (6.13) for the potential ψ is the same as Eq. (4.7) of Ref. 37 for one of the Maxwell field components (and is separable). We emphasize the distinction in the role played by ψ in Teukolsky's treatment (where $\psi = \rho^{-2}\varphi_2$ and thus gives one NP component of the Maxwell tensor) and the present formulation [where ψ gives all three NP components via the operations (5.6)]. For a discussion of solutions to Eq. (6.13), see Refs. 37, 41, and 42.

The field tensor components (5.6), with ψ in the product form $\psi = e^{-i\omega t}e^{im\varphi}S(\theta)R(r)$ and with the substitutions (6.11) and (6.12), reduce to

$$
\begin{aligned}
\varphi_0 =& \{[\Delta^{-1}(r^2+a^2)]^2\omega^2 + \Delta^{-2}a^2m^2 - 2\Delta^{-2}a(r^2+a^2)m\omega + 2i\Delta^{-2}M(r^2-a^2)\omega \\
&- 2i\Delta^{-2}a(r-M)m - \partial_{rr}{}^2 + 2i\Delta^{-1}[am-(r^2+a^2)\omega]\,\partial_r\}\bar{\psi}, \\
\varphi_1 =& -2^{-1/2}(r-ia\cos\theta)^{-1}(i\Delta^{-1}a\sin\theta(r^2+a^2)\omega^2 + i\Delta^{-1}am^2\csc\theta \\
&- i\Delta^{-1}\csc\theta[a^2(1+\sin^2\theta)+r^2]m\omega + i\Delta^{-1}(r^2+a^2)\omega\cot\theta \\
&+ 2\Delta^{-1}(r-ia\cos\theta)^{-1}aMr\omega\sin\theta - i\Delta^{-1}am\cot\theta \\
&+ \Delta^{-1}\csc\theta(r-ia\cos\theta)^{-1}(\Sigma-2Mr)m - (r-ia\cos\theta)^{-1}\cot\theta + \partial_{r\theta}{}^2 \\
&+ [a\omega\sin\theta - m\csc\theta + \cot\theta - i(r-ia\cos\theta)^{-1}a\sin\theta]\,\partial_r \\
&+ \{i\Delta^{-1}[(r^2+a^2)\omega - am] - (r-ia\cos\theta)^{-1}\}\,\partial_\theta)\bar{\psi}, \\
\varphi_2 =& -\tfrac{1}{2}(r-ia\cos\theta)^{-2}[a^2\omega^2\sin^2\theta + m^2\csc^2\theta - 2am\omega + 2a\omega\cos\theta - \csc^2\theta + \partial_{\theta\theta}{}^2 + (2a\omega\sin\theta - 2m\csc\theta + \cot\theta)\partial_\theta]\bar{\psi}.
\end{aligned}
\tag{6.14}
$$

The "monopole" field in the Kerr geometry is given by[36]

$$\varphi_1 = (r-ia\cos\theta)^{-2}, \quad \varphi_0 = \varphi_2 = 0.$$

In the general case, the solutions for $S(\theta)$ and $R(r)$ must be numerical; for the static, axisymmetric case ($m=\omega=0$), the arbitrary 2^l pole fields have been obtained analytically.[42]

It has been shown by Brill and Cohen[43] that the space-time exterior to a slowly rotating spherical body is just the Kerr metric, to first order in the rotation parameter; hence the fields (6.14), keeping first order in a, may be used for applications to pulsar electrodynamics or any other phenomena associated with rotating neutron stars.

Applications of the method to other space-times of astrophysical interest are in progress and will be published elsewhere.[44]

VII. DISCUSSION

The flat-space method of electromagnetic Hertz potentials has been generalized to all curved space-times. The covariant formulation of this procedure has provided the framework for an extension of the Debye potential scheme to an astrophysically interesting class of spaces, where it gives a new, direct, and practical method for constructing Maxwell fields by solving one decoupled linear scalar wave equation. This formulation allows realistic problems in relativistic astrophysics associated with neutron stars, pulsars, black holes, and global (cosmological) phenomena to be investigated by direct computation.

In each space-time already studied explicitly, it appears that the most general Maxwell field (always excepting the monopole) is obtained by this procedure [i.e., that the fields (5.6) corresponding to the general solution of Eq. (5.5) are complete]. We conjecture therefore that it is the case in every space-time in which Eq. (5.5) is valid; the development of a formal proof is in progress.

The wave equation (5.5) has proved to be separable in all spaces thus far examined in detail. Stewart and Walker[38] have shown, using a method developed by Held,[45] that their wave equations, valid for all vacuum Type D spaces, are separable in certain of these spaces, without using coordinates; their argument is couched purely in a new null-vector formalism[46] similar to NP. Presumably the question of the separability of Eq. (5.5) [which may readily be written in this new "GHP" (Geroch-Held-Penrose) formalism] could be addressed by these methods.

The proof of the completeness of the fields (5.6) and the separability of the wave equation (5.5) would yield the result that the construction of the general Maxwell field in this class of space-times is reduced by Eqs. (5.5) and (5.6) to the solution of ordinary differential equations.

Results strictly analogous to the spin-1 results of this paper have been obtained for zero-rest-mass fields with other physically interesting values of spin and will be presented elsewhere.

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APPENDIX A

The aim of this appendix is to illustrate the procedure of taking equations in the differential-forms notation of Sec. III and writing them explicitly in a definite choice of Cartan frame. This will enable the reader unfamiliar with forms to derive such equations as Eqs. (6.2) and (6.6).

As a simple application we write the homogeneous Maxwell equations $df=0$ [the first of Eq. (3.1)] in the Cartesian coordinate frame in Minkowski space, which is $\omega^0=dt$, $\omega^1=dx$, $\omega^2=dy$, $\omega^3=dz$. In terms of the basis 2-forms the Maxwell 2-form is

$$f=f_{01}dt\wedge dx+f_{02}dt\wedge dy+f_{03}dt\wedge dz+f_{12}dx\wedge dy+f_{13}dx\wedge dz+f_{23}dy\wedge dz\,, \tag{A1}$$

with the physical correspondence

$$E_i=f_{i0},\;\; B_1=f_{23},\;\; B_2=f_{31},\;\; B_3=f_{12}\,. \tag{A2}$$

The definition of the exterior derivative[14-19] gives

$$\begin{aligned} df=&f_{01,y}\,dy\wedge dt\wedge dx+f_{01,z}\,dz\wedge dt\wedge dx+f_{02,x}\,dx\wedge dt\wedge dy+f_{02,z}\,dz\;\; dt\wedge dy\\ &+f_{03,x}\,dx\wedge dt\wedge dz+f_{03,y}\,dy\wedge dt\wedge dz+f_{12,t}\,dt\wedge dx\wedge dy+f_{12,z}\,dz\wedge dx\wedge dy\\ &+f_{13,t}\,dt\wedge dx\wedge dz+f_{13,y}\,dy\wedge dx\wedge dz+f_{23,t}\,dt\wedge dy\wedge dz+f_{23,x}\,dx\wedge dy\wedge dz\\ =&\,0\,. \end{aligned} \tag{A3}$$

Examination of the 012 component, for example (by antisymmetrically combining terms), gives

$$\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}+\frac{\partial B_3}{\partial t}=0\,, \tag{A4}$$

just the z component of $\vec{\nabla}\times\vec{E}+\partial\vec{B}/\partial t=0$; the other components of Eq. (A3) are just the remaining three homogeneous Maxwell equations.

Next the wave equation (2.8) will be derived from Eq. (3.7) by the procedure outlined at the end of Sec. III. Since Hodge duals of forms will be taken repeatedly, it is convenient to summarize this operator on a basis of orthonormal forms on a space-time:

$$\begin{aligned} &*\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3=1,\\ &*1=-\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3,\\ &*\omega^0\wedge\omega^1=\omega^2\wedge\omega^3,\quad *\omega^0=\omega^1\wedge\omega^2\wedge\omega^3,\quad *\omega^1\wedge\omega^2\wedge\omega^3=\omega^0,\\ &*\omega^0\wedge\omega^2=-\omega^1\wedge\omega^3,\quad *\omega^1=\omega^0\wedge\omega^2\wedge\omega^3,\quad *\omega^0\wedge\omega^2\wedge\omega^3=\omega^1,\\ &*\omega^0\wedge\omega^3=\omega^1\wedge\omega^2,\quad *\omega^2=\omega^0\wedge\omega^3\wedge\omega^1,\quad *\omega^0\wedge\omega^3\wedge\omega^1=\omega^2,\\ &*\omega^1\wedge\omega^2=-\omega^0\wedge\omega^3,\quad *\omega^3=\omega^0\wedge\omega^1\wedge\omega^2,\quad *\omega^0\wedge\omega^1\wedge\omega^2=\omega^3,\\ &*\omega^1\wedge\omega^3=\omega^0\wedge\omega^2,\\ &*\omega^2\wedge\omega^3=-\omega^0\wedge\omega^1. \end{aligned} \tag{A5}$$

where

$$ds^2=-(\omega^0)^2+(\omega^1)^2+(\omega^2)^2+(\omega^3)^2\,.$$

To simplify the calculation we may take $P=P_E\omega^0\wedge\omega^1$, $G=(2P_E/r)\omega^0$, $W=0$; the equation for $P=P_M\omega^2\wedge\omega^3$, $G=0$, $W=(2P_M/r)\omega^0$ is then obtained just by taking the Hodge dual since Δ and $*$ commute. The choice of frame is $\omega^0=dt$, $\omega^1=dr$, $\omega^2=r\,d\theta$, $\omega^3=r\sin\theta\, d\varphi$. First we compute $d\delta P$, with the notation $P_E=\psi$, so that $P=\psi\omega^0\wedge\omega^1$:

$$\begin{aligned} &*P=\psi\omega^2\wedge\omega^3=\psi r^2\sin\theta\, d\theta\wedge d\varphi,\\ &d{*P}=\psi_{,t}r^2\sin\theta\, dt\wedge d\theta\wedge d\varphi+(\psi r^2)_{,r}\sin\theta\, dr\wedge d\theta\wedge d\varphi=\psi_{,t}\,\omega^0\wedge\omega^2\wedge\omega^3+r^{-2}(\psi r^2)_{,r}\,\omega^1\wedge\omega^2\wedge\omega^3,\\ &*d*P=\psi_{,t}\,\omega^1+r^{-2}(\psi r^2)_{,r}\,\omega^0=\psi_{,t}\,dr+r^{-2}(\psi r^2)_{,r}\,dt,\\ &d{*d*}P=\psi_{,tt}\,dt\wedge dr+\psi_{,t\theta}d\theta\wedge dr+\psi_{,t\varphi}d\varphi\wedge dr\\ &\qquad +[r^{-2}(\psi r^2)_{,r}]_{,r}\,dr\wedge dt+r^{-2}(r^2\psi_{,\theta})_{,r}\,d\theta\wedge dt+r^{-2}(r^2\psi_{,\varphi})_{,r}\,d\varphi\wedge dt\,. \end{aligned}$$

Thus

$$\begin{aligned} d\delta P=&\{\psi_{,tt}-[r^{-2}(\psi r^2)_{,r}]_{,r}\}\,\omega^0\wedge\omega^1-r^{-3}(r^2\psi_{,\theta})_{,r}\,\omega^0\wedge\omega^2\\ &-(r^3\sin\theta)^{-1}(r^2\psi_{,\varphi})_{,r}\,\omega^0\wedge\omega^3-r^{-1}\psi_{,t\theta}\omega^1\wedge\omega^2-(r\sin\theta)^{-1}\psi_{,t\varphi}\omega^1\wedge\omega^3\,. \end{aligned} \tag{A6}$$

In similar fashion we compute δdP:

$$
\begin{aligned}
P &= \psi dt \wedge dr, \\
dP &= \psi_{,\theta} d\theta \wedge dt \wedge dr + \psi_{,\varphi} d\varphi \wedge dt \wedge dr = r^{-1}\psi_{,\theta}\omega^2 \wedge \omega^0 \wedge \omega^1 + (r\sin\theta)^{-1}(\psi_{,\varphi})\omega^3 \wedge \omega^0 \wedge \omega^1, \\
*dP &= r^{-1}\psi_{,\theta}\omega^3 - (r\sin\theta)^{-1}\psi_{,\varphi}\omega^2 = \psi_{,\theta}\sin\theta\, d\varphi - (\sin\theta)^{-1}\psi_{,\varphi} d\theta, \\
d*dP &= \psi_{,\theta t}\sin\theta\, dt \wedge d\varphi + \psi_{,\theta r}\sin\theta\, dr \wedge d\varphi + (\psi_{,\theta}\sin\theta)_{,\theta} d\theta \wedge d\varphi \\
&\quad - (\sin\theta)^{-1}(\psi_{,\varphi t}) dt \wedge d\theta - (\sin\theta)^{-1}(\psi_{,\varphi r}) dr \wedge d\theta - (\sin\theta)^{-1}(\psi_{,\varphi\varphi}) d\varphi \wedge d\theta \\
&= r^{-1}\psi_{,\theta t}\omega^0 \wedge \omega^3 + r^{-1}\psi_{,\theta r}\omega^1 \wedge \omega^3 + (r^2\sin\theta)^{-1}(\psi_{,\theta}\sin\theta)_{,\theta}\omega^2 \wedge \omega^3 \\
&\quad - (r\sin\theta)^{-1}(\psi_{,\varphi t})\omega^0 \wedge \omega^2 - (r\sin\theta)^{-1}(\psi_{,\varphi r})\omega^1 \wedge \omega^2 - (r^2\sin^2\theta)^{-1}(\psi_{,\varphi\varphi})\omega^3 \wedge \omega^2, \\
\delta dP &= -[(r^2\sin\theta)^{-1}(\psi_{,\theta}\sin\theta)_{,\theta} + (r^2\sin^2\theta)^{-1}(\psi_{,\varphi\varphi})]\,\omega^0 \wedge \omega^1 \\
&\quad + r^{-1}\psi_{,\theta r}\omega^0 \wedge \omega^2 + (r\sin\theta)^{-1}(\psi_{,\varphi r})\omega^0 \wedge \omega^3 + r^{-1}\psi_{,\theta t}\omega^1 \wedge \omega^2 + (r\sin\theta)^{-1}(\psi_{,\varphi t})\omega^1 \wedge \omega^3 .
\end{aligned}
\tag{A7}
$$

Also, for $G = (2\psi/r)\omega^0$, we have

$$
\begin{aligned}
dG &= 2(\psi r^{-1})_{,r} dr \wedge dt + 2r^{-1}\psi_{,\theta} d\theta \wedge dt + 2r^{-1}\psi_{,\varphi} d\varphi \wedge dt \\
&= -2(\psi r^{-1})_{,r}\omega^0 \wedge \omega^1 - 2r^{-2}\psi_{,\theta}\omega^0 \wedge \omega^2 - 2(r^2\sin\theta)^{-1}(\psi_{,\varphi})\omega^0 \wedge \omega^3 .
\end{aligned}
\tag{A8}
$$

The left-hand side of Eq. (3.7) is obtained by adding Eqs. (A6) and (A7) and is found to be

$$
\begin{aligned}
\Delta P = d\delta P + \delta dP &= [\psi_{,tt} - (r^{-2}(\psi r^2)_{,r})_{,r} - (r^2\sin\theta)^{-1}(\psi_{,\theta}\sin\theta)_{,\theta} - (r^2\sin^2\theta)^{-1}(\psi_{,\varphi\varphi})]\,\omega^0 \wedge \omega^1 \\
&\quad - 2r^{-2}\psi_{,\theta}\omega^0 \wedge \omega^2 - 2(r^2\sin\theta)^{-1}(\psi_{,\varphi})\omega^0 \wedge \omega^3 .
\end{aligned}
\tag{A9}
$$

Equating this to dG from Eq. (A8) gives

$$
\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = 0 \tag{A10}
$$

for the $\omega^0 \wedge \omega^1$ component; the remaining components are identically satisfied and hence put no further condition upon ψ. Equation (A10) is just the flat-space wave equation (2.8); the Schwarzschild equation (6.2) and the Friedmann equation (6.6) are derived in a similar manner.

APPENDIX B

In this appendix the dictionary for translating null-frame notation into the NP notation is presented for use in Sec. V. In addition the Cartan-frame method for computing Ricci rotation coefficients is sketched and is advocated as a simple method of computing the spin coefficients for any reader intending to make applications of Eqs. (5.5) and (5.6) in specific space-times.

The Ricci rotation coefficients of the frame, which appear, for example, in Eq. (4.6) through the use of Eq. (5.1), are denoted in NP notation[5] by the following:

$$
\begin{aligned}
&\gamma_{121} = \epsilon + \bar\epsilon, && \gamma_{141} = \bar\kappa, && \gamma_{241} = -\pi, \\
&\gamma_{122} = \gamma + \bar\gamma, && \gamma_{142} = \bar\tau, && \gamma_{242} = -\nu, \\
&\gamma_{123} = \bar\alpha + \beta, && \gamma_{143} = \bar\rho, && \gamma_{243} = -\mu, \\
&\gamma_{124} = \alpha + \bar\beta, && \gamma_{144} = \bar\sigma, && \gamma_{244} = -\lambda, \\
&\gamma_{131} = \kappa, && \gamma_{231} = -\bar\pi, && \gamma_{341} = \bar\epsilon - \epsilon, \\
&\gamma_{132} = \tau, && \gamma_{232} = -\bar\nu, && \gamma_{342} = \bar\gamma - \gamma, \\
&\gamma_{133} = \sigma, && \gamma_{233} = -\bar\lambda, && \gamma_{343} = \bar\alpha - \beta, \\
&\gamma_{134} = \rho, && \gamma_{234} = -\bar\mu, && \gamma_{344} = \bar\beta - \alpha,
\end{aligned}
\tag{B1}
$$

and are antisymmetric in the first pair of indices.

The intrinsic frame derivatives, which occur in the same equations, are denoted by

$$
\omega_1 = D,\ \ \omega_2 = \Delta,\ \ \omega_3 = \delta,\ \ \omega_4 = \bar\delta. \tag{B2}
$$

The correspondence between the numerical indexing and the labeling by l, n, m, and $\bar m$ of covariant tetrad indices, which have been used here interchangeably, is just $1, 2, 3, 4 \leftrightarrow l, n, m, \bar m$, respectively.

The Cartan procedure for computing Ricci rotation coefficients of a tetrad [hence the spin coefficients via Eq. (B1)] is now presented. It is first necessary to lower coordinate indices on the tetrad vectors to obtain the dual 1-forms $-n_\mu dx^\mu$, $-l_\mu dx^\mu$, $\bar m_\mu dx^\mu$, and $m_\mu dx^\mu$, where μ runs over the coordinate indices. These frame 1-forms we denote by ω^i, $i = 1, 2, 3, 4$, where i runs over frame (tetrad) indices. Then Cartan's first structural equations give

$$
d\omega^i = -\omega^i{}_j \wedge \omega^j , \tag{B3}
$$

where the connection 1-forms $\omega^i{}_j$ may be expanded along the basis 1-forms ω^i to give

$$
\omega^i{}_j = \gamma^i{}_{jk}\,\omega^k, \tag{B4}
$$

and the $\gamma^i{}_{jk}$ are just the Ricci rotation coefficients. Hence Eq. (B3) may be rewritten

$$d\omega^i = -\gamma^i{}_{jk}\,\omega^k \wedge \omega^j\,. \tag{B5}$$

Unfortunately the $\gamma^i{}_{jk}$'s have no special symmetry in the indices j and k, so that they cannot always be found from Eq. (B5) by inspection. This difficulty is solved by introducing auxiliary quantities $C_{kj}{}^i$, which are by definition antisymmetric in j and k, so that they may be found by inspection from

$$d\omega^i = \tfrac{1}{2} C_{jk}{}^i\, \omega^j \wedge \omega^k\,. \tag{B6}$$

What remains is to relate the $\gamma^i{}_{jk}$ to the known quantities $C_{jk}{}^i$. From Eqs. (B5) and (B6) we have

$$(\gamma^i{}_{jk} + \tfrac{1}{2} C_{kj}{}^i)\omega^j \wedge \omega^k = 0\,,$$

which by the antisymmetry of $\omega^j \wedge \omega^k$ gives

$$\gamma^i{}_{jk} + \tfrac{1}{2} C_{kj}{}^i = \gamma^i{}_{kj} + \tfrac{1}{2} C_{jk}{}^i\,.$$

Therefore

$$\gamma^i{}_{jk} - \gamma^i{}_{kj} = C_{jk}{}^i\,. \tag{B7}$$

Lowering the i index [which must be done with the tetrad matrix $\eta_{ij} = \eta^{ij}$ given by Eq. (5.2)] and permuting the indices in Eq. (B7) results in

$$\gamma_{ijk} = -\tfrac{1}{2}(C_{ijk} - C_{kij} - C_{jki})\,, \tag{B8}$$

where use has been made of the antisymmetry properties of the C's $(C_{kji} = -C_{jki})$ and the γ's $(\gamma_{ijk} = -\gamma_{jik})$. Equation (B8) is the desired result and gives the spin coefficients through the inverted form of Eq. (B1), which is

$$\begin{aligned}
&\alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}), \quad \lambda = -\gamma_{244}, \quad \rho = \gamma_{134},\\
&\beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}), \quad \mu = -\gamma_{243}, \quad \sigma = \gamma_{133},\\
&\gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342}), \quad \nu = -\gamma_{242}, \quad \tau = \gamma_{132},\\
&\epsilon = \tfrac{1}{2}(\gamma_{121} - \gamma_{341}), \quad \pi = -\gamma_{241}, \quad \kappa = \gamma_{131}\,.
\end{aligned} \tag{B9}$$

The use of Eqs. (B6), (B8), and (B9) in conjunction is highly recommended as a labor-saving scheme for computing the spin coefficients.

The reader who wishes to apply Eqs. (5.5) and (5.6) to new space-times will also require an understanding of the principal directions of the Weyl tensor (see, e.g., Refs. 7, 47, or 48) in order to align the tetrad vector l; the remaining tetrad elements are chosen to satisfy the quasi-orthonormal conditions $l\cdot n = -1$, $m\cdot \bar{m} = 1$, all other scalar products vanish.

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