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General relativity with spin and torsion and its deviations from Einstein's theory*

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The field equations of general relativity with spin and torsion (U_4 theory) are considered to describe correctly the gravitational properties of matter on a microphysical level. By an averaging procedure one arrives at a macroscopic field equation, which under normal matter densities coincides with Einstein's equation of conventional general relativity. For very high matter densities, even if the spins are randomly distributed, Einstein's equation breaks down and U_4 theory must be applied. It is shown how the singularity theorems of Penrose and Hawking must be modified to apply in U_4 theory. All known cosmological models in U_4 theory which prevent singularities are shown to violate an energy condition of a singularity theorem.

I. U_4 THEORY AND ITS MACROSCOPIC AVERAGE

General relativity with spin and torsion (U_4 theory)[1] is as consistent with experiment as conventional general relativity (GR), because present technology does not suffice to distinguish between the predictions of the two theories. Therefore, U_4 theory must be considered seriously as an alternative. (For reviews see, e.g., Ref. 2.)

In U_4 theory, spacetime is described by a non-Riemannian geometry. The non-Riemannian part of the affine connection, or *torsion tensor* $S_{ij}{}^k \equiv \Gamma^k_{[ij]}$ $(i, j, \ldots = 0, 1, 2, 3;$ square brackets denote antisymmetrization), is linked with the spin angular momentum of matter $\tau_{ij}{}^k$. The field equations of U_4 theory are

$$R_{ij} - \tfrac{1}{2} g_{ij} R_k{}^k = k \Sigma_{ij} , \tag{1}$$

$$S_{ij}{}^k + \delta^k_i S_{jl}{}^l - \delta^k_j S_{il}{}^l = k \tau_{ij}{}^k , \tag{2}$$

where k is the relativistic gravitational constant, δ^j_i the Kronecker delta, g_{ij} the metric tensor with signature $(+, -, -, -)$, $R_{ij} = R_{kij}{}^k$, and $R_{ijk}{}^l$ the curvature tensor of the Riemann-Cartan connection

$$\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + S_{ij}{}^k - S_j{}^k{}_i + S^k{}_{ij} , \tag{3}$$

and $\left\{ {k \atop ij} \right\}$ is the Christoffel symbol of the metric. Σ^{ij} and $\tau_{ij}{}^k$ are the canonical energy-momentum and spin angular momentum tensors of matter, respectively.[3]

If one substitutes (2) in (1), after some computation one arrives at the combined field equation[1e] which has a pseudo-Einsteinian form:

$$R^{ij}(\{\ \}) - \tfrac{1}{2} g^{ij} R_k{}^k(\{\ \}) = k \tilde{\sigma}^{ij} , \tag{4}$$

$$\begin{aligned} \tilde{\sigma}^{ij} \equiv \sigma^{ij} + k\big[&-4 \tau^{ik}{}_{[l} \tau^{jl}{}_{k]} - 2 \tau^{ikl} \tau^j{}_{kl} + \tau^{kli} \tau_{kl}{}^j \\ &+ \tfrac{1}{2} g^{ij} (4 \tau_m{}^k{}_{[l} \tau^{ml}{}_{k]} + \tau^{mkl} \tau_{mkl})\big] , \end{aligned} \tag{5}$$

where $\{\ \}$ means that the quantities have been computed from the Riemannian part, $\left\{ {k \atop ij} \right\}$, of the affine connection and are the same as in general relativity. The *combined energy-momentum* tensor $\tilde{\sigma}^{ij}$ on the right-hand side, however, contains spin correction terms implicitly in σ^{ij} and explicitly in the bracket; these terms are not present in GR. σ^{ij} is the metric (and symmetric) energy-momentum tensor defined according to Hilbert's variational prescription familiar from ordinary general relativity.

There are several alternative ways[1e,4-6] of splitting up $\tilde{\sigma}^{ij}$ in (4). For example, it is possible to introduce on the right-hand side of (4) the *canonical* instead of the *metric* energy-momentum tensor according to the prescription

$$\sigma^{ij} = \Sigma^{ij} - \nabla^*_k (\tau^{ijk} - \tau^{jki} + \tau^{kij}) . \tag{6}$$

Here $\nabla^*_k = \nabla_k + 2 S_{kl}{}^l$, where ∇_k is the covariant derivative with respect to the affine connection Γ^i_{kl}.

It is crucial to note that *spin* in U_4 theory is canonical spin, that is, the *intrinsic* spin of elementary particles, not the so-called spin of galaxies or planets. (Kopczyński[7] takes a different point of view.)

In the microscopic domain of matter, (4) and (5) should be valid. But the quantities σ^{ij} and $\tau_{ij}{}^k$ are microscopically fluctuating. Therefore, in order to obtain an equation for bulk matter, one should compute a spacetime average of (4) and (5), just as one does in deriving the macroscopic Maxwell equations. This means that an "infinitesimal" volume element must contain a large number of atoms or elementary particles. Usually, the average of the spin and the spin gradient will vanish, but the same will *not* be true for the spin-squared terms in (5). [For example, the last term in (5), which is locally isotropic in a geodesic coordinate frame with respect to the Christoffel symbols $\left\{ {k \atop ij} \right\}$ does not vanish.] Thus, after averaging, one obtains a result like

$$\langle \tilde{\sigma}^{ij} \rangle = \langle \sigma^{ij} \rangle + k \langle \tfrac{1}{2} g^{ij} \tau^{mkl} \tau_{mkl} \rangle \\ + \text{spin terms of the same order of magnitude.} \tag{7}$$

That is, *even for macroscopically vanishing spin,* we do not recover exactly Einstein's field equation. Rather, we get an energy-momentum tensor corrected by spin-squared terms which are negligible at normal matter densities (see Sec. II).

We claim that the field equations (1) and (2) or the combined field equation (4) are, at a classical level, the correct microscopic gravitational field equations. Einstein's field equation ought to be considered a macroscopic phenomenological equation of limited validity, obtained by averaging Eq. (4). Thus, we would propose that U_4 theory is a more natural starting point for a quantization program. The final word on U_4 theory must come from experiment, of course.

II. CRITICAL MASS DENSITY AND SIMPLE COSMOLOGICAL MODELS

We would like to get some feeling for the magnitude of the effects involved in macrophysics. We use a semiclassical model of a spin fluid.[8] Let the momentum density of the fluid be p_i, its spin density be s_{ij}, and its pressure be Π, and let its elements be moving with 4-velocity u^k. We assume that the momentum density and the spin density are *transported* with the velocity u^k. We obtain the energy-momentum tensor and the spin tensor by going over from a momentary rest system of a fluid element to an arbitrary moving system:

$$\Sigma_i{}^j = (p_i c + \Pi u_i) u^j - \delta_i^j \Pi \,, \tag{8}$$

$$\tau_{ij}{}^k = s_{ij} u^k \,, \tag{9a}$$

$$s_{ij} u^j = 0 \,. \tag{9b}$$

For convenience, we drop the averaging signs on $\langle \Sigma_i{}^j \rangle$ and $\langle \tau_{ij}{}^k \rangle$, but bear in mind that all equations which follow are macroscopic ones.

The identification of the convective energy-momentum and spin angular momentum tensors on the right-hand sides of (8) and (9a) with the corresponding canonical ones is by no means trivial. It is suggested by the fact that their respective conservation theorems look alike, but a final proof can only be given by establishing a Lagrangian formulation for a semiclassical spin fluid.

If we define the rest mass density $\rho \equiv p_k u^k$ and the square of the spin $s^2 \equiv 2 s_{ij} s^{ij}$, we get for the combined energy-momentum tensor, from (5), (6), (8), (9), and angular momentum conservation,[5,6]

$$\tilde{\sigma}^{ij} = (\rho c^2 + \Pi - \tfrac{1}{2} k c^2 s^2) u^i u^j - (\Pi - \tfrac{1}{4} k c^2 s^2) g^{ij} \\ - 2c (u_k u^l + \delta_k^l) \overset{\{\}}{\nabla}_l (s^{k(i} u^{j)}) \,, \tag{10}$$

where $\overset{\{\}}{\nabla}$ is the covariant derivative corresponding to the Christoffel connection. The combined rest energy density producing the metric field then turns out to be

$$\tilde{\sigma}^{ij} u_i u_j = \rho c^2 - \tfrac{1}{4} k c^2 s^2 + 2 c s^{ij} \overset{\{\}}{\nabla}_{[i} u_{j]} \,. \tag{11}$$

Note that $\overset{\{\}}{\nabla}_{[i} u_{j]}$, the curl of the velocity, is an exterior derivative, independent of the connection. The last term may also be written in the form $2 c u_i \overset{\{\}}{\nabla}_k s^{ik}$.

As an example let us suppose the spin fluid consists of neutrons with mass m and spin $\frac{1}{2}\hbar$. As noted above, we must imagine that the "infinitesimal" volume element of the fluid already contains many neutrons. The particle number density is

$$n = \frac{\rho}{m} = (s^2/\hbar^2)^{1/2} \,. \tag{12}$$

Observe that (12) is valid whether or not the spins of the neutrons are aligned. In the rest system of a fluid element, with the help of (11) and (12), we get the estimate

$$\tilde{\sigma}^{00} = \rho c^2 [1 - (\rho/\bar{\rho})] + \vec{s} \cdot \operatorname{curl} \vec{v} \,, \tag{13a}$$

where

$$\bar{\rho} = \frac{m^2}{k \hbar^2} \approx 10^{54}\ \mathrm{g\,cm^{-3}} \,, \tag{13b}$$

$\vec{s}$ is the spin 3-vector, and $\vec{v}$ is the fluid 3-velocity. The last term in (13a) vanishes for a spin fluid without vorticity and also vanishes wherever the spin-density fluctuates over a shorter characteristic length than the local fluid vorticity. This term may dominate in cosmological models with both aligned spins and vorticity.

Depending on the equation of state $\Pi = \Pi(\rho)$, Eq. (13a) tells us that beyond the huge matter density $\bar{\rho}$ gravitational behavior of macroscopic matter is heavily influenced by spin terms,[4] if general relativity works at all under such hypothetical circumstances. This is consistent with a result of Kerlick,[9] who has shown that for neutron stars (where $\rho \ll \bar{\rho}$) the effects of torsion are totally negligible.

Extremely high matter densities may have been present in the early stages of the universe. For this reason, Kopczyński,[10,7] Trautman,[11] Stewart and Hájíček,[12] Tafel,[13] von der Heyde,[5] and Kerlick[14] have worked out different cosmological models incorporating torsion. These models did not develop singularities when the spins of matter were assumed to be aligned and pressure was neglected. According to our arguments in Sec. I above, it is not necessary to assume alignment of the spins, since the spin-square terms which prevent the singularity do not vanish even for randomly directed spins. Models with pressure[7] are of no particular interest if we assume the usual equation of state for collapsing nuclear matter, because the pressure becomes negligible at the high temperatures near the big bang. The nonsingular models without pressure all have a maximum matter density of the order of $\bar{\rho}$ estimated in (13).

Why is it possible to prevent singularities from occurring in U_4 theory? We would like to have a general criterion rather than to rely on specific cosmological models.[15]

III. VIOLATION OF AN ENERGY CONDITION IN NONSINGULAR COSMOLOGIES WITH TORSION

The singularity theorems of Penrose and Hawking[16] show that under very general assumptions singularities cannot be prevented in general relativity. These theorems can be extended to U_4 theory very easily, as the following discussion will show.

It is convenient to consider three classes of curves in a U_4 manifold:

Autoparallels $\mathcal{A}$ (straightest lines) are curves along which the tangent vector to the curve is transported parallel to itself, under the transport law associated with the connection Γ_{ij}^{k}.

Geodesics $\mathcal{G}$ (shortest or longest lines) are curves of extremal length according to the metric tensor g_{ij}. They are also curves whose tangent vector is transported parallelly according to the transport law of the Christoffel connection $\{{}^{k}_{ij}\}$.

Trajectories $\mathcal{T}$ are the paths of particles with or without spin, and are in general neither $\mathcal{A}$ nor $\mathcal{G}$ but must be derived from the field equations or conservation laws.

Spinless massive particles travel along timelike $\mathcal{G}$. This can be derived from the conservation laws in U_4 theory. Photons travel along null $\mathcal{G}$, since Maxwell's vacuum field equations are the same as in general relativity. This means that neither spinless particles nor photons feel or produce torsion. The causal structure of a U_4 manifold is the same as a Riemannian one. Thus, timelike or null *geodesic* incompleteness is as valid a criterion for singularities in manifolds with torsion as in manifolds which are torsion-free.

Now, the pseudo-Einsteinian form of Eq. (4) allows us to generalize the singularity theorems by substituting the combined energy-momentum $\tilde{\sigma}^{ij}$ for the canonical energy-momentum tensor of general relativity. That is

$$(\tilde{\sigma}^{ij} - \tfrac{1}{2}g^{ij}\tilde{\sigma}_{k}{}^{k})\xi_i \xi_j \geq 0 \tag{14}$$

for all timelike vectors ξ^i. Thus, U_4 theory introduces a different energy-momentum tensor of matter into the singularity theorems (see also Ref. 17).

The left-hand side of (14), using the velocity vector u^i, can be calculated easily from (11) and the trace of (10). It turns out to be

$$(\tilde{\sigma}^{ij} - \tfrac{1}{2}g^{ij}\tilde{\sigma}_{k}{}^{k})u_i u_j = \tfrac{1}{2}\rho c^2 + \tfrac{3}{2}\Pi - \tfrac{1}{2}kc^2 s^2 + 2cs^{ij}\overset{\{\}}{\nabla}_{[i}u_{j]} . \tag{15}$$

The cited cosmological models which prevent singularities have all been constructed from the matter tensors (8) and (9) or specializations therefrom. Since (15) is a consequence of (8) and (9), it applies to all these models. Furthermore, in the models in question, the last term of (15) vanishes, and the spin squared and the mass density depend upon the age of the universe. As soon as the spin density reaches a value such that

$$\rho c^2 + 3\Pi(\rho) < kc^2 s^2 , \tag{16}$$

(15) becomes negative. For the spin fluid of these cosmological models, spin is proportional to the matter density in the same way as in (12), basically as a consequence of the angular momentum theorem.[5] Thus, (15) becomes negative at the critical density $\bar{\rho}$, when we neglect the pressure.

Consequently, we are able to understand the possible prevention of singularities in U_4 theory from a unified point of view. Let us collect these results in the following proposition.

Proposition. The singularity theorem of Hawking and Penrose in Ref. 16 (p. 266) applies to U_4 theory upon the substitution

$$\underset{\text{(energy-momentum tensor of GR)}}{\sigma_{ij}(\{\,\})} \quad \rightarrow \quad \underset{\text{(combined energy-momentum tensor of } U_4 \text{ theory).}}{\tilde{\sigma}_{ij}} \tag{17}$$

If the quantity $(\tilde{\sigma}^{ij} - \frac{1}{2}g^{ij}\tilde{\sigma}_k{}^k)\xi_i\xi_j$ becomes negative for any timelike unit vector ξ^i, the energy condition is violated and a singularity may be prevented.

The question of the singularity behavior of cosmological models in U_4 theory has now changed from geometrical reasoning to a question about the behavior of matter, and in particular its combined energy-momentum tensor at very high densities. It could well be that in the models above, singularities are prevented because a semiclassical description of matter is used which is not appropriate under those circumstances.

Given the combined energy-momentum tensor of matter, the proposition above will tell us whether singularities may be prevented in U_4 theory. Thus, future investigations will have to concentrate on finding the combined energy-momentum tensor of matter near the big bang.

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