Space-Time Position Operators

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A formalism is presented giving the development of $\psi(\bar{x})$ in time with the interpretation that $\psi(\bar{x})$ is the probability amplitude for observing an event at a space-time point $\bar{x}$. No properties other than the four space-time coordinates are associated with an event. A Hilbert space is defined in which $\psi(\bar{x})$ is the result of a scalar product. The space-time position operators defined in this Hilbert space have no association with particle properties, such as mass. These operators cannot be defined in the Hilbert space spanned by solutions of a Schrödinger equation, since the operators lead out of the Hilbert subspace belonging to a given mass. It is shown that state vectors in Hilbert space that are eigenvectors of $P_\mu P^\mu$ produce position amplitudes satisfying the Klein-Gordon equation. The relation between this Hilbert space and the one introduced by Dirac is discussed. Spin is not considered.

I. DIFFICULTIES WITH RELATIVISTIC QUANTUM MECHANICS

THE current formulation of relativistic quantum mechanics is basically a generalization of classical mechanics. It seeks operators to replace classical quantities. Because it reached its present stage of development by first passing through nonrelativistic mechanics, time is treated as a parameter while operators corresponding to space coordinates are required. Although relativistic quantum mechanics today is covariant, this distinction between time and space coordinates is contrary to the spirit of Einstein's classical theory of relativity.

The current theory calls for three space-coordinate operators whose time dependence (in the Heisenberg picture) corresponds to the drawing out of the particle world line. Although these operators are readily identified in nonrelativistic quantum mechanics, the search for them in the relativistic theory has run into surprising difficulties.[1–5] These difficulties may be summarized as follows. Although Newton and Wigner[2] have determined a unique set of space position operators satisfying a very reasonable set of requirements, these operators do not transform in a simple way under Lorentz transformations involving the time axis (boosts). Barut and Malin[3] have pointed out that the position probability distribution, obtained from the coefficients in the expansion of the state function in terms of the eigenfunctions of the Newton-Wigner operators, does not transform like the fourth component of a four-vector. This question disturbed Wigner and Philips enough to motivate Philips to include a derivation of covariant position operators in his dissertation.[5] He found that, in order to have his operators transform properly, it was necessary to remove the requirement that a position eigenfunction, displaced relative to another eigenfunction, be orthogonal to it. This is, of course, quite unsatisfactory.

A number of other attempts[4] have been made to resolve this difficulty within the framework of the present theory. None has been completely satisfactory. (A possible exception to this is the paper by Johnson. He discusses space-time operators similar to those used here, but he also introduces an additional proper-time variable.) For this reason, a reformulation of relativistic quantum mechanics is begun in this paper. Four space-time position operators will be found that do transform properly under all Lorentz transformations. The discussion in this paper will be limited to a zero-spin noninteracting particle.

II. SPACE-TIME PROBABILITY DISTRIBUTION

We will take very seriously Einstein's requirement that the nature of time be very similar to that of the space coordinates. For this reason, an operator corresponding to the time coordinate will be added to those for the space coordinates.

Instead of generalizing classical mechanics, we shall keep in mind Feynman's picture[6] of quantum electrodynamics where space-time is viewed as a whole, and the universe is made up of particles that propagate from one space-time point to another. Particles interact with each other at space-time points where we find Feynman vertices.

The process of looking at space at each instant of time and watching the development of systems with time puts time on a different basis than space. To avoid this, we will view all of space-time at once. This calls for the replacement of the space probability distribution function that develops with time by a space-time

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probability distribution. To illustrate the physical interpretation of such a probability distribution, we shall consider a thought experiment. Since this paper is limited to a noninteracting particle, a difficulty appears immediately. An interaction is necessary to locate a particle. To remedy this, it is necessary to consider the limit of vanishing interaction.

Let us consider as an example, a μ^- particle ignoring all interactions except the one allowing it to decay to an electron and two neutrinos. A detection apparatus might include a detector capable of finding one of the decay products, either an electron or neutrino, either immediately after the decay or in some other way that allows the time and position of the decay to be determined. Although this detector must interact strongly with a decay product, it must have no interaction with the original muon. In addition, the experimental apparatus must include a space-time coordinate frame for determining the space coordinates and time of a decay—perhaps a frame consisting of rods and clocks constructed in the manner prescribed by Einstein.

Let us assume that means are available for creating μ^- states repeatedly at known times. Then the space location and time interval since creation can be measured and plotted on a four-dimensional graph for each time the particle is detected. After a large number of detections, the points will become dense enough to determine a space-time probability distribution $\rho(\bar{x})$, where $\bar{x}$ is a four-vector locating a space-time point.[7] This probability distribution will decay exponentially with time with the half-life of a muon. We want to consider the limit where the interaction of the particle (the muon) with other particles becomes negligibly small. In this limit, the rate of decay of the distribution with time becomes smaller and smaller. At the same time, the experiment must be repeated a larger and larger number of times to establish the probability distribution to a given accuracy. We can define the position measurement for any state with this apparatus by introducing a very weak interaction between the muon and the electron-neutrino field and taking the limit as the coupling constant approaches zero.

Although experiments done in the laboratory involve particles with specified masses, the position-measuring experiment just described does not simultaneously measure the mass. It merely locates a space-time "event" (the point of decay) and measures the coordinates of this event with the aid of a coordinate frame. We shall find it convenient to divorce these events from a specified mass and to refer to a "one-event" state rather than to a "particle" state. As soon as the event occurs, the state disappears and a new one takes its place.

If we wish to determine the customary probability distribution in space at a time t from the data obtained by the measurement described above, we can do this by considering the distribution of points between the hyperplanes at t and $t+dt$. The interval dt must be so small that the variation of the density of points in the vicinity of an arbitrary space point is negligible from time t to $t+dt$. The space probability distribution multiplied by $d\mathbf{x}$ will then be proportional to the number of points contained in an elemental space volume of size $d\mathbf{x}$ lying between the hyperplanes at t and $t+dt$. It will also be proportional to $\rho(\bar{x})dt$.

The space-time distribution function $\rho(\bar{x})$ must transform as a scalar since the number of points in space-time volume d^4x is proportional to $\rho(\bar{x})d^4x$. This number of points will not change when viewed from a different inertial frame and the volume size d^4x will not change. The space probability distribution $\rho(\bar{x})dt$ will transform like the fourth component of a four-vector.

III. HILBERT SPACE FOR RELATIVISTIC QUANTUM MECHANICS

The Hilbert space introduced by Dirac and currently used in quantum mechanics consists of state vectors that move with time (Schrödinger picture) or operators that move with time (Heisenberg picture). This Hilbert space at a given instant of time can be used to obtain probabilities at that time. This formalism is not consistent with our purpose of treating time and space on a similar footing.

To remedy this defect, we extend this Hilbert space so that it contains eigenvectors or eigenkets of a time operator as well as three space-position operators. Now instead of allowing the state ket to move until it reaches the desired time, we must project the state ket on the time eigenket belonging to the required eigenvalue. Thus the amplitudes $\psi(\bar{x})$ for finding a space-time event at a point $\bar{x}$ is given by the scalar product of the state ket $|\psi\rangle$ with the position ket $|\bar{x}\rangle$, $\langle\bar{x}|\psi\rangle$ where $\bar{x}$ is a space-time four-vector with components

$$x^0=ct,\quad x^1=x,\quad x^2=y,\quad \text{and}\quad x^3=z. \tag{1}$$

This Hilbert space will contain state kets belonging to all possible masses (including imaginary ones) where Dirac's Hilbert space is limited to states belonging to a given non-negative real mass. Of course, states of particles found in nature do have a given mass and must therefore lie on a subspace belonging to that mass. However, we shall see that space-time position operators are not contained in mass subshell and that this is the reason for the failure of past efforts to find position operators in Dirac's Hilbert space. Projecting the space position operators on to a mass subshell in a reasonable manner will produce the Newton-Wigner operators, although they are not true position operators.

We limit ourselves to discussing the states of a single particle with spin zero and leave higher spin states to a later paper.

[7] Four-vectors will be indicated by bars over letters and three-vectors by boldface letters.

IV. POSITION OPERATORS

As mentioned in Sec. III, it is convenient to define a Hilbert space containing state kets $|\psi\rangle$ and position kets $|\bar{x}\rangle$ such that the wave function is given by the scalar product

$$\psi(\bar{x})=\langle\bar{x}|\psi\rangle. \tag{2}$$

Each point in space-time has a ket $|\bar{x}\rangle$ associated with it, and the set of all these kets span the Hilbert space. They are taken to be orthogonal and normalized so that

$$\langle\bar{x}|\bar{x}'\rangle=\delta(\bar{x}-\bar{x}'). \tag{3}$$

As we shall see later, these kets are not eigenkets of the mass operator and, therefore, cannot be contained in a Hilbert space confined to solutions of a Schrödinger-type equation with a given mass.

An arbitrary ket in the Hilbert space may be expanded in terms of the position kets so that

$$|\psi\rangle=\int|\bar{x}\rangle\langle\bar{x}|\psi\rangle d^4x, \tag{4}$$

where

$$d^4x=dx^0dx^1dx^2dx^3. \tag{5}$$

Thus one form of the unit operator is given by

$$1=\int|\bar{x}\rangle\langle\bar{x}|d^4x. \tag{6}$$

The scalar product of two state vectors is given by

$$\langle\psi|\psi\rangle=\int\langle\psi|\bar{x}\rangle\langle\bar{x}|\psi\rangle d^4x=\int\psi^*(\bar{x})\psi(\bar{x})d^4x. \tag{7}$$

Since the $|\bar{x}\rangle$ kets span the Hilbert space, the four operators X^μ ($\mu=0$, 1, 2, 3) can be defined by

$$X^\mu|\bar{x}\rangle=x^\mu|\bar{x}\rangle. \tag{8}$$

Since we have already assumed that the eigenvalues of these four operators may be used to label a ket, we have required that

$$[X^\mu,X^\nu]=0. \tag{9}$$

These operators correspond to space-time position operators in the sense that their eigenvalues are the numbers obtained from a measurement of the coordinates of an event. The fact that they commute implies that the four coordinates may be measured simultaneously.

V. DISPLACEMENT AND ROTATION OPERATORS

As was mentioned above, to every point in space-time there corresponds a vector or ket in Hilbert space. The kets representing space-time points separated by an infinitesimal displacement l in the μth direction can be related by a displacement generator P_μ defined by

$$P_\mu|\bar{x}\rangle=i\lim_{l\to 0}l^{-1}(|\bar{x}+\bar{\imath}_\mu l\rangle-|\bar{x}\rangle), \tag{10}$$

where $\bar{\imath}_\mu$ is the unit vector in the μth direction of space-time. The i is inserted to make the operator self-adjoint. The meaning of P_μ becomes more clear when we take the adjoint of this equation and project an arbitrary ket $|\psi\rangle$ on it to obtain

$$\langle\bar{x}|P_\mu|\psi\rangle=-i\lim_{l\to 0}l^{-1}(\langle\bar{x}+\bar{\imath}_\mu l|\psi\rangle-\langle\bar{x}|\psi\rangle) =-i\partial_\mu\psi(\bar{x}), \tag{11}$$

where

$$\partial_\mu=\partial/\partial x^\mu. \tag{12}$$

In a similar manner, the rotation operator $M_\mu{}^\nu$ can be defined by

$$M_\mu{}^\nu|\bar{x}\rangle=i\lim_{\omega\to 0}\omega^{-1}(|\bar{x}+\delta\bar{x}\rangle-|\bar{x}\rangle), \tag{13}$$

so that

$$\langle\bar{x}|M_\mu{}^\nu|\psi\rangle=-i(\partial/\partial\omega)\psi(\bar{x}), \tag{14}$$

where ω is the angle of rotation from μth to the νth axis, and $\delta\bar{x}$ is the displacement of a point rotated through the angle ω.

The P_μ and $M_\mu{}^\nu$ operators, in a flat space-time, are the generators of the inhomogeneous Lorentz or Poincaré group. Their commutation relations were worked out long ago[8,9]; they are

$$[P_\mu,P_\nu]=0, \tag{15}$$

$$[P_\sigma,M_{\mu\nu}]=i(P_\mu g_{\nu\sigma}-P_\nu g_{\mu\sigma}), \tag{16}$$

and

$$[M_{\nu\mu},M_{\gamma\eta}]=i(M_{\nu\gamma}g_{\mu\eta}+M_{\mu\eta}g_{\nu\gamma}-M_{\nu\eta}g_{\mu\gamma}-M_{\mu\gamma}g_{\nu\eta}), \tag{17}$$

where

$$-g_{00}=g_{11}=g_{22}=g_{33}=1. \tag{18}$$

As mentioned above, the measuring apparatus used to determine the coordinates of an event includes a frame of reference. If a second observer using a different frame of reference measures these coordinates for the same event and obtains the numbers y^ν, they must be related to the numbers x^ν for the first observer by the relation

$$y^\nu=a^\nu{}_\mu x^\mu, \tag{19}$$

where $a^\nu{}_\mu$ is the coefficient in the Lorentz transformation from one frame to the other. The amplitude the second observer obtains is

$$\psi'(\bar{y})=\langle\bar{y}|\psi\rangle \tag{20}$$

and must equal the amplitude obtained by the first observer since we are discussing spin-zero systems.

[8] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[9] V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1946), Eqs. (3a), (3b), and (6). Note that they use a metric that is the negative of the one used in this paper and defined in our Eq. (18).

Since the observers are measuring the same event, we have

$$|\bar{x}\rangle \equiv |\bar{y}\rangle \quad (21)$$

although the numbers x^μ and y^μ are different. The position operators for the second observer are the Y^μ's satisfying

$$Y^\mu|\bar{y}\rangle = y^\mu|\bar{y}\rangle. \quad (22)$$

But according to Eqs. (8), (19), and (21), we have

$$Y^\mu|\bar{y}\rangle = a^\mu{}_\nu x^\nu|\bar{x}\rangle = a^\mu{}_\nu X^\nu|\bar{x}\rangle. \quad (23)$$

Since $|\bar{y}\rangle \equiv |\bar{x}\rangle$ is an arbitrary position ket,

$$Y^\mu = a^\mu{}_\nu X^\nu. \quad (24)$$

In this sense, the position operators transform like the components of a four-vector.

If the frame of the second observer is rotated through an angle ω from the μth toward the νth axis, then, according to the definition of $M_\mu{}^\nu$ in Eq. (13), a ket $|\bar{y}'\rangle$ belonging to a space-time event whose coordinates relative to the second frame are the same as the point $\bar{x}$ relative to the first frame is related to $|\bar{x}\rangle$ by[10]

$$|\bar{y}'\rangle = \exp(-i\omega M_\mu{}^\nu)|\bar{x}\rangle. \quad (25)$$

In this case, we write

$$Y^\eta|\bar{y}'\rangle = x^\eta|\bar{y}'\rangle. \quad (26)$$

Substituting from Eq. (25), multiplying by $\exp(i\omega M_\mu{}^\nu)$, and using Eq. (24) shows that

$$Y^\eta = \exp(-i\omega M_\mu{}^\nu)X^\eta \exp(i\omega M_\mu{}^\nu) = a^\eta{}_\gamma X^\gamma. \quad (27)$$

If ω is infinitesimal, the exponents can be expanded and $a^\eta{}_\gamma$ becomes $\delta^\eta{}_\gamma + \omega\delta_\mu{}^\eta\delta_{\gamma\nu} - \omega\delta_\nu{}^\eta\delta_{\gamma\mu}$. The last equation then reduces to

$$[M_{\mu\nu}, X_\eta] = iX_\nu g_{\eta\mu} - iX_\mu g_{\eta\nu}, \quad (28)$$

where all indices have been lowered. In a similar way, the commutation relations between X^ν's and P^μ's can be shown to be

$$[X^\nu, P^\mu] = ig^{\nu\mu}. \quad (29)$$

The commutation relations obtained when μ and ν take on the values 1, 2, or 3 are the same as those present in the current form of relativistic theory. This equation for $\mu=\nu=0$ is new and is the relation between the time and energy operators. It implies a limit on simultaneous measurability of time and energy.

Since the P^μ operators commute, they may have a set of common eigenkets so that

$$P^\mu|\bar{p}\rangle = p^\mu|\bar{p}\rangle. \quad (30)$$

This equation may be projected on to $|\bar{x}\rangle$ to obtain, with the aid of Eq. (11),

$$\langle\bar{x}|P^\mu|\bar{p}\rangle = -i\partial^\mu\langle\bar{x}|\bar{p}\rangle = p^\mu\langle\bar{x}|\bar{p}\rangle. \quad (31)$$

[10] If the time axis is not involved in the rotation, ω is just the spatial angle. If μ or ν is zero, we are dealing with a "boost" and ω is $-\tanh^{-1}(v/c)$, where v is the speed of the second frame relative to the first.

This set of four equations is solved by

$$\langle\bar{x}|\bar{p}\rangle = e^{i\bar{p}\cdot\bar{x}}, \quad (32)$$

and this will be taken to be the expansion coefficient for the expansion of $|\bar{p}\rangle$ in terms of $|\bar{x}\rangle$'s. Thus another form of the unit operator is

$$1 = \int |\bar{p}\rangle\langle\bar{p}|\, d^4p(2\pi)^{-4}. \quad (33)$$

VI. MASS-SHELL–NEWTON-WIGNER OPERATORS

The ket $|\psi\rangle$ is specified when all of its projections $\langle\bar{x}|\psi\rangle$ on the $|\bar{x}\rangle$ kets are given. Thus a knowledge of $|\psi\rangle$ is equivalent to the knowledge of the wave function $\psi(\bar{x})$ at all points of space-time. In actual practice, $\psi(\bar{x})$ is known at all space-time points if it and its first time derivative throughout space are known at one given time. We shall see that this smaller amount of knowledge is sufficient because $|\psi\rangle$ is known to lie on a mass shell. This is indicated by the presence of the rest-mass parameter in the wave equation. Thus the knowledge that a state ket lies on a mass shell plus the values of $\langle\bar{x}|\psi\rangle$ and its time derivative at one time is equivalent to the knowledge of $\langle\bar{x}|\psi\rangle$ at all space-time points.

The rest mass is an invariant in classical relativity theory. Thus we anticipate that it will be the eigenvalue of an invariant operator. The only invariant available to us is[8,9] $P_\mu P^\mu$. Let us define $|\psi k\rangle$ to be an eigenket of $P_\mu P^\mu$ so that

$$P_\mu P^\mu|\psi k\rangle = -k^2|\psi k\rangle. \quad (34)$$

Projecting this equation on to $|\bar{x}\rangle$ and making use of Eq. (11) gives

$$\partial_\mu\partial^\mu\langle\bar{x}|\psi k\rangle = k^2\langle\bar{x}|\psi k\rangle. \quad (35)$$

This is the Klein-Gordon equation if we make the identification

$$k = mc/\hbar. \quad (36)$$

Thus we see that the wave equation is a consequence of restricting the state ket to a given mass shell in Hilbert space.

The commutation relation between X^ν and $P_\mu P^\mu$ can easily be obtained from Eq. (29) and is

$$[X^\nu, P_\mu P^\mu] = 2iP^\nu. \quad (37)$$

Since $P_\mu P^\mu$ does not commute with X^ν, a localized state of a particle (with a given mass) cannot be constructed. In fact all masses, including imaginary ones, would be necessary to construct an X^ν eigenstate. Thus it follows that a position measurement that determines the position of a particle at a point such as the idealized one described in Sec. II is physically impossible because of the restricted number of mass states available in our universe.

Nevertheless, position measurements are made on particles. They are unable, however, to locate the position with zero uncertainty. We can attempt to find a position operator for these measurements by projecting the position eigenkets into a mass subshell with the aid of the projection operator

$$Q_{kr}=\tfrac{1}{2}\int |\bar{p}_{kr}\rangle\langle\bar{p}_{kr}|\,\epsilon_k^{-1}d\mathbf{p}(2\pi)^{-4}, \tag{38}$$

where

$$p_{kr}^j=p^j,\ p_{kr}^0=r\epsilon_k,\ \epsilon_k=(p_jp^j+k^2)^{1/2}\ (j=1,2,3) \tag{39}$$

and

$$d\mathbf{p}=dp_1dp_2dp_3. \tag{40}$$

We can speculate that a physically possible "position" measurement may throw a particle into a state $Q_{kr}|\bar{x}\rangle$. However, if we take the scalar product of this ket with another such ket, the result is $\langle\bar{x}'|Q_{k'r'}Q_{kr}|\bar{x}\rangle$. From Eq. (38), we have

$$Q_{k'r'}Q_{kr}=\tfrac{1}{4}\int\int |\bar{p}'_{k'r'}\rangle\langle\bar{p}'_{k'r'}|\bar{p}_{kr}\rangle \times\langle\bar{p}_{kr}|\,\epsilon_{k'}^{-1}\epsilon_k^{-1}d\mathbf{p}'d\mathbf{p}(2\pi)^{-8}. \tag{41}$$

The unit operator in Eq. (6) and Eq. (32) may be used to show that

$$\langle\bar{p}'|\bar{p}\rangle=\int\langle\bar{p}'|\bar{x}\rangle\langle\bar{x}|\bar{p}\rangle d^4x=(2\pi)^4\delta(\bar{p}'-\bar{p}). \tag{42}$$

Substituting this into Eq. (41) and integrating over $d\mathbf{p}'$ gives

$$Q_{k'r'}Q_{kr}=\tfrac{1}{4}\int |\bar{p}_{kr}\rangle\langle\bar{p}_{kr}|\,\epsilon_k^{-2}\delta(\epsilon_{k'}-\epsilon_k)d\mathbf{p}(2\pi)^{-4}\delta_{r,r'}. \tag{43}$$

As a function of k^2,

$$\delta(\epsilon_{k'}-\epsilon_k)=2\epsilon_k\delta(k'^2-k^2). \tag{44}$$

Thus Eq. (43) reduces to

$$Q_{k'r'}Q_{kr}=Q_{kr}\delta(k'^2-k^2)\delta_{r,r'} \tag{45}$$

and

$$\langle\bar{x}'|Q_{k'r'}Q_{kr}|\bar{x}\rangle=\langle\bar{x}'|Q_{kr}|\bar{x}\rangle\delta(k'^2-k^2)\delta_{r,r'} =\tfrac{1}{2}\int \exp[i\bar{p}_{kr}\cdot(\bar{x}'-\bar{x})]\epsilon_k^{-1}d\mathbf{p}(2\pi)^{-4} \times\delta(k'^2-k^2)\delta_{r,r'}. \tag{46}$$

The last integral is not a δ function in $x^{j'}-x^j$ and so the $Q_k|\bar{x}\rangle$ kets are not orthogonal. It is clear, however, that the integral would become a δ function if $x^0=x^{0'}=0$ and an additional factor ϵ_k could be introduced into the integrand. Thus it appears that the kets

$$|\mathbf{x}kr\rangle=Q_{kr}(4\pi E_p)^{1/2}|0,x^1,x^2,x^3\rangle, \tag{47}$$

where

$$E_p=(P_jP^j+k^2)^{1/2} \tag{48}$$

and where $\mathbf{x}$, which has the three components x^1, x^2, and x^3, can serve as Schrödinger-picture "position" eigenkets. Then we have

$$\langle\mathbf{x}'k'r'|\mathbf{x}kr\rangle=\Pi_j\delta(x^{j'}-x^j)\delta(k'^2-k^2)\delta_{r,r'}. \tag{49}$$

These kets are eigenkets of an operator X_{kr}^j satisfying

$$\langle\mathbf{x}''k''r''|(X_{kr}^j-x^jQ_{k'r'})|\mathbf{x}kr\rangle=0. \tag{50}$$

With the definition of $|\mathbf{x}kr\rangle$ in Eq. (47), this equation may be arranged to read

$$\langle\mathbf{x}''k''r''|Q_{kr}(4\pi)^{1/2}(X_{kr}^jE_p^{1/2}-E_p^{1/2}Q_{k'r'}X^j)|0\mathbf{x}\rangle=0. \tag{51}$$

$X_{k'r'}^j$ commutes with Q_{kr} since it is in the mass shell. A relation between $X_{k'r'}^j$ and the other operators can be determined more easily by defining A by

$$X_{kr}^j=Q_{kr}(X^j+A). \tag{52}$$

Then substituting this into Eq. (51) shows that A must satisfy

$$AE_p^{1/2}+[X^j,E_p^{1/2}]=0. \tag{53}$$

Evaluating the commutator with the aid of Eq. (29) gives

$$A=-iP^j/2E_p^2 \tag{54}$$

and

$$X_{kr}=Q_{kr}(X^j-iP^j/2E_p^2). \tag{55}$$

Newton and Wigner's[2] Eq. (11) is essentially the momentum representation of this operator. (The difference is a factor Q_{kr}.)

Since the Newton-Wigner position kets $|\mathbf{x}kr\rangle$ lie on an energy shell, they are possible particle states. They are not confined to a single point in space but have a space-time position amplitude given by

$$\langle\bar{x}'|\mathbf{x}kr\rangle=\langle\bar{x}'|Q_{kr}(4\pi E_p)^{1/2}|0,\mathbf{x}\rangle. \tag{56}$$

Substituting Q_{kr} from Eq. (38) gives

$$\langle\bar{x}'|\mathbf{x}kr\rangle=\pi^{1/2}\int \exp[i\bar{p}_{kr}\cdot(\bar{x}'-\bar{x})]\epsilon_k^{-1/2}d\mathbf{p}(2\pi)^{-3},\quad x^0=0. \tag{57}$$

If $x^{0'}$ is zero, this expression reduces to the integral identified by Newton and Wigner [Ref. 2, Eq. (9a)], so that

$$\langle 0,\mathbf{x}'|\mathbf{x}kr\rangle\propto(k/|\mathbf{x}'-\mathbf{x}|)^{5/4}H_{5/4}^{(1)}(ik|\mathbf{x}'-\mathbf{x}|). \tag{58}$$

It decays exponentially as $e^{-k|\mathbf{x}'-\mathbf{x}|}$ at large separation distances and behaves like $|\mathbf{x}'-\mathbf{x}|^{-5/2}$ at small distances.

VII. WAVE-FUNCTION SCALAR PRODUCT

As we have already seen, the state ket for a particle with a given mass must be selected from those that are eigenkets of $P_\mu P^\mu$ with the eigenvalue $-(mc/h)^2$, i.e., they lie on a mass shell. All other kets in the Hilbert space are not available as state kets. The position eigenkets do not lie on a mass shell and are not possible state kets for a particle. The normalization of a particle with a positive state ket $|\psi k1\rangle$ can be conveniently accounted for by using the mass-shell projection operator so that

$$|\psi k1\rangle = Q_{k1}|\psi\rangle. \quad (59)$$

The scalar product of two particle state kets $Q_{kr}|\psi\rangle$ and $Q_{k'r'}|\psi'\rangle$ may be written, with the aid of Eq. (45), as

$$\langle\psi'|Q_{k'r'}Q_{kr}|\psi\rangle = \langle\psi'|Q_k|\psi\rangle\delta(k'^2-k^2)\delta_{r,}$$
$$= \tfrac{1}{2}\int\langle\psi'|\bar{p}_{kr}\rangle\langle\bar{p}_{kr}|\psi\rangle\epsilon_k^{-1}d\mathbf{p}(2\pi)^{-4} \times\delta(k'^2-k^2)\delta_{r,r'}, \quad (60)$$

where the expression for Q_{kr} in Eq. (38) has been employed. This corresponds to Wigner's[8] Eq. (59a) for the scalar product of two state functions. It is important to realize, however, that scalar products such as the one above correspond to integrations over time as well as space when coordinate functions are used as can be seen by inserting the unit operator in Eq. (6). The infinity that occurs when k'^2 equals k^2 is present in the integration over all space-time since a wave function on a mass shell cannot decay with time. This is clear from Eq. (37), where we see that X^0 and the mass operator do not commute. Thus confining a state to a mass shell and to positive energies means that its wave function will extend through all time.

VIII. SPIN-ZERO HAMILTONIAN

Because of the δ function that enters $|\psi k1\rangle$ limiting it to the mass shell, it is not convenient to directly relate it to kets in Dirac's Hilbert space. However, it is possible to obtain the Hamiltonian H that Dirac would write down. For this purpose, we recall that the time dependence of the wave function in Dirac's notation is found by operating with $e^{-iHt/\hbar}$ on the state ket and taking the scalar product of the resulting ket and a space position ket. To write the wave function in this form, we first project Eq. (59) on to $\langle\bar{x}|$ and obtain

$$\langle\bar{x}|\psi k1\rangle = \langle\bar{x}|Q_{k1}|\psi\rangle = \langle\mathbf{x}0|\exp(iP_0x^0)Q_{k1}|\psi\rangle. \quad (61)$$

The operator P_0 is a time displacement generator and is analogous to the time derivative appearing in the Schrödinger equation. It is equivalent, however, to an operator involving space displacements (the Hamiltonian) when it operates on a ket known to be on a mass shell. This operator can be obtained from Eq. (34) rewritten as

$$[(P^0)^2-E_p^2]|\psi k\rangle = 0, \quad (62)$$

where E_p is defined in Eq. (48). This Eq. (62) factors into two equations

$$P^0|\psi kr\rangle = rE_p|\psi kr\rangle, \quad r=\pm 1. \quad (63)$$

Since Q_{k1} in Eq. (61) is on a positive-energy mass shell, P_0 can be replaced by $-E_p$. Since E_p commutes with Q_{k1}, Eq. (61) becomes

$$\langle\bar{x}|\psi k1\rangle = \langle\mathbf{x}0|Q_{k1}\exp(-iE_px^0)|\psi\rangle. \quad (64)$$

If we recognize $Q_{k1}|\mathbf{x}0\rangle$ as the relativistic extension of the Schrödinger-picture position ket for Dirac, this equation gives the expression in Dirac form, where

$$H = chE_p. \quad (65)$$

There is one difficulty with this as we have seen. The kets $Q_{k1}|\mathbf{x}0\rangle$ are not orthogonal. This can be remedied by introducing the Newton-Wigner kets defined in Eq. (47) and writing

$$\langle\bar{x}|\psi k1\rangle = \langle\mathbf{x}k1|\exp(-iE_px^0)|\phi\rangle, \quad (66)$$

where

$$|\phi\rangle = (4\pi E_p)^{-1/2}|\psi\rangle. \quad (67)$$

Now the position kets are orthogonal, but the Dirac state ket $|\phi\rangle$ differs from the true state ket by the factor $(4\pi E_p)^{-1/2}$.

IX. TRANSFORMATION PROPERTIES OF NEWTON-WIGNER OPERATORS

The eigenkets $|\mathbf{x}k1\rangle$ presented in Eq. (47) are defined relative to some coordinate frame since they contain E_p, and x^0 is set equal to zero. The operator Q_{k1} projects into a subspace belonging to an eigenvalue of the invariant $P_\mu P^\mu$ and the positive sign of the energy. The sign of the energy, however, is also an invariant for p kets lying inside the light cone (real mass). Thus for real masses any of the operators of the Poincaré group will commute with Q_{k1}.

Equation (25) gives the proper change in the ket when a point is rotated through an angle ω from the point $\bar{x}$. We can now see if the Newton-Wigner position eigenkets satisfy this condition. With the definition of $|\mathbf{x}k1\rangle$ in Eq. (47), it is clear that the Newton-Wigner eigenket belonging to a rotated point with coordinates $\bar{x}'$, $|\mathbf{x}'k1\rangle$ is given by

$$|\mathbf{x}'k1\rangle = Q_{k1}(4\pi E_p)^{1/2}\exp(-i\omega M_\mu{}^\nu)|0,x^1,x^2,x^3\rangle, \quad (68)$$

where 0 is the value of x^0. If $\mu=j$ and $\nu=l$ where j and l are 1, 2, or 3, then $M_j{}^l$ commutes with E_p, and Eq. (68) reduces to

$$|\mathbf{x}'k1\rangle = \exp(-i\omega M_j{}^l)|\mathbf{x}k1\rangle, \quad (69)$$

as Eq. (25) says it should. However, if either μ or ν are zero, this commutation cannot be made and Eq. (25) is not satisfied. Since Newton-Wigner eigenkets are defined with $x^0=0$, we might not expect Eq. (25) to hold if the rotation moves the point out of the $x^0=0$

plane. This can be avoided by considering the $\mathbf{x}=0$ point. This does not help, however, since

$$[E_p, \exp(-i\omega M_j^0)]|\bar{x}\rangle \neq 0, \quad \bar{x}=0. \tag{70}$$

Thus the Newton-Wigner operators do not transform under "boosts" in the same way as their eigenvalues should transform if these eigenvalues are to correspond to the position measurements that we are able to perform in the laboratory.

On the other hand, the kets $Q_{k1}|\bar{x}\rangle$ do transform properly. They are, however, not orthogonal. This was pointed out by Philips.[5]

X. CURRENT FOUR-VECTOR

If $\psi^*\psi$ is the space-time frequency distribution for events, the space probability distribution ρ_s at a given time is given by

$$\rho_s(\bar{x}) = \psi^*(\bar{x})\psi(\bar{x})dx^0 \Big/ \int_{\mathbf{x}'} \psi^*\psi dx^0 d\mathbf{x}', \tag{71}$$

where dx^0 is the thickness of a constant-time hyperplane. This quantity ρ_s can be obtained from repeated applications of the experiment described in Sec. II by counting the number of events occurring in a very small volume of size $d\mathbf{x}$ around the point $\mathbf{x}$ in the time interval of length dx^0 around x^0 and dividing this number by the total number of events occurring throughout all space in this time interval.

To identify the expression for the spatial current, we must obtain an equation of continuity containing $\psi^*\psi dx^0$ as the probability distribution. In Eqs. (62) and (63), we noted that $|\psi k\rangle$ could conveniently be broken into the two parts belonging to positive- and negative-energy states $|\psi k1\rangle$ and $|\psi k, -1\rangle$. Each component satisfies Eq. (63). Projecting this equation on to $|\bar{x}\rangle$ gives

$$i\partial_0\psi_r(\bar{x}) = rE_\nabla\psi_r(\bar{x}), \tag{72}$$

where

$$\psi_r = \langle\bar{x}|\psi kr\rangle \tag{73}$$

and

$$E_\nabla = (-\nabla^2+k^2)^{1/2}. \tag{74}$$

The continuity equation corresponding to the ρ_s defined in Eq. (71) is found if we multiply Eq. (72) by ψ_r^* and subtract the complex-conjugate equation to obtain

$$i\partial_0(\psi_r^*\psi_r) = r(\psi_r^*E_\nabla\psi_r - \psi_r E_\nabla\psi_r^*). \tag{75}$$

The expression for E_∇ can now be expanded in powers of ∇^2/k^2. When this power series is substituted into the above equation, each term will have a factor of the form

$$\psi_r^*\nabla^{2n}\psi_r - \psi_r\nabla^{2n}\psi_r^* = \nabla\cdot[\psi_r^*\nabla\nabla^{2(n-1)}\psi_r - (\nabla\psi_r^*)\nabla^{2(n-1)}\psi_r + (\nabla^2\psi_r^*)\nabla\nabla^{2(n-2)}\psi_r - (\nabla\nabla^2\psi_r^*)\nabla^{2(n-2)}\psi_r \cdots - (\nabla\nabla^{2(n-1)}\psi_r^*)\psi_r]. \tag{76}$$

Substituting this expression into Eq. (75) and dividing by $\int\psi^*\psi d\mathbf{x}$ gives the continuity equation

$$(\partial/\partial t)\rho_s = -\nabla\cdot\mathbf{j}, \tag{77}$$

where the current is identified as

$$\mathbf{j} = -ric\left(\int\psi^*\psi d\mathbf{x}\right)^{-1}\{(2k)^{-1}[\psi_r^*\nabla\psi_r - (\nabla\psi_r^*)\psi_r] + (8k^3)^{-1}[\psi_r^*\nabla\nabla^2\psi_r - (\nabla\psi_r^*)\nabla^2\psi_r + (\nabla^2\psi_r^*)\nabla\psi_r - (\nabla\nabla^2\psi_r^*)\psi_r] + (16k^5)^{-1}[\psi_r^*\nabla\nabla^2\nabla^2\psi_r - (\nabla\psi_r^*)\nabla^2\nabla^2\psi_r + (\nabla^2\psi_r^*)\nabla\nabla^2\psi_r - (\nabla\nabla^2\psi_r^*)\nabla^2\psi_r + (\nabla^2\nabla^2\psi_r^*)\nabla\psi_r - (\nabla\nabla^2\nabla^2\psi_r^*)\psi_r] + \cdots\}. \tag{78}$$

XI. MOMENTUM PROBABILITY DISTRIBUTION

Although the space-time position amplitude has been defined to be $\langle\bar{x}|\psi\rangle$, the momentum probability amplitude has not yet been given. The momentum amplitude is determined when the position amplitude is defined since momentum measuring thought experiments using position measurements can be devised. We shall choose to identify the momentum operators through Ehrenfest's theorem. In the case of noninteracting particles, this theorem states that the rate of change of the mean momentum with time must vanish.

Let us now determine the expression for the mean value of the operator $\mathbf{P}$, a vector operator with components P^j, at a given instant of time. This requires an expansion of the state ket in terms of eigenkets of $\mathbf{P}$ and X^0 normalized so that

$$\langle\bar{x}|x^{0\prime}\mathbf{p}\rangle = \delta(x^{0\prime}-x^0)e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{79}$$

It is possible to form these eigenkets because $\mathbf{P}$ and X^0 commute. If an experiment is performed to simultaneously measure the physical quantities corresponding to $\mathbf{P}$ and X^0, the amplitude for obtaining the values $\mathbf{p}$ and x^0 is $\langle x^0\mathbf{p}|\psi\rangle$. From this it is clear that the mean value of $\mathbf{P}$ at the time x^0, $\langle\mathbf{p}\rangle_{x^0}$, is given by

$$\langle\mathbf{p}\rangle_{x^0} = \langle\psi|Q_{x^0}\mathbf{P}|\psi\rangle/\langle\psi|Q_{x^0}|\psi\rangle, \tag{80}$$

where Q_{x^0} is the projection operator

$$Q_{x^0} = \int |x^0\mathbf{p}\rangle\langle x^0\mathbf{p}|d\mathbf{p}(2\pi)^{-3}. \tag{81}$$

This projection operator can just as well be written in the form

$$Q_{x^0} = \int |\bar{x}\rangle\langle\bar{x}|d\mathbf{x}, \tag{82}$$

where the time integration has been omitted.

Let us now evaluate the time derivative of this average, $\langle\mathbf{p}\rangle_{x^0}$. It is clear, if Eq. (82) is used in writing out $\langle\psi|Q_{x^0}|\psi\rangle$, that

$$\langle\psi|Q_{x^0}|\psi\rangle=\int\psi^*(\bar{x})\psi(\bar{x})d\mathbf{x}. \tag{83}$$

The result of substituting Eq. (76) into Eq. (75) showed that $\psi^*\psi$ satisfies a continuity equation. Thus $\int\psi^*(\bar{x})\psi(\bar{x})d\mathbf{x}$ is independent of time. As a result, differentiating Eq. (80) with respect to x^0 and multiplying by i gives

$$i\frac{d}{dx^0}\langle\mathbf{p}\rangle_{x^0}=\frac{\langle\psi|i\partial_0 Q_{x^0}\mathbf{P}|\psi\rangle}{\langle\psi|Q_{x^0}|\psi\rangle}. \tag{84}$$

Operating on Eq. (82) with $i\partial_0$ gives

$$i\partial_0 Q_{x^0}=\int(i\partial_0|\bar{x}\rangle)\langle\bar{x}|d\mathbf{x}+\int|\bar{x}\rangle i\partial_0\langle\bar{x}|d\mathbf{x}. \tag{85}$$

With Eq. (10) in mind, we can see that

$$i\partial_0\langle\bar{x}|=(-i\partial_0|\bar{x}\rangle)^\dagger=-(P_0|\bar{x}\rangle)^\dagger=-\langle\bar{x}|P_0. \tag{86}$$

Substituting this into Eq. (86) and this, in turn, into Eq. (84), gives

$$i\frac{d}{dx^0}\langle\mathbf{p}\rangle_{x^0}=\frac{\langle\psi|[P_0,Q_{x^0}]\mathbf{P}|\psi\rangle}{\langle\psi|Q_{x^0}|\psi\rangle}. \tag{87}$$

Now P_0 and Q_{x^0} will not, in general, commute since Q_{x^0} is a function of X^0, and X^0 fails to commute with P_0. However, P_0 does commute with $\mathbf{P}$ and can operate to the right directly on $|\psi\rangle$ or to the left on $\langle\psi|$. If $|\psi\rangle$ lies on a mass shell, Eq. (63) shows that P^0 can be replaced by rE_p defined in Eq. (48). Since E_p does not involve P_0 and the other components of $\bar{P}$ commute with X^0, $[E_p,Q_{x^0}]$ will vanish. Thus we have

$$(d/dx^0)\langle\mathbf{p}\rangle_{x^0}=0,\quad |\psi\rangle \text{ on a mass shell}. \tag{88}$$

This shows that the mean value of $\mathbf{P}$ satisfies Einstein's equation for a noninteracting particle, and therefore $\mathbf{P}$ must be proportional to the three-vector momentum operator. Further investigation along the lines of the usual treatment of quantum mechanics shows that $\mathbf{P}$ is precisely the three-vector momentum operator. Since P^0 is the fourth component of the four-vector $\bar{P}$, it must be the energy operator. It is not the Hamiltonian, however, since the Hamiltonian must be a function of $\mathbf{P}$ and not P^0. The usual quantum-mechanical arguments can now be used to show that $\langle\bar{p}|Q_{k1}|\psi\rangle$ is the momentum amplitude for a particle in state $Q_{k1}|\psi\rangle$.

XII. FIELD-THEORY OPERATORS

It is now possible to identify the field creation and annihilation operators in this extended Hilbert space. Let us define the operators $\Phi_{kr}(\bar{x})$ so that they operate on the vacuum (no-particle) state to give

$$\langle 0|\Phi_{k1}(\bar{x})=(2\pi)^{1/2}\langle\bar{x}|Q_{k1} \tag{89}$$

and

$$\Phi_{k,-1}(\bar{x})|0\rangle=(2\pi)^{1/2}Q_{k,-1}|\bar{x}\rangle. \tag{90}$$

In addition,

$$\Phi_k(\bar{x})=\sum_r\Phi_{kr}(\bar{x}). \tag{91}$$

The commutation relations for these operators can be identified by taking their vacuum expectation values. Thus, using Eq. (91) and its adjoint, we have

$$\langle 0|[\Phi_{k'}^\dagger(\bar{x}'),\Phi_k(\bar{x})]|0\rangle =2\pi\langle\bar{x}'|Q_{k',-1}Q_{k,-1}|\bar{x}\rangle-2\pi\langle\bar{x}|Q_{k1}Q_{k'1}|\bar{x}'\rangle. \tag{92}$$

This can be reduced with the aid of Eq. (45) to

$$\langle 0|[\Phi_{k'}^\dagger(\bar{x}'),\Phi_k(\bar{x})]|0\rangle=2\pi\delta(k^2-k'^2) \times[\langle\bar{x}'|Q_{k,-1}|\bar{x}\rangle-\langle\bar{x}|Q_{k1}|\bar{x}'\rangle]. \tag{93}$$

Substituting the expression for Q_{kr} in Eq. (38) gives

$$\langle 0|[\Phi_{k'}^\dagger(\bar{x}'),\Phi_k(\bar{x})]|0\rangle=-\delta(k^2-k'^2) \times\sum_r r\int\exp[i\bar{p}_{kr}\cdot(\bar{x}'-\bar{x})] \times d\mathbf{p}(2\epsilon_k)^{-1}(2\pi)^{-3} =-i\Delta(\bar{x}'-\bar{x})\delta(k^2-k'^2), \tag{94}$$

where $\Delta(\bar{x}'-\bar{x})$ is a standard singular function found in field theory.[11] If the vacuum state is normalized to unity, we have

$$[\Phi_{k'}(\bar{x}'),\Phi_k^\dagger(\bar{x})]=-[\Phi_k^\dagger(\bar{x}),\Phi_{k'}(\bar{x}')] =i\Delta(\bar{x}'-\bar{x})\delta(k^2-k'^2). \tag{95}$$

In a similar manner, it can be shown that

$$[\Phi_{k'}(\bar{x}'),\Phi_k(\bar{x})]=[\Phi_{k'}^\dagger(\bar{x}'),\Phi_k^\dagger(\bar{x})]=0. \tag{96}$$

These are the commutation relations for complex field operators found in many texts,[11] except for the factor $\delta(k^2-k'^2)$ resulting from our use of a continuous mass spectrum.

An expansion in terms of momentum state operators can be found if we first replace Q_{kr} by the expression in Eq. (38) to obtain

$$\langle 0|\Phi_{k1}(\bar{x})=(2\pi)^{1/2}\int\exp(i\bar{p}_{k1}\cdot\bar{x}) \times\langle\bar{p}_{k1}|d\mathbf{p}(2\epsilon_k)^{-1}(2\pi)^{-4} \tag{97}$$

and

$$\Phi_{k,-1}(\bar{x})|0\rangle=(2\pi)^{1/2}\int\exp(-i\bar{p}_{k,-1}\cdot\bar{x}) \times|\bar{p}_{k,-1}\rangle d\mathbf{p}(2\epsilon_k)^{-1}(2\pi)^{-4}. \tag{98}$$

[11] K. Nishijima, *Fields and Particles* (W. A. Benjamin, Inc., New York, 1969), pp. 47–48.

Momentum creation operators can be defined by

$$a_{kr}^{\dagger}(\mathbf{p})|0\rangle=(2\epsilon_k)^{-1/2}|\bar{p}_{kr}\rangle. \quad (99)$$

This equation and its adjoint together with Eqs. (97) and (98) allow us to write

$$\Phi_{k1}(\bar{x})=(2\pi)^{1/2}\int \exp(i\bar{p}_{k1}\cdot\bar{x}) \times a_{k1}(\mathbf{p})d\mathbf{p}(2\epsilon_k)^{-1/2}(2\pi)^{-4} \quad (100)$$

and

$$\Phi_{k,-1}(\bar{x})=(2\pi)^{1/2}\int \exp(i\bar{p}_{k,-1}\cdot\bar{x}) \times a_{k,-1}^{\dagger}(\mathbf{p})d\mathbf{p}(2\epsilon_k)^{-1/2}(2\pi)^{-4}. \quad (101)$$

It is clear that these operators have the space and time dependence required to satisfy the Klein-Gordon equation. They are, therefore, the continuous mass spectrum analogs of the standard field-theory operators for spin-zero particles.

The commutation relation for the momentum field operators can again be readily obtained from Eq. (99) and its adjoint. The vacuum expectation value for the commutation is

$$\langle 0|[a_{k'r'}(\mathbf{p}'),a_{kr}^{\dagger}(\mathbf{p})]|0\rangle=(2\epsilon_k)^{-1}\langle\bar{p}'_{k'r'}|\bar{p}_{kr}\rangle =\delta_{r'r}\delta(\mathbf{p}'-\mathbf{p})\delta(k'^2-k^2). \quad (102)$$

Thus,

$$[a_{k'r'}(\mathbf{p}'),a_{kr}^{\dagger}(\mathbf{p})]=\delta_{r'r}\delta(\mathbf{p}'-\mathbf{p})\delta(k'^2-k^2). \quad (103)$$

Similarly,

$$[a_{k'r'}(\mathbf{p}'),a_{kr}(\mathbf{p})]=[a_{k'r'}^{\dagger}(\mathbf{p}'),a_{kr}^{\dagger}(\mathbf{p})]=0. \quad (104)$$

Again, these are the standard relations[11] except for factors $\delta(k'^2-k^2)$.

It is now possible to determine the spread in space and time of the state $\Phi_k^{\dagger}(\bar{x})|0\rangle$. From Eqs. (89)–(91) and their adjoints, it is clear that

$$\langle\bar{x}'|\Phi_k^{\dagger}(\bar{x})|0\rangle=(2\pi)^{1/2}\langle\bar{x}'|Q_{k1}|\bar{x}\rangle. \quad (105)$$

With the definition of Q_{k1} in Eq. (38), Eq. (105) reduces to

$$\langle x'|\Phi_k^{\dagger}(\bar{x})|0\rangle=(2\pi)^{1/2} \times\tfrac{1}{2}\int \exp[i\bar{p}_{kr}\cdot(\bar{x}'-\bar{x})]\epsilon_k^{-1}d\mathbf{p}(2\pi)^{-4}. \quad (106)$$

This integral can be identified with the function Δ^+ (see Ref. 11, p. 36), so that

$$\langle\bar{x}'|\Phi_k(\bar{x})|0\rangle=(2\pi)^{-1/2}i\Delta^+(\bar{x}'-\bar{x}). \quad (107)$$

This function contains a δ-function singularity on the light cone and decreases exponentially outside of it.

If the state ket on the positive mass shell is given by Eq. (59), then

$$\langle 0|\Phi_k(\bar{x})|\psi\rangle=(2\pi)^{1/2}\langle\bar{x}|Q_{k1}|\psi\rangle=(2\pi)^{1/2}\langle\bar{x}|\psi k1\rangle. \quad (108)$$

XIII. DISCUSSION

We have seen that efforts to define position eigenstates that are (1) orthogonal, (2) possible particle states, and (3) transform properly under Lorentz transformations have failed. It is necessary, therefore, to give up one of these conditions. Newton and Wigner[2] gave up condition (3). Philips[5] gave up (1). We have proposed here the giving up of condition (2). Our proposal has added the advantage that a time operator appears and places time and space on the same footing, as Einstein would have preferred.

To accomplish this goal for spin-zero noninteracting particles, we have been forced to alter some of the usual quantum-mechanical elements. These are (1) a Hilbert space spanned by the representations of the spatial rotation and displacement group containing operators corresponding to physical measurements, and (2) vectors (kets) $|\Psi\rangle$ and $|\mathbf{x}\rangle$ corresponding, respectively, to the state of a one-particle system and eigenkets of the space-position measurement operators so that $\langle\mathbf{x}|\psi\rangle$ is the space-position amplitude, and (3) a wave equation containing the particle rest mass to determine the development of $|\psi\rangle$ with time. These have been replaced by (1) a Hilbert space spanned by the representations of the inhomogeneous Lorentz group (spin zero) containing operators corresponding to physical measurements, (2) kets $|\psi\rangle$ and $|\bar{x}\rangle$ corresponding, respectively, to the state of a one-*event* system and eigenkets of the *space-time* position measurement operators so that $\langle\bar{x}|\psi\rangle$ is the space-time position amplitude, and (3) the requirement that a particle state ket must lie on a mass shell.

Field operators in this extended Hilbert space with the same commutation relations as those of standard quantum field-theoretic treatments have been defined. They satisfy the Klein-Gordon equation. Our formalism allows the determination of the extent in space and time of such states as $\Phi^{\dagger}(\bar{x})|0\rangle$. The identification of these field operators together with the proof that $\langle\bar{x}|\psi\rangle$ satisfies the Klein-Gordon equation puts this formalism in contact with the more standard treatments.

This formalism reduces to the usual nonrelativistic quantum theory since the Klein-Gordon equation reduces to the Schrödinger equation and the current expression in Eq. (78) reduces to the usual one in the limit as $c\rightarrow\infty$.

The displacement generators of the Poincaré group have been identified as the energy-momentum operators, and their commutation relations with the space-time operators have been found. The commutators for the space operators are the same as those in the more standard theory while an additional energy-time commutation relation has been discovered.

The state ket $|\psi\rangle$ is for a single event and exists in a Hilbert space containing all masses from zero to infinity as well as imaginary ones. Such a Hilbert space is necessary to define the eigenkets of space-time operators. A physical particle, however, is found to be limited to states lying on a specified mass shell. Thus only a very limited part of the Hilbert space is available for particle states. This raises the question of why states that are superpositions of different mass eigenkets are not found in nature. The formalism presented here sheds no light on this question.

It is a straightforward matter to extend this formalism to include states of nonzero spin. This extension will be presented in another paper.

The above Hilbert space has been used to derive the Schrödinger equation for a *single* particle. It is, in a sense, a subspace of a Hilbert space containing vectors belonging to an arbitrary number of events. A discussion of multiple event states will be left to another paper. It is clear, however, that the Green's functions for the Schrödinger equations derived for one-particle states can be used to join points of interaction between particles at Feynman vertices.[6] If a field-operator approach is preferred, these operators can be found according to the procedure described in Sec. XII.

It is clear that the eigenvalues of the space-time position operators will determine the location of Feynman vertices, the points where different kinds of particles interact. This does not seem surprising since a space-time position measurement will involve the interaction at a vertex of some probing particle and the particle being observed.

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Evidence for Heavy-Particle Production Processes at Energies above 2×10^{11} eV

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Cosmic-ray flux measurements in the energy region 10^{10}–10^{14} eV obtained by calorimeters on the satellites Proton I and II have shown results that are at variance with previous data. While a single power law provides an approximate fit to the all-particle spectrum, the primary proton flux falls sharply at energies above $\sim5\times10^{11}$ eV, indicating that at high energies protons become progressively scarcer in the primary flux. The cross section for particle production by protons on carbon is found to rise by 20% in the interval between 2×10^{10} and 10^{12} eV. Assuming that, in the energy region of interest, (1) the real proton flux is given by a single power law, and (2) the nuclear composition remains constant, we show that the satellite flux measurements can be explained by an energy-loss mechanism in the calorimeter, the loss being a function of the energy per nucleon rather than the total energy. Furthermore, this "X" process has a cross section of the right magnitude to account for the p-carbon cross-section measurements. The X process could be described in terms of particle production or dissociation of the primary protons.

I. INTRODUCTION

Measurements of the primary cosmic-ray flux and the p-carbon cross sections at high energies performed by the artificial earth satellites of the Proton series[1–3] have yielded results at variance with other data and with currently held beliefs.

The detector used by Grigorov *et al.* consisted of pairs of ionization calorimeters,[1] each three nuclear mean free paths long, together with suitable triggering and particle-counting hardware. Carbon and polyethylene targets could be inserted in the path of the incident primary particles. These instruments were flown in Protons I, II, and III, and in November of 1968, a fourth satellite, Proton IV, carrying more advanced instrumentation, was launched.[4]

The results of the measurements on the cosmic-ray flux in the energy range 10^{10}–2×10^{14} eV show[2] an integral spectrum for the total particle flux that the experimenters fitted by a single power law with exponent $\gamma=1.74\pm0.06$.

The proton flux is found to behave in a surprising way. While its behavior is similar to that of the all-

[1] N. L. Grigorov *et al.*, Kosmich. Issled. Akad. Nauk SSSR **5**, 383 (1967).
[2] N. L. Grigorov *et al.*, Kosmich. Issled. Akad. Nauk SSSR **5**, 395 (1967).
[3] N. L. Grigorov *et al.*, Kosmich. Issled. Akad. Nauk SSSR **5**, 420 (1967).
[4] *Space Daily*, Nov. 21, 1968, p. 88.