Stability of Reissner-Nordström black holes*

Vincent Moncrief
Department of Mathematics, University of California, Berkeley, California 94720
(Received 2 May 1974)

Extending some previous work, we here consider the even-parity perturbations of the Reissner-Nordström family of black holes. Gauge-invariant functions of the gravitational and electromagnetic perturbations are defined in terms of the standard Regge-Wheeler expansion functions, and the coupled wave equations obeyed by these quantities are given. The wave equations are decoupled by a simple transformation, and the decoupled equations are shown to admit no unstable normal-mode solutions obeying the appropriate boundary conditions.

I. INTRODUCTION

The results of a recent paper[1] concerning the stability of the Reissner-Nordström family of black holes against odd-parity perturbations are here extended to the even-parity perturbations. Specifically, we show that unstable normal-mode solutions of the wave equations for the gauge-invariant perturbation functions (defined below) do not exist. We have derived this result from a Hamiltonian analysis of the perturbation equations which is completely analogous to that described in Ref. 1. Here, however, only the main results are presented. The details of the derivation will be left for a subsequent paper.

Our main technique has been first to derive a variational integral for the coupled gravitational and electromagnetic perturbations by taking the second variation of the appropriate, exact variational integral. This method follows closely the analysis given by Taub[2] except that we have worked with the Hamiltonian rather than the Lagrangian form of the variational integrals. We have expanded the perturbation functions in the Regge-Wheeler[3,4] tensor harmonics and then performed a canonical transformation from the Regge-Wheeler variables to a new set which is more naturally adapted to the gauge symmetry of the perturbed Einstein-Maxwell equations. Of the new variables two conjugate pairs are simultaneously invariant under electromagnetic and coordinate gauge transformations. All of the remaining canonical variables are either gauge-dependent or vanish by virtue of the initial-value constraints. The stability argument consists primarily of showing that the Hamiltonian for the perturbations is a positive-definite and conserved function of the gauge-invariant perturbation variables.

To simplify the statement of our results we shall here confine our attention to the gauge-invariant functions and the coupled wave equations which they obey. These equations are decoupled by a simple transformation and the stability result is rederived by a direct analysis of the decoupled equations. We here consider only modes with harmonic index $L \geqslant 2$. These modes admit both electromagnetic and gravitational radiation.

Other studies of the Reissner-Nordström perturbations have recently been made by Zerilli,[5] by Chitre, Price, and Sandberg,[6] and by Johnston, Ruffini, and Zerilli.[7]

II. PERTURBATION ANALYSIS

The unperturbed metric is

$$ds^2 = -N^2 dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (1)$$

where

$$N^2 = e^{-2\lambda} = 1 - (2m/r) + e^2/r^2 \qquad (2)$$

and $|e| \leqslant m$. We consider perturbations exterior to the event horizon at $r = r_+ = m + (m^2 - e^2)^{1/2}$. Our notation for the perturbation functions is the same here as in Ref. 1. The perturbed three-metric h_{ij} and the perturbed lapse and shift functions N' and N_i' are expanded in even-parity Regge-Wheeler harmonics.[3,4] This expansion introduces the functions h_1, H_2, K, and G (for h_{ij}) and H_0, H_1, and h_0 (for N' and N_i'). A similar expansion is introduced for the momenta p^{ij} conjugate to h_{ij}. The perturbed electromagnetic variables A_μ' and $\mathcal{E}'^i$ are expanded as

$$\begin{aligned} A_0' &= a_0(r,t) Y_{LM}, \\ A_i' &= \left((a_1 + a_{2,r}) Y_{LM},\ a_2 \frac{\partial Y_{LM}}{\partial\theta},\ a_2 \frac{\partial Y_{LM}}{\partial\varphi} \right), \\ \mathcal{E}'^i &= \left(f_1 \sin\theta\ Y_{LM},\ \frac{f_2 + f_{1,r}}{L(L+1)} \sin\theta \frac{\partial Y_{LM}}{\partial\theta},\right. \\ &\qquad \left. \frac{f_2 + f_{1,r}}{\sin\theta L(L+1)} \frac{\partial Y_{LM}}{\partial\varphi} \right). \end{aligned} \qquad (3)$$

The parametrization has been chosen so that $(a_1(r,t), f_1(r,t))$ and $(a_2(r,t), f_2(r,t))$ are canonically conjugate pairs and so that the electromagnetic constraint becomes

$$\mathcal{E}'^{i}{}_{,i}=-f_2\sin\theta Y_{LM}=0\,. \quad (4)$$

In adapting the choice of canonical variables to the coordinate gauge transformations we are led to introduce the functions

$$q_1=4re^{-4\lambda}k_2+L(L+1)rk_1 \quad (5)$$

and

$$F=f_1+eL(L+1)G\,, \quad (6)$$

where

$$\begin{aligned} k_1&=K+re^{-2\lambda}G_{,r}-(2/r)e^{-2\lambda}h_1\,,\\ k_2&=\tfrac{1}{2}e^{2\lambda}[H_2-(1+r\lambda_{,r})K-rK_{,r}]\,. \end{aligned} \quad (7)$$

The functions q_1 and F are gauge-invariant as are their conjugate momenta. The complete canonical transformation and the derivation of Hamilton's equations will be given elsewhere. Hamilton's equations for q_1, F and their momenta may be combined to yield a pair of coupled second-order equations. In terms of

$$Q=(q_1/\Lambda)(L-1)^{1/2}(L+2)^{1/2} \quad (8)$$

and

$$H=F-(2e/r)(q_1/\Lambda)\,, \quad (9)$$

where

$$\Lambda=(L-1)(L+2)+(6m/r)-(4e^2/r^2)\,, \quad (10)$$

we have obtained

$$\Box\begin{pmatrix}H\\Q\end{pmatrix}+Ne^{-\lambda}V\begin{pmatrix}H\\Q\end{pmatrix}=-Ne^{-\lambda}ST\begin{pmatrix}H\\Q\end{pmatrix}, \quad (11)$$

where

$$\Box=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^{*2}}\,,$$

$$T=\begin{pmatrix}+3m & -2e(L-1)^{1/2}(L+2)^{1/2}\\ -2e(L-1)^{1/2}(L+2)^{1/2} & -3m\end{pmatrix}, \quad (12)$$

$$V=+\frac{Ne^{-\lambda}}{(r\Lambda)^2}\left(\frac{8e^2}{r^2}-\frac{6m}{r}\right)^2+\frac{8e^2Ne^{-\lambda}}{r^4\Lambda}+\frac{L(L+1)(L-1)(L+2)}{r^2\Lambda}+\frac{3m}{r^3}+\frac{4e^2}{r^4\Lambda}\left(2-\frac{6m}{r}+\frac{4e^2}{r^2}\right),$$

and

$$S=\frac{L(L+1)}{r^3\Lambda}+\frac{2Ne^{-\lambda}}{r^3\Lambda^2}\left[(L-1)(L+2)+\frac{4e^2}{r^2}\right]-\frac{1}{r^3\Lambda}\left(\frac{2m}{r}-\frac{2e^2}{r^2}\right) \quad (13)$$

and where r^* is defined by

$$\frac{dr}{dr^*}=Ne^{-\lambda}=1-2m/r+e^2/r^2\,. \quad (14)$$

Clearly, an orthogonal transformation which diagonalizes T decouples the two wave equations. Let A be the orthogonal matrix for which

$$ATA^{\rm tr}=\begin{pmatrix}+\sigma & 0\\ 0 & -\sigma\end{pmatrix} \quad (15)$$

with

$$\sigma=[9m^2+4e^2(L-1)(L+2)]^{1/2}\,. \quad (16)$$

Then R_+ and R_- defined by

$$\begin{pmatrix}R_+\\R_-\end{pmatrix}=A\begin{pmatrix}H\\Q\end{pmatrix} \quad (17)$$

obey

$$\Box\begin{pmatrix}R_+\\R_-\end{pmatrix}+Ne^{-\lambda}\begin{pmatrix}V_+ & 0\\ 0 & V_-\end{pmatrix}\begin{pmatrix}R_+\\R_-\end{pmatrix}=0 \quad (18)$$

where

$$\begin{aligned} V_+&=V+\sigma S\,,\\ V_-&=V-\sigma S\,. \end{aligned} \quad (19)$$

The scattering potentials $U_\pm\equiv Ne^{-\lambda}V_\pm$ vanish rapidly as $r\to\infty$ and as $r\to r_+$ ($r^*\to-\infty$). Thus the normal-mode solutions (with time dependence $e^{i\omega t}$) of Eqs. (18) have the asymptotic behavior

$$\begin{aligned} R_\pm&\sim C_\pm\exp[i\omega(t-r^*)] \quad \text{as } r\to\infty\,,\\ R_\pm&\sim D_\pm\exp[i\omega(t+r^*)] \quad \text{as } r\to r_+ \;(r^*\to-\infty)\,, \end{aligned} \quad (20)$$

where $C_\pm$ and $D_\pm$ are certain constants and where we have imposed the boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves at the event horizon.

III. STABILITY

An unstable normal-mode solution is one for which the frequency ω has a negative imaginary part. Such solutions grow exponentially in time. From the asymptotic forms (20) we see that such solutions decay exponentially in r^* (at constant t)

as $|r^*| \to \infty$. If $R_\pm$ are unstable solutions we have from (18)

$$-\omega^2 R_\pm - \frac{\partial^2 R_\pm}{\partial r^{*2}} + U_\pm R_\pm = 0. \tag{21}$$

Therefore,

$$\omega^2 \int_{-\infty}^{\infty} dr^* |R_\pm|^2 = -\int_{-\infty}^{\infty} dr^* \left(R_\pm^* \frac{\partial^2 R_\pm}{\partial r^{*2}} - U_\pm |R_\pm|^2 \right)$$
$$= \int_{-\infty}^{\infty} dr^* \left(\left| \frac{\partial R_\pm}{\partial r^*} \right|^2 + U_\pm |R_\pm|^2 \right), \tag{22}$$

since, by the exponential decay of $R_\pm$ for large $|r^*|$, no boundary terms survive in the integration by parts. From (22) it follows that ω^2 is real and therefore (by the instability assumption) that ω is purely imaginary. However, $\omega^2 < 0$ is clearly impossible if U_+ and U_- are non-negative functions of r in the range $r_+ \le r \le \infty$. By straightforward algebra one can show that U_+ and U_- are indeed non-negative [on (r_+, ∞)] for each value of $L \ge 2$ and for all e and m such that $|e| \le m$. Consequently the assumption of unstable normal-mode solutions obeying the specified boundary conditions leads to a contradiction.

A similar analysis can be given for the $L = 1$ modes in which only electromagnetic radiation can occur. The $L = 0$ perturbations are spherically symmetric and thus are tangent to the Reissner-Nordström family of solutions. They merely allow for small changes of the charge and mass parameters.

ACKNOWLEDGMENTS

I thank Professor A. H. Taub for a number of valuable conversations concerning variational methods and stability problems in general relativity.

*Work supported in part by the National Science Foundation under Grant No. GP 31358.

[1]V. Moncrief, Phys. Rev. D 9, 2707 (1974).
[2]A. Taub, Commun. Math. Phys. 15, 235 (1969).
[3]T. Regge and J. A. Wheeler, Phys. Rev. 108, 1063 (1957).
[4]C. Vishveshwara, Phys. Rev. D 1, 2870 (1970).
[5]F. Zerilli, Phys. Rev. D 9, 860 (1974).
[6]D. M. Chitre, R. H. Price, and V. D. Sandberg, Phys. Rev. Lett. 31, 1018 (1973).
[7]M. Johnston, R. Ruffini, and F. Zerilli, Phys. Rev. Lett. 31, 1317 (1973).

PHYSICAL REVIEW D VOLUME 10, NUMBER 4 15 AUGUST 1974

Electromagnetic scattering from a black hole and the glory effect*

Bahram Mashhoon
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 4 March 1974)

The scattering of electromagnetic radiation by a black hole is discussed and further results on this problem are presented. It is shown that the backward glory effect is absent in the Schwarzschild field as well as in the Kerr case when plane electromagnetic waves are incident along the axis of symmetry of the field. A cosmological distribution of Kerr black holes could result in the polarization of the cosmic background radiation for which a crude estimate is given.

I. INTRODUCTION

Evidence for the existence of a black hole may be obtained through the detection of electromagnetic radiation that is scattered from it. Though optical means are not very promising at present, the scattering problem is of interest since a collapsed object that is not accreting fresh matter scatters radiation merely by its gravitational field. This problem has been partially analyzed in a previous publication.[1] The purpose of this paper is to extend that analysis, to show that the backward glory effect is absent in the Schwarzschild field, and to give an estimate of the effect of a cosmological distribution of black holes on the polarization of the background radiation.

It has been shown[2] that the electromagnetic field equations can be cast into the form of Maxwell's equations in flat spacetime but in a "medium" with dielectric and permeability tensors[3] (ϵ_{ij}) and (μ_{ij}),