as $|r^*|\to\infty$. If $R_\pm$ are unstable solutions we have from (18)

$$-\omega^2 R_\pm - \frac{\partial^2 R_\pm}{\partial r^{*2}} + U_\pm R_\pm = 0\,. \qquad (21)$$

Therefore,

$$\omega^2\int_{-\infty}^{\infty} dr^* |R_\pm|^2 = -\int_{-\infty}^{\infty} dr^*\left(R_\pm^* \frac{\partial^2 R_\pm}{\partial r^{*2}} - U_\pm |R_\pm|^2\right)$$
$$= \int_{-\infty}^{\infty} dr^*\left(\left|\frac{\partial R_\pm}{\partial r^*}\right|^2 + U_\pm |R_\pm|^2\right), \qquad (22)$$

since, by the exponential decay of $R_\pm$ for large $|r^*|$, no boundary terms survive in the integration by parts. From (22) it follows that ω^2 is real and therefore (by the instability assumption) that ω is purely imaginary. However, $\omega^2<0$ is clearly impossible if U_+ and U_- are non-negative functions of r in the range $r_+ \le r \le \infty$. By straightforward algebra one can show that U_+ and U_- are indeed non-negative [on (r_+,∞)] for each value of $L \ge 2$ and for all e and m such that $|e| \le m$. Consequently the assumption of unstable normal-mode solutions obeying the specified boundary conditions leads to a contradiction.

A similar analysis can be given for the $L=1$ modes in which only electromagnetic radiation can occur. The $L=0$ perturbations are spherically symmetric and thus are tangent to the Reissner-Nordström family of solutions. They merely allow for small changes of the charge and mass parameters.

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Electromagnetic scattering from a black hole and the glory effect*

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The scattering of electromagnetic radiation by a black hole is discussed and further results on this problem are presented. It is shown that the backward glory effect is absent in the Schwarzschild field as well as in the Kerr case when plane electromagnetic waves are incident along the axis of symmetry of the field. A cosmological distribution of Kerr black holes could result in the polarization of the cosmic background radiation for which a crude estimate is given.

I. INTRODUCTION

Evidence for the existence of a black hole may be obtained through the detection of electromagnetic radiation that is scattered from it. Though optical means are not very promising at present, the scattering problem is of interest since a collapsed object that is not accreting fresh matter scatters radiation merely by its gravitational field. This problem has been partially analyzed in a previous publication.[1] The purpose of this paper is to extend that analysis, to show that the backward glory effect is absent in the Schwarzschild field, and to give an estimate of the effect of a cosmological distribution of black holes on the polarization of the background radiation.

It has been shown[2] that the electromagnetic field equations can be cast into the form of Maxwell's equations in flat spacetime but in a "medium" with dielectric and permeability tensors[3] (ϵ_{ij}) and (μ_{ij}), and a gravitational vector potential $\vec{G}$, where

$$\epsilon_{ij} = \mu_{ij}$$
$$= -(-g)^{1/2}\frac{g^{ij}}{g_{00}}, \quad (1)$$

$$G_i = -\frac{g_{0i}}{g_{00}}. \quad (2)$$

The constitutive relations of the medium are then

$$D_i = \epsilon_{ij} E_j - (\vec{G}\times\vec{H})_i , \quad (3)$$

$$B_i = \mu_{ij} H_j + (\vec{G}\times\vec{E})_i . \quad (4)$$

The Maxwell's equations can then be written in the form

$$\frac{1}{i}\vec{\nabla}\times\vec{F} = \frac{\partial}{\partial t}\vec{S} + 4\pi\vec{J}, \quad (5)$$

$$\vec{\nabla}\cdot\vec{S} = 4\pi\rho , \quad (6)$$

where $\vec{F}$ is the Kramers vector $\vec{F} = \vec{E} + i\vec{H}$ and $\vec{S}$ is given by

$$S_i = \epsilon_{ij} F_j + i(\vec{G}\times\vec{F})_i . \quad (7)$$

The conformal invariance of Maxwell's equations is made manifest in this formulation. Indeed, it can easily be seen[1] that experiments using electromagnetic waves can determine the metric only up to a conformal factor.

This method of treating electromagnetic phenomena is of interest since, besides its explicit conformal invariance, methods familiar from the flat-spacetime theory can be easily adapted to the curved spacetime of Einstein's theory. Hence it can be shown[2] that no double refraction can occur in a gravitational field. The problem of the dispersion of electromagnetic waves in a gravitational field becomes tractable as well.[1] Another result on the glory effect will be given in the present paper. The spherical symmetry of the Schwarzschild field and the axial symmetry of the Kerr field lead to the conclusion that the backward glory effect is absent in the Schwarzschild case and also for electromagnetic waves incident on a Kerr black hole along its axis of rotation.

II. SCHWARZSCHILD BLACK HOLE

Let A be the scattering matrix for plane electromagnetic waves in the field of an uncharged black hole. It has been shown[1] that this matrix is diagonal in the circular polarization basis. That is, as referred to local polarization bases, an incident right circularly polarized wave is scattered into an outgoing right circularly polarized wave, and similarly for left circularly polarized radiation. A_{RR} and A_{LL} are the corresponding amplitudes for these processes. For a Schwarzschild black hole

$$A_{RR} = A_{LL} = \frac{1}{4ik}(1+\cos\theta)Q(\mu k, \cos\theta), \quad (8)$$

$$Q = \sum_{J\geq 1}(2J+1)[R_J^{1/2}\exp(2i\delta_J) - 1]Q_J(\cos\theta). \quad (9)$$

The differential scattering cross section is then

$$\frac{d\sigma_{\text{scatt}}}{d\Omega} = \frac{1}{16k^2}(1+\cos\theta)^2|Q|^2, \quad (10)$$

and the absorption cross section is given by

$$\sigma_{\text{abs}} = \pi k^{-2}\sum_{J\geq 1}(2J+1)(1-R_J). \quad (11)$$

In these formulas k is the wave number, μ is the mass of the black hole, and θ is the angle between the incident and outgoing propagation vectors. $Q_J(z)$, $J = 1, 2, \ldots,$ are polynomials of degree $J-1$ and are completely determined by the requirements that $Q_J(1) = 1$ and

$$\int_{-1}^{1}(1+z)^2 Q_J(z)Q_{J'}(z)dz = \frac{8}{2J+1}\delta_{JJ'}. \quad (12)$$

R_J and δ_J are the reflection coefficient and the phase shift for a spherical wave of angular momentum J, respectively. Approximate formulas can be deduced for these functions[1,4] but exact expressions are not available. Thus their dependence on the frequency of the wave has been determined numerically. Figure 1 gives R_J as a function of μk for $J = 1, 2$. Similar graphs for δ_J are presented in Fig. 2. More details on R_J and δ_J are given in the Appendix. It can easily be seen that the polarization properties of an incident plane wave are unaffected by passing through the gravitational field of the Schwarzschild black hole.

It follows from the Appendix that both the forward differential scattering cross section and σ_{scatt} are divergent due to the long-range gravitational interaction between the wave and the black hole. A well-known property of a black body is that for

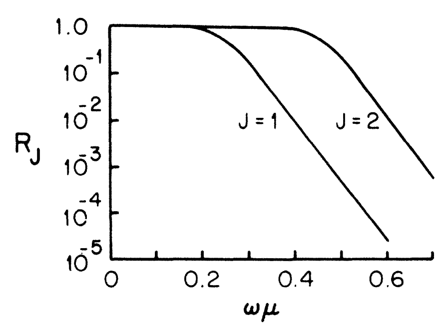

FIG. 1. Plot of the reflection coefficient R_J, $J = 1, 2$, for the scattering of the 2^J-pole radiation from a Schwarzschild black hole vs the product of the mass of the black hole and the frequency of radiation.

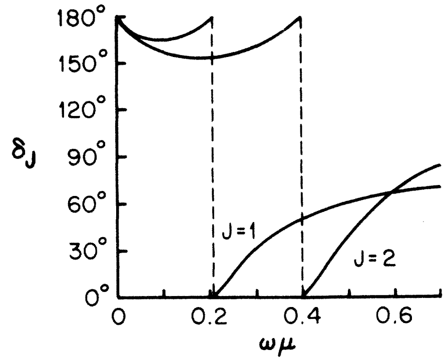

FIG. 2. Plot of the phase shift δ_J, $J=1,2$, for the scattering of the 2^J-pole radiation from a Schwarzschild black hole vs the product of the mass of the black hole and the frequency of radiation.

$k\to\infty$, $\sigma_{\text{scatt}}\to\sigma_{\text{abs}}$. This is not the case for a black hole, however, since for $k\mu\gg 1$, $\sigma_{\text{abs}}=27\pi\mu^2$.

The analysis of null geodesics in the Schwarzschild field has revealed the possible existence of the glory effect in black-hole physics.[5] That is, when electromagnetic waves are incident on a Schwarzschild black hole, the (classical) differential scattering cross section is infinite in the backward direction. In this approach the backscattered rays can be thought of as coming from "rings of brightness" located at[5] $b/\mu - 3^{3/2}=0.151$, $0.00028,\ldots$, where b is the impact parameter. These correspond to rays that take a $\pi, 3\pi, \ldots$ loop around a black hole and return to the source. Each of the branches makes two equal contributions to the classical differential cross section when θ is very close to π. These contributions interfere, however, in the wave picture.[6] It can be shown that the backscattered waves should interfere destructively so that there is no backward glory effect. In fact $d\sigma_{\text{scatt}}/d\Omega=0$ for $\theta=\pi$ according to the wave picture. This can be seen using a theorem[7] in classical electromagnetic theory which states that if the scatterer is rotationally symmetric about the axis of incidence of the plane wave, then the diagonal elements of the scattering matrix (in the circular polarization basis) are zero in the backward direction and the off-diagonal elements are zero in the forward direction. The result can be applied to the scattering by a black hole since the gravitational field can be replaced by a scattering (and absorbing) "medium" in flat spacetime. This conclusion can easily be drawn from the photon picture as well since the rotational symmetry of the scatterer about the axis of incidence implies the conservation of angular momentum about this axis. Thus, the helicity of the backscattered photon should be opposite to that of the incident circularly polarized photon. This contradicts the fact that helicity is conserved in black-hole scattering. Hence the amplitude for backscattering is zero.

III. KERR BLACK HOLE

The scattering amplitude matrix is diagonal in the Kerr case, and thus when plane waves are incident along the rotation axis of the black hole the differential cross section is zero in the backward direction.[8] The diagonal elements are not equal in general and this leads to the result that the polarization properties of the incident wave are affected by the black hole.[1] For an unpolarized incident plane wave, the final polarization is given by

$$p=\left|\frac{|A_{RR}|^2-|A_{LL}|^2}{|A_{RR}|^2+|A_{LL}|^2}\right|, \tag{13}$$

which is proportional to a/μ, where $J_{\text{BH}}=\mu a$ is the angular momentum of the black hole. When $a\ll\mu$, Eq. (5) for waves of frequency $\omega=k$ can be written to first order in a/μ in the Kerr field as

$$\left(\frac{1}{i}\vec{\nabla}-\omega\vec{G}\right)\times\vec{F}=-i\omega n\vec{F}, \tag{14}$$

where n and $\vec{G}$ are given in Schwarzschild isotropic coordinates by

$$n(r)=(r+\tfrac{1}{2}\mu)^3/r^2(r-\tfrac{1}{2}\mu), \tag{15}$$

$$\vec{G}(\vec{r})=2\frac{\vec{J}_{\text{BH}}\times\vec{r}}{r(r-\frac{1}{2}\mu)^2}. \tag{16}$$

Here $n>1$ has the interpretation of the index of refraction for the Schwarzschild field. It diverges at the stationary limit and $n\to 1$ as $r\to\infty$. The vector potential also diverges at the stationary limit but for $r\gg\mu$ it assumes the familiar form

$$\vec{G}\simeq 2\frac{\vec{J}_{\text{BH}}\times\vec{r}}{r^3}.$$

Thus, to first order in a/μ, the photon interacts with a "gravitational magnetic field" which is produced by the "gravitational magnetic dipole moment" (due to the "mass current") of the rotating black hole. It follows that photons of opposite initial helicities are scattered differently in the Kerr field. This is to be compared with the fact that there is a spin-spin part to the interaction of a test particle of internal angular momentum $\vec{J}$ with a black hole which to lowest order in r/μ is of the tensor type[9,10]

$$r^{-5}[3(\vec{J}_{\text{BH}}\cdot\vec{r})(\vec{J}\cdot\vec{r})-r^2\vec{J}_{\text{BH}}\cdot\vec{J}]. \tag{17}$$

The polarizing property of a Kerr black hole is probably maintained for very low frequencies $\omega\mu\ll 1$.

It is of interest to consider the possible influence of black holes on the polarization properties of the microwave background radiation. After the recombination era when the radiation decouples from matter, its polarization properties could be affected by the gravitational field of black holes. Let us assume that initially unpolarized radiation undergoes single scattering by Kerr black holes and is then incident on a polarimeter. Let $A_{RR}(\hat{k}_f, \hat{k}_i)$ be the amplitude for scattering of a right circularly polarized plane wave incident on a Kerr black hole in the $\hat{k}_i(\theta_i, \phi_i)$ direction and received in the $\hat{k}_f(\theta_f, \phi_f)$ direction by the polarimeter. The coordinate system is chosen such that θ is measured from the rotation axis of the black hole. A similar definition holds for $A_{LL}(\hat{k}_f, \hat{k}_i)$. For a given $\hat{k}_f$, the degree of circular polarization of the observed radiation is given by[11]

$$P(\omega_0, \hat{k}_f) = \tfrac{1}{2}\int_{\tau_R}^{\tau_0}\int p(|A_{RR}|^2 + |A_{LL}|^2)N(\tau, \hat{k}_f)d\tau\, d\Omega_i\,, \tag{18}$$

where p is given by (13) and N is the density of black holes which are assumed to be identical and rotating in the same direction. The integrations in (18) are to be carried out over the incident direction and the cosmic time from the recombination era to the present. The scattering amplitudes (and hence p) in (18) are functions of the frequency $\omega = [1+z(\tau)]\omega_0$, where z is the red shift and ω_0 the observed frequency of the radiation.

A crude estimate of the importance of P for black holes of mass μ can be obtained by considering $\delta \equiv 4\pi\mu^2 \int_{\tau_R}^{\tau_0} N d\tau$ since for $\omega\mu \gg 1$ the scattering cross section is proportional to μ^2. For a uniform spatial distribution of black holes of density close to the present critical density and using the cosmologically flat Einstein–de Sitter model we find that δ is given by

$$\delta \simeq \mu H_0[(1+z_R)^{3/2} - 1]. \tag{19}$$

Thus δ increases linearly with μ and for $z_R \sim 10^3$ and $\mu \sim 10^8 M_\odot - 10^{11} M_\odot$ we get $\delta \simeq 0.5\times10^{-10}$–$0.5\times10^{-7}$. The smallness of δ indicates that the effect of black holes on the polarization of the background radiation is probably negligible, though it is difficult at present to draw definite conclusions about the effect of a cosmological distribution of collapsed objects. Thus, the possible existence of this polarizing (or depolarizing, if the radiation is initially polarized) effect should be kept in mind.

There are other processes which could lead to a polarization of the background radiation. Especially attractive is the suggestion by Rees[12] that anisotropic expansion of the universe before the radiation was last scattered would lead to temperature anisotropy which on repeated Thomson scattering could result in the linear polarization of the primeval radiation. The present experimental upper limit on the degree of linear polarization of the background radiation at a wavelength of 3.2 cm is[13] $\sim 6\times10^{-4}$. It should be borne in mind, however, that the galaxy probably makes a contribution to this measurement of polarization.[13]

In the simple Rees model the polarization of the Rayleigh-Jeans part of the background radiation spectrum is independent of frequency. This provides a test for the model. It is therefore of interest to have experimental determination of the dispersion of polarization of the background radiation.

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APPENDIX

The reflection coefficient R_J and the phase shift δ_J are defined for the scattering of the 2^J-pole radiation from a Schwarzschild black hole. It can be shown[1] that the scattering problem is reduced to the one-dimensional equation

$$\frac{d^2\psi}{dx^2} + (k^2 - U)\psi = 0\,, \tag{A1}$$

with

$$U = \frac{J(J+1)}{r^2}\left(1 - \frac{2\mu}{r}\right)$$

and

$$x = r + 2\mu \ln\left(\frac{r}{2\mu} - 1\right) + c\,,$$

where c is a constant. The scattering amplitude is then defined to be $-(-1)^J R_J^{1/2}\exp(2i\delta_J)$ for $c=0$. The presence of $c \neq 0$ has no measurable effect on the cross section for plane electromagnetic waves.[14] If $x \to x + c$, $\delta_J \to \delta_J + kc$ and R_J does not change. It is simple to prove that if $z \neq -1$,

$$(1+z)\sum_{J\geq 1}(2J+1)Q_J(z) = 8\,\delta(1-z)\,. \tag{A2}$$

Therefore, for $0 < \theta < \pi$,

$$(1+\cos\theta)Q = (1+\cos\theta)\sum_{J\geq 1}(2J+1)R_J^{1/2}\exp(2i\delta_J)Q_J(\cos\theta)\,; \tag{A3}$$

hence $x \to x + c$ implies that $(1 + \cos\theta)Q$ is only multiplied by a phase factor $\exp(2ikc)$ and so the differential scattering cross section is independent of c.

It is of interest to investigate the behavior of δ_J in the classical limit for $J \gg k\mu \gg 1$. Let $\phi = (1 - 2\mu/r)^{1/2}\psi$; then

$$\frac{d^2\phi}{dr^2} + \left\{ \left(k^2 + \frac{\mu^2}{r^4} \right)\left(1 - \frac{2\mu}{r} \right)^{-2} + \frac{1}{r^2}\left[\frac{2\mu}{r} - J(J+1) \right]\left(1 - \frac{2\mu}{r} \right)^{-1} \right\}\phi = 0, \quad \text{(A4)}$$

and for a beam of photons ($r \gg \mu$) this equation reduces to

$$\frac{d^2\phi}{dr^2} + \left[k^2 + \frac{4\mu k^2}{r} - \frac{J(J+1)}{r^2} + O\left(\frac{1}{r^3} \right) \right]\phi = 0. \quad \text{(A5)}$$

This is similar to the radial wave equation for the Coulomb problem.[14] Hence the phase shift for $J \gg k\mu \gg 1$ is given by

$$\delta_J \simeq \eta_J + 2k\mu \ln(4k\mu). \quad \text{(A6)}$$

where the Coulomb phase shift $\eta_J \simeq -2k\mu \ln(J + \frac{1}{2})$ in this case. It is now simple to derive the Rutherford formula for the classical small-angle scattering. It is easily seen[15] that most of the contribution to the sum in (9) comes from the values of J near J_0, $J_0 + \frac{1}{2} = 4k\mu/\theta$. Then the scattering amplitude turns out to be

$$-(4\mu/\theta^2)\exp(is),$$

with

$$s = 4k\mu(1 + \ln\theta).$$

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