Structure of Regge Poles and Their Residues at $t=0$ for General Two-Body-to-Two-Body Reactions

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A convenient method is introduced to analyze the requirements imposed by the analyticity of the full helicity amplitudes on the structure of Regge poles and their residues at $t=0$ for two-body-to-two-body reactions with general masses and spins. This method enables us to visualize the structure of daughter trajectories and conspirators clearly. Also, in practice, this method enables us to make the following derivation easily for reactions with arbitrary spins. (1) The most singular parts of the daughter and conspirator residues at $t=0$ are calculated for unequal-mass–unequal-mass reactions and unequal-mass–equal-mass reactions. Then, through factorization, the nonvanishing parts of the daughter and conspirator residues are obtained for the equal-mass–equal-mass reactions. They are identified with a one-Lorentz-pole expansion. (2) In calculating the daughter and conspirator residues, the analyticity requirements of both the t-channel and the s-channel helicity amplitudes are satisfied. Therefore, the conspiracy equations are shown to be satisfied explicitly. (3) The restrictions on the slopes of the daughter trajectories are also obtained. Their independence of the external masses and spins is shown. (4) The restrictions on the slopes of the conspirators are also calculated. We obtain an interesting new result: For a trajectory of quantum number M, at $t=0$, the trajectories $\alpha_+(t)$ and $\alpha_-(t)$ are equal, and likewise their derivatives up to the $(M-1)$th. Before carrying out all these calculations, all the t factors of the Regge residue have to be determined. By introducing a quantum number M in the unequal-mass–unequal-mass reactions, the t factors of the parent as well as the daughter residues are uniquely determined using the conventional method of analyticity and factorization. This quantum number M is identified to be the $O(4)$ M in the equal-mass–equal-mass reactions. We note that if the definition of the quantum number M is not affected by the coincidence of $\alpha(t)$ with an integer at $t=0$, then the trajectory $\alpha(0)$ will choose sense if $M<\alpha(0)$ and choose nonsense if $M>\alpha(0)$. At the end of the paper, a discussion is given on the implications for the group-theoretical approach to the Regge-pole theory.

INTRODUCTION

THE Regge trajectory $\alpha(t)$ and its residue $\beta(t)$ in equal-mass spinless reactions are real analytic functions having only the dynamical cuts starting at threshold.[1] Both $\alpha(t)$ and $\beta(t)$ are analytic at $t=0$. In the general two-to-two reactions, the Regge-pole structure at $t=0$ is complicated by two features: high spins and unequal masses. In the unequal-mass cases,[2] the high-energy expansion of each Regge term is singular at $t=0$. This behavior is in contrast with the analyticity property of the full amplitude. Consequently, an infinity of integer-spaced trajectories must exist in order to cancel the singularities at $t=0$. These are the daughter trajectories.[2] In the case of high spins, additional singularities exist in the residue functions $\beta_{\lambda\mu}(t)$; these singularities can be found by first analyzing the kinematic singularities[3–5] of the full amplitude $f_{\lambda\mu}(t)$ and then imposing factorization[6] on $\beta_{\lambda\mu}(t)$. Factorization is a result of unitarity and the simple-pole assumption. However, this analytic approach appears to miss two important properties: first, the conspiracy relations[7,8] and how they are satisfied; and second, the additional $O(4)$ symmetry at $t=0$ in the equal-mass reactions.[9–11] With the one-Lorentz-pole assumption at $t=0$ in the equal-mass reaction, many specific results can be obtained—for example, the existence of a parity doublet and the daughter trajectories, and the way they

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collaborate to satisfy the conspiracy relations. However, the one-Lorentz-pole approach cannot be used directly for unequal-mass cases. Therefore, these two approaches (Lorentz-pole and analyticity) were used in a complementary way[12]: first using the one-Lorentz-pole assumption for equal-mass–equal-mass reactions (*EE* reactions), and then using factorization and analyticity for unequal-mass–unequal-mass reactions (*UU* reactions) and unequal-mass–equal-mass reactions (*UE* reactions). This approach has been used to make experimental predictions for some reactions.[13] However, in all these works, the t factors are determined for the residues of *only the leading one or two* trajectories. How the daughter trajectories sum in general reactions has never been fully understood. Only in models has this been dealt with completely.[14,15] But they all have the drawback of specifying too much, e.g., (1) they have parallel trajectories, (2) they can only specify that a trajectory's coupling does or does not vanish at the equal-mass vertex at $t=0$. There is no way of recovering a trajectory and its coupling from $t=0$.

Recently, attempts were made to use the original analyticity and factorization method to study the problem thoroughly. The analysis involved can be divided into two categories.

(1) Studying the t factors of the residues of the leading one or two trajectories: The existence of conspirators in the *UU* reactions can be established in this way.[16] Results have been obtained for many specific reactions.[17] A neat and general solution was obtained by Frampton[18] for the parent trajectories.

(2) Calculating the daughter and conspirator residues explicitly: It is found that the most singular part of the daughter residues in the *UU* and the *UE* reactions can uniquely determine the nonvanishing parts of the daughter residues in the *EE* reactions. It is shown that the result is just a one-Lorentz-pole expansion. For the spinless case this was done by Taylor[19] using parallel daughter trafectories, and later by Bronzan and Jones[20] without assuming parallel daughter trajectories. For the total spin-one case, it was done by Drago and Di Vecchia[21] and by Bronzan.[21] This is a major breakthrough in understanding the problem.

Weiss[22] takes a different approach. Assuming that particles of arbitrarily high spin exist, he shows that to satisfy all the conspiracy relations of all the equal-mass reactions with arbitrarily high spins, a one-Lorentz-pole expansion is one of the two possible solutions. However, the other solution can only be eliminated by considering first the *UU* and the *UE* reactions.[19,20] Using analyticity in the *UU* and the *UE* reactions, conditions on the slope of daughter trajectories can also be obtained.[23] However, the methods so far used are quite complicated and are almost impossible to generalize to high spins. Also because of the complexity, it is not very easy to see the structure.

In this paper, we introduce a convenient method of analyzing order by order the analyticity requirements of the full helicity amplitudes on the structure of Regge poles and their residues at $t=0$ for two-body-to-two-body reactions of general masses and spins. A qualitatively different property develops for reactions of total spin greater than 1: The most singular part of the original helicity residues of the daughter trajectories cannot uniquely be determined in the *UE* and the *EE* reactions. With the help of the constraint equations only the new daughter residues $\hat{\beta}_{\mu,s\lambda}$ defined in Eq. (3.38) can be uniquely determined. Our main purpose is to give a clear and complete picture of the structure; therefore, many results previously derived in the spinless and total-spin-1 case are also included in the paper.[24] The organization of the paper is as follows.

In Sec. I, we demonstrate the method in the spinless case. We show the relation between the number of additional zeros in the residue at $t=0$ and the number of arbitrary parameters needed to determine the most singular parts of the daughter residues. Using this method, the most singular parts of the daughter residues are calculated in the *UU* and *UE* reactions, then through factorization the daughter residues in the *EE* reaction are obtained. They are identified with the one-Lorentz-pole expansion. The conditions on the derivatives of the daughter trajectories are obtained. These conditions are shown to be consistent in the *UU* and the *UE* reactions.

In Sec. II, the t factors of the parent as well as the daughter trajectories are derived. The results are summarized in Fig. 1. As indicated in Refs. 14 and 18, the

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quantum number M is introduced in the UU reaction to fix the t factors of the residues.[25] The results for the parent trajectories are the same as Frampton's and the method used is the same as in Ref. 6. We include the discussions here for completeness and for discussions in Sec. III.

In Sec. III, we calculate essentially the same things as in Sec. I but for reactions of high spins. The structure here is more complex. The conspirator and its daughters must be included in order to satisfy the analyticity of the t-channel amplitudes $f_{cA,Db}{}^t$. We show that for trajectories of quantum number M, the functions $\alpha_{\kappa,+}(t)$ and $\alpha_{\kappa,-}(t)$ are equal up to the $(M-1)$th derivative at $t=0$. In the UE reactions, the residues are explicitly constructed so that they satisfy the further analyticity requirement of the s-channel helicity amplitudes $f_{cc,ab}{}^s$ in addition to that of $f_{cA,Db}{}^t$. Therefore, the way the conspiracy relations are satisfied is explicitly shown.

In Appendix A, we list some useful formulas. In Appendix B, we elaborate the Andrews-Gunson[26] method of calculating the expansion coefficients of the E functions. In Appendix C, we show that the conditions on the slopes of the daughter trajectories in the UE reactions are consistent with those in the UU reactions for general spins.

I. METHOD OF ANALYZING STRUCTURE OF REGGE TRAJECTORIES AND RESIDUES REQUIRED BY ANALYTICITY AND FACTORIZATION AT $t=0$: SPINLESS REACTIONS

A. Residues of Daughter Trajectories at $t=0$

The Regge-pole contribution to a spinless amplitude is given by

$$f(s,t)=(1-e^{-i\pi\alpha}/\sin\pi\alpha)\beta(t)\mathcal{P}_{0,0}{}^{\alpha}(z_t)\,, \qquad (1.1)$$

where

$$\mathcal{P}_{0,0}{}^{\alpha}(z_t)\equiv\tan\pi\alpha\; Q_{-\alpha-1}(z_t)\,,$$

$$z_t\equiv\Big[2st+t^2-t\sum_i m_i^2+(m_d^2-m_b^2)\times(m_c^2-m_a^2)\Big]\Big/\mathcal{T}_{ac}\mathcal{T}_{bd}\,,$$

$$\mathcal{T}_{ac}{}^2\equiv[t-(m_a+m_c)^2][t-(m_a-m_c)^2]=4t(p_t)^2\,,$$

$$\mathcal{T}_{bd}{}^2\equiv[t-(m_b+m_d)^2][t-(m_b-m_d)^2]=4t(p_t')^2\,.$$

The t factor of the residue is $\beta(t)\sim(p_tp_t')^{\alpha}$:

$$\beta^{UU}(t)\sim t^{-\alpha} \qquad (1.2)$$

and

$$z_t-1=st/s_1+O(t)\,,$$

where

$$s_1=\tfrac{1}{2}|(m_a{}^2-m_c{}^2)(m_b{}^2-m_d{}^2)|^{1/2}. \qquad (1.3)$$

For large energy s, the function $Q_{-\alpha-1}(z_t)$ can be expanded in terms of (z_t-1). The $t^{-\alpha}$ singularity of $\beta(t)$ is canceled only by the leading term $(z_t-1)^{\alpha}$, which gives an s^{α} term in $f(s,t)$. All the lower-order terms, $(z_t-1)^{\alpha-1}$, $(z_t-1)^{\alpha-2}$, etc., give terms like $s^{\alpha}(st)^{-1}$, $s^{\alpha}(st)^{-2}$, etc., which contribute poles to $f(s,t)$ at $t=0$. This violates the analyticity of $f(s,t)$. The daughter trajectories,[2] $\alpha_\kappa=\alpha-\kappa$, where $\kappa=1,2,\cdots$, are introduced such as to cancel these singularities at $t=0$—that is, to make Eq. (1.1) become

$$f(s,t)=\gamma(t)t^{-\alpha}\frac{1}{\cos\pi\alpha}(1-e^{-i\pi\alpha})\times\sum_{\text{all }\kappa=0}^{\infty}a^{\kappa}(t)Q_{-\alpha_\kappa-1}(z_t)\,, \qquad (1.4)$$

so that there are no lower-order terms in (z_t-1) except $(z_t-1)^{\alpha}$. Therefore, the daughter-residue a's are required to satisfy

$$\sum_{\text{all }\kappa=0}^{\infty}a^{\kappa}(t=0)Q_{-\alpha_\kappa-1}(z_t)=[\tfrac{1}{2}(z_t-1)]^{\alpha}, \qquad (1.5)$$

where the factor $\frac{1}{2}$ in front of (z_t-1) is just for convenience in the definition of a^{κ}. Notice that the a^{κ}'s are regular in t. Therefore, all the daughters have the same t factor as the parent. Also notice that even though $f(s,t)$ has only one term in $(z_t-1)^{\alpha}$, $f(s,t)$ does have all terms s^{α}, $s^{\alpha-1}$, $s^{\alpha-2}$, etc., at $t=0$. Obviously the daughters must have the same phase as well as the same (parity)×(J parity) as the parent, so the odd daughters must have opposite parity to that of the parent. Also, all the daughters must have the same quantum numbers as the parent. The expansion coefficients a^{κ} of Eq. (1.5) can be calculated using the method given by Andrews and Gunson.[26] We give the detailed calculation in Appendix B. The result is

$$a^{\kappa}=g_{0,0}{}^{\kappa,\alpha}\quad\text{with}\quad \alpha-\alpha_\kappa=\kappa$$
$$=\pi^{-1}\tan\pi\alpha[\Gamma(\alpha+1)]^2(-)^{\kappa}(2\alpha_\kappa+1)\times[\Gamma(\kappa+1)\Gamma(\alpha+\alpha_\kappa+2)]^{-1}, \qquad (1.6)$$

where the general definition of $g_{0,0}{}^{\kappa,\alpha}$ is given by Eq. (B15) in Appendix B.

For EU reactions, similar arguments go through. The only difference is in the kinematics. That is,

$$\beta^{UE}(t)\sim(\sqrt{t})^{-\alpha} \qquad (1.7)$$

and

$$z_t=(s\sqrt{t})/s_2+O(t)\,, \qquad (1.8)$$

where $s_2=\frac{1}{2}m|m_a{}^2-m_c{}^2|$. Since $Q_{-\alpha-1}(z_t)$ contains only terms of z^{α}, $z^{\alpha-2}$, $z^{\alpha-4}$, etc., only even daughters are needed so that

$$\sum_{\text{even }\kappa=0}^{\infty}b^{\kappa}Q_{-\alpha_\kappa-1}(z_t)=(\tfrac{1}{2}z_t)^{\alpha}. \qquad (1.9)$$

Actually, the Pauli principle and the conservation of G parity imply that the odd trajectories and the even

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trajectories cannot both couple to the same equal-mass identical-particle or particle-antiparticle pair, because the even and the odd trajectories have exactly the same quantum numbers but opposite parity. In a UE reaction, where only the odd trajectories can couple to the equal-mass pair, their residues will have the following t factor:

$$\beta^{UE}(t)\sim(\sqrt{t})^{-\alpha+1}. \qquad (1.10)$$

The equation corresponding to Eq. (1.9) is

$$\sum_{\text{odd }\kappa=1}^{\infty} b^{\kappa}Q_{-\alpha_\kappa-1}(z_t)=(\tfrac{1}{2}z_t)^{\alpha-1}. \qquad (1.11)$$

By factorization,

$$\beta^{EE}(t)\beta^{UU}(t)=[\beta^{UE}(t)]^2, \qquad (1.12)$$

and from Eqs. (1.2) and (1.10), we obtain

$$\beta^{EE}(t)\sim t, \qquad (1.13)$$

for odd trajectories. Therefore, the odd daughters decouple from an equal-mass pair at $t=0$ irrespective of whether the internal quantum number allows the coupling or not. We shall see in Sec. II that our solution here corresponds to the $M=0$ solution. From Eqs. (B26) and (B30) in Appendix B, the solution to Eq. (1.9) is

$$\begin{aligned}b^{\kappa}(t=0)&=h_{00}{}^{\kappa,\alpha}\\&=\pi^{-1}\tan\pi\alpha\Gamma(\alpha+1)(\tfrac{1}{2})^{\kappa}(2\alpha_\kappa+1)\Gamma(\alpha_\kappa+1)\\&\quad\times[\Gamma(\kappa+1)\Gamma(2\alpha_\kappa+2)]^{-1}\\&\quad\quad\times F(-\kappa,\alpha_\kappa+1;2\alpha_\kappa+2;2)\\&=\tan\pi\alpha\Gamma(\alpha+1)(-)^{\kappa/2}(\tfrac{1}{2})^{\alpha+\alpha_\kappa+1}(2\alpha_\kappa+1)\\&\quad\times[\Gamma(\kappa+1)\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{3}{2})\Gamma(-\tfrac{1}{2}\kappa+\tfrac{1}{2})]^{-1},\end{aligned} \qquad (1.14)$$

with $\alpha-\alpha_\kappa=\kappa$. Notice that b^{κ} is zero if κ is a positive odd integer.

For the EE reactions, we can find the nonvanishing part of the daughter residues by factorization:

$$\begin{aligned}c^{\kappa}(t=0)&=(b^{\kappa})^2/a^{\kappa}\\&=\pi\tan\pi\alpha(\tfrac{1}{2})^{2(\alpha+\alpha_\kappa+1)}(2\alpha_\kappa+1)\Gamma(\alpha+\alpha_\kappa+2)\\&\quad\times[\Gamma(\kappa+1)]^{-1}[\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{3}{2})\Gamma(-\tfrac{1}{2}\kappa+\tfrac{1}{2})]^{-2}.\end{aligned} \qquad (1.15)$$

Notice that $c^{\kappa}=0$ if κ is a positive odd integer. This is consistent with Eq. (1.13). Checking with Eqs. (A9) and (A10), one finds that

$$\sum_{\text{even }\kappa=0}^{\infty} c^{\kappa}(t=0)Q_{-\alpha_\kappa-1}(z_t)$$

is just proportional to an $M=0$ $O(4)$ pole. Therefore, via factorization, the analyticity requirements in unequal-mass reactions necessarily lead to the one-Lorentz-pole solution. This was first shown in Refs. 19 and 20.

Notice that we have just discussed a case where the residue can have the most singular form allowed by the kinematics given in Eqs. (1.2) and (1.7). If there happen to be additional zeros of t in the residues (say, in the UU reaction), then

$$\beta(t)\sim t^{-\alpha+n}, \qquad (1.16)$$

where n is a positive integer. Then the analyticity of the full amplitude will allow lower-order terms in z_t-1 on the right-hand side of Eq. (1.5), i.e.,

$$\begin{aligned}\sum_{\text{all }\kappa=0}^{\infty} a^{\kappa}(t=0)Q_{-\alpha_\kappa-1}(z_t)&=[\tfrac{1}{2}(z_t-1)]^{\alpha}+d_1[\tfrac{1}{2}(z_t-1)]^{\alpha-1}\\&\quad+\cdots+d_n[\tfrac{1}{2}(z_t-1)]^{\alpha-n},\end{aligned} \qquad (1.17)$$

where the d's are not determined. So the daughter residues are determined up to the first n arbitrary residues. With the g's given in Eq. (B15),

$$a^{\kappa}=g_{0,0}{}^{\kappa,\alpha}+d_1g_{0,0}{}^{\kappa,\alpha-1}+\cdots+d_ng_{0,0}{}^{\kappa,\alpha-n}. \qquad (1.18)$$

Similarly, in the UE reaction, if

$$\beta(t)\sim(\sqrt{t})^{-\alpha}t^{n}, \qquad (1.19)$$

then Eq. (1.11) changes to

$$\begin{aligned}\sum_{\text{even }\kappa=0}^{\infty} b^{\kappa}(t=0)Q_{-\alpha_\kappa-1}(z_t)&=(\tfrac{1}{2}z_t)^{\alpha}+d_1(\tfrac{1}{2}z_t)^{\alpha-2}\\&\quad+\cdots+d_n(\tfrac{1}{2}z_t)^{\alpha-2n},\end{aligned} \qquad (1.20)$$

and with the h's given in Eq. (B26),

$$b^{\kappa}(t=0)=h_{0,0}{}^{\kappa,\alpha}+d_1h_{0,0}{}^{\kappa,\alpha-2}+\cdots+d_nh_{0,0}{}^{\kappa,\alpha-2n}. \qquad (1.21)$$

All these observations will be useful in analyzing restrictions on the slopes of the trajectories.

B. Slopes of Daughter Trajectories at $t=0$

We see now that the analyticity requirement on the full amplitude uniquely determines the positions of the daughter trajectories $\alpha_\kappa(0)$ and the most singular part of their residues in terms of $\alpha(0)$. Obviously, there are also restrictions on the slopes of the trajectories and the less singular parts of the residues. We shall show that $\alpha_\kappa'(0)$ is determined in terms of $\alpha(0)$, $\alpha'(0)$, and $\alpha_1'(0)$. We do not discuss the higher-order derivatives here.

The condition on $\alpha_\kappa'(0)$ comes from the following terms. From the representation

$$\begin{aligned}Q_{-\alpha_\kappa-1}(z_t)&=\frac{1}{2}\frac{[\Gamma(-\alpha_\kappa)]^2}{\Gamma(-2\alpha_\kappa)}[\tfrac{1}{2}(z_t-1)]^{\alpha_\kappa(t)}\\&\quad\times F(-\alpha_\kappa,-\alpha_\kappa;-2\alpha_\kappa;2/(1-z_t)),\end{aligned} \qquad (1.22)$$

we see that all the $\alpha(t)$-dependent power of $z_t{}^{\alpha_\kappa}$ is in $[\tfrac{1}{2}(z_t-1)]^{\alpha_\kappa(t)}$. We can expand this $\alpha_\kappa(t)$ in Eq. (1.22):

$$\begin{aligned}Q_{-\alpha_\kappa-1}(z_t)&=\bar{Q}_{-\alpha_\kappa-1}(z_t)\\&\quad\times\{1+[t\alpha'(0)+t^2\alpha''(0)+\cdots]\ln\tfrac{1}{2}(z_t-1)\\&\quad+[t\alpha'(0)+t^2\alpha''(0)+\cdots]^2\\&\quad\quad\times[\ln\tfrac{1}{2}(z_t-1)]^2+\cdots\},\end{aligned} \qquad (1.23)$$

where $\bar{Q}_{-\alpha_\kappa-1}(z_t)$ is the same as $Q_{-\alpha_\kappa-1}(z_t)$ except that the $\alpha_\kappa(t)$ of $[\frac{1}{2}(z_t-1)]^{\alpha_\kappa(t)}$ in Eq. (1.22) is replaced by $\alpha_\kappa(0)$.

Let us consider the term with $t\alpha'(0)\ln\frac{1}{2}(z_t-1)$ in the UU reaction. Because of this extra factor of t, the constraint equation due to analyticity will be of the form of Eq. (1.17), instead of Eq. (1.6):

$$\sum_{\text{all }\kappa=0}^{\infty} a^\kappa(0)\alpha_\kappa'(0)Q_{-\alpha_\kappa-1}(z_t)\ln\tfrac{1}{2}(z_t-1) =\alpha'(0)[\tfrac{1}{2}(z_t-1)]^{\alpha(0)}\ln\tfrac{1}{2}(z_t-1) +d_1[\tfrac{1}{2}(z_t-1)]^{\alpha(0)-1}\ln\tfrac{1}{2}(z_t-1). \quad (1.24)$$

Since the $a^\kappa(0)$'s are known, we can calculate d_1:

$$d_1=\alpha'(0)a^0(0)P_1{}^\alpha+\alpha_1'(0)a^1P_0{}^{\alpha_1} =[\alpha'(0)-\alpha_1'(0)]a^0(0)P_1{}^\alpha =[\alpha'(0)-\alpha_1'(0)]\times\tfrac{1}{2}\alpha, \quad (1.25)$$

where $P_1{}^\alpha$ is the coefficient of $[\frac{1}{2}(z_t-1)]^{\alpha-1}$ in $Q_{-\alpha-1}(z_t)$, and $P_0{}^{\alpha_1}$ is the coefficient of $[\frac{1}{2}(z_t-1)]^{\alpha_1}$ in $Q_{-\alpha_1-1}(z_t)$. The relations $a^0P_1{}^{\alpha_0}=+\frac{1}{2}\alpha$ and $a^0P_1{}^\alpha+a^1P_0{}^{\alpha_1}=0$ are used. So the calculation of the condition on $\alpha_\kappa'(0)$ amounts to calculating the coefficients of the following expansions:

$$\sum_{\text{all }\kappa=0}^{\infty} a^\kappa(0)\alpha_\kappa'(0)Q_{-\alpha_\kappa-1}(z_t)=\alpha'(0)[\tfrac{1}{2}(z_t-1)]^{\alpha(0)} +\tfrac{1}{2}\alpha[\alpha'(0)-\alpha_1'(0)][\tfrac{1}{2}(z_t-1)]^{\alpha(0)-1}. \quad (1.26)$$

Using Eqs. (1.17), (1.6), and (B15), we obtain the solution to Eq. (1.26):

$$a^\kappa(0)\alpha_\kappa'(0)=\alpha'(0)g_{0,0}{}^{\kappa,\alpha} +\tfrac{1}{2}\alpha[\alpha'(0)-\alpha_1'(0)]g_{0,0}{}^{\kappa,\alpha-1}. \quad (1.27)$$

From Eqs. (1.6) and (B15), which gives

$$g_{0,0}{}^{\kappa,\alpha-1}/g_{0,0}{}^{\kappa,\alpha}=-\kappa(2\alpha-\kappa+1)/\alpha^2, \quad (1.28)$$

we obtain

$$\alpha_\kappa'(0)-\alpha'(0)=[\alpha_1'(0)-\alpha'(0)] \times\kappa[2\alpha(0)-\kappa+1]/2\alpha(0). \quad (1.29)$$

This is so-called mass formula for $M=0$ trajectories derived in Ref. 23. Similar calculations can be made for $[t\alpha'(0)]^n[\ln\frac{1}{2}(z_t-1)]^n$. It is left to the reader to show that the results are consistent with Eq. (1.29).

Obviously, the restriction on the slopes of trajectories ought to be independent of the external masses. Here we are going to calculate the restrictions on the slopes of the daughter trajectories in the UE reactions; then we shall see if they are consistent with those obtained in UU reactions. If they were not, the daughters might be forced to be parallel to the parent. However, we shall show that the restrictions are, in fact, consistent. By an argument similar to that for the UU reaction, we find the equation for the slopes of the daughter trajectories

$$b^\kappa a_\kappa'(0)=h_{0,0}{}^{\kappa,\alpha}\alpha'(0)+d_1h_{0,0}{}^{\kappa,\alpha-2}, \quad \kappa=0,2,4,\cdots. \quad (1.30)$$

In Eq. (1.30),

$$d_1=[\alpha'(0)P_2{}^\alpha h_{0,0}{}^{0,\alpha}+\alpha_2'(0)P_0{}^{\alpha_2}h_{0,0}{}^{2,\alpha}] =[-\alpha'(0)+\alpha_2'(0)]P_0{}^{\alpha_2}h_{0,0}{}^{2,\alpha} =[\alpha_2'(0)-\alpha'(0)] \times\alpha(\alpha-1)/8(2\alpha-1), \quad (1.31)$$

$$h_{0,0}{}^{\kappa,\alpha-2}/h_{0,0}{}^{\kappa,\alpha}=\kappa(2\alpha-\kappa+1)/2(2\alpha-1), \quad (1.32)$$

so Eq. (1.32) becomes

$$\alpha_\kappa'(0)-\alpha'(0)=[\alpha_2'(0)-\alpha'(0)] \times\kappa(2\alpha-\kappa+1)/2(2\alpha-1). \quad (1.33)$$

To check consistency, we substitute the $\kappa=2$ solution from Eq. (1.29), i.e., $\alpha_2'-\alpha'=(\alpha_1'-\alpha')(2\alpha-1)/\alpha$, into Eq. (1.33):

$$\alpha_\kappa'(0)-\alpha'(0)=[\alpha_1'(0)-\alpha'(0)]\kappa(2\alpha-\kappa+1)/2\alpha.$$

This is just Eq. (1.29). Therefore, the consistency is confirmed. We see that the daughters are not forced to be parallel to the parent. In Sec. III and Appendix C, we shall demonstrate that Eq. (1.29) is correct for arbitrary M and independent of external masses,[27] except for $M=1$, where Eq. (1.29) is correct if we replace the α's by the corresponding sums for the parity doublet, $\alpha_++\alpha_-$.

II. t FACTORS OF PARENT AND DAUGHTER RESIDUES AT $t=0$

From the discussion in Sec. I, we see that the calculation of the daughter residues depends crucially on the t factors of the residues. Therefore, we have to discuss the t factors of the residues first.

A. Helicity Formalism

Throughout the paper we shall use the helicity formalism.[28] We first review some of the well-known properties of the helicity amplitudes.[29] Since we are concerned with Regge poles in the t channel, it is natural to use the t-channel helicity amplitudes

$$f_{cA,Db}{}^t\equiv(\sqrt{2}\sin\tfrac{1}{2}\theta_t)^{|\mu-\bar\mu|}(\sqrt{2}\cos\tfrac{1}{2}\theta_t)^{|\mu+\bar\mu|}\bar f_{\bar\mu,\mu}{}^t, \quad (2.1)$$

$$\bar f_{\bar\mu,\mu}{}^t\equiv\sum_J(2J+1)F_{\bar\mu,\mu}{}^J\bar d_{\mu,\bar\mu}{}^J(z_t), \quad (2.2)$$

where

$$\mu\equiv D-b, \quad \bar\mu\equiv c-A,$$

and

$$\bar d_{\mu,\bar\mu}{}^J(z_t)\equiv d_{\mu,\bar\mu}{}^J(z_t)(\sqrt{2}\sin\tfrac{1}{2}\theta_t)^{-|\bar\mu-\mu|}\times(\sqrt{2}\cos\tfrac{1}{2}\theta_t)^{-|\bar\mu+\mu|}.$$

[27] See Eqs. (3.81)–(3.83) and Appendix C.
[28] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[29] The Reggeization formalism is from M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964), Appendix B; we use the same convention as in Refs. 4 and 6.

To be specific, we consider the case $\bar{\mu}\geqslant\mu\geqslant 0$ and use the μ's as the total helicities for the unequal-mass vertex. The parity-conserving helicity amplitudes are

$$f_{\bar{\mu},\mu}{}^{t,\pm}\equiv\bar{f}_{\bar{\mu},\mu}{}^{t}\pm\eta\bar{f}_{-\bar{\mu},\mu}{}^{t}=\sum_J(2J+1)F_{\bar{\mu},\mu}{}^{J,\pm}(t)\bar{d}_{\mu,\bar{\mu}}{}^{J,+}(z_t)+\sum_J(2J+1)F_{\bar{\mu},\mu}{}^{J,\mp}\bar{d}_{\mu,\bar{\mu}}{}^{J-}(z_t) \quad (2.3)$$

and

$$F_{\bar{\mu},\mu}{}^{J\pm}\equiv F_{\bar{\mu},\mu}{}^{J}\pm\eta_J F_{-\bar{\mu},\mu}{}^{J}, \quad (2.4)$$

where

$$\eta=(-)^{\mu+\bar{\mu}}\eta_c\eta_A(-)^{s_c+s_d}$$

and

$$\eta_J=\eta_c\eta_A(-)^{s_c+s_d}.$$

The amplitudes F^{J+} contain only the normal spin-parity states $[P(-)^J=+]$; the amplitudes F^{J-} contain only the abnormal spin-parity states $[P(-)^J=-]$.

B. *UU* Reactions

The full t-channel helicity amplitudes f^t of the UU reactions are analytic at $t=0$. But the $\bar{f}^t$, which are analytic in s, do have singularities at $t=0$, due to the vanishing of $\sin\frac{1}{2}\theta_t$ or $\cos\frac{1}{2}\theta_t\sim\sqrt{t}$ at $t=0$. If[30] $(m_a{}^2-m_c{}^2)\times(m_b{}^2-m_d{}^2)>0$, then $\theta_t=0$ at $t=0$. Therefore, the kinematic singularities at $t=0$ are

$$\bar{f}_{\bar{\mu},\mu}{}^{t,\pm}\sim(\sqrt{t})^{-|\bar{\mu}+\mu|}, \quad (2.5)$$

$$\bar{f}_{-\bar{\mu},\mu}\sim(\sqrt{t})^{-|\bar{\mu}-\mu|}, \quad (2.6)$$

and

$$f_{\bar{\mu},\mu}{}^{t}\sim(\sqrt{t})^{-|\bar{\mu}+\mu|}. \quad (2.7)$$

Notice that the most singular parts of both $f_{\bar{\mu},\mu}{}^{t,+}$ and $f_{\bar{\mu},\mu}{}^{t,-}$ are from $\bar{f}_{-\bar{\mu},\mu}{}^{t}$, so they are correlated.

1. *t Factors of Parent Trajectory*

From Eqs. (2.5)–(2.7),[18,25] the residue $\beta_{\bar{\mu},\mu}{}^{\pm}(t)$ of $F_{\bar{\mu},\mu}{}^{J,\pm}(t)$ for a Regge trajectory α has the following singularity:

$$\beta_{\bar{\mu},\mu}{}^{\pm}(t)\equiv\gamma_{\bar{\mu},\mu}{}^{\pm}(t)(\sqrt{t})^{-(\bar{\mu}+\mu)}t^{-\alpha+\bar{\mu}}=\gamma_{\bar{\mu},\mu}{}^{\pm}(t)t^{-\alpha}(\sqrt{t})^{\bar{\mu}-\mu}, \quad (2.8)$$

where the γ's are analytic in t. The first part of the singularity is from the full helicity amplitudes, and the second part is from partial-wave projection. The singularity in Eq. (2.8) is the most singular behavior allowed by analyticity. Considering the following residues:

$$\beta_{\bar{\mu},\bar{\mu}}{}^{\pm}=\gamma_{\bar{\mu},\bar{\mu}}{}^{\pm}t^{-\alpha}, \quad (2.9)$$

$$\beta_{\mu,\mu}{}^{\pm}=\gamma_{\mu,\mu}{}^{\pm}t^{-\alpha}, \quad (2.10)$$

$$\beta_{\bar{\mu},\mu}{}^{\pm}=\gamma_{\bar{\mu},\mu}{}^{\pm}t^{-\alpha}(\sqrt{t})^{\bar{\mu}-\mu}, \quad (2.11)$$

we clearly see that they do not satisfy factorization, i.e., it is not the case that

$$(\beta_{\bar{\mu}\mu}{}^{\pm})^2=\beta_{\bar{\mu}\bar{\mu}}{}^{\pm}\beta_{\mu\mu}{}^{\pm}. \quad (2.12)$$

Some analytic zeros in t must be introduced to either $\beta_{\mu\mu}{}^{\pm}$ or $\beta_{\bar{\mu}\bar{\mu}}{}^{\pm}$ or both.[6] After applying this discussion to all possible μ and $\bar{\mu}$, one will find that there is always *one and only one* diagonal residue that can resume its most singular form $t^{-\alpha}$. As indicated in Refs. 14 and 18, a quantum number M (always positive by convention) can be uniquely defined for a Regge trajectory α, as follows: A Regge trajectory α has a quantum number M if its coupling to the helicity state $\bar{\mu}=\mu=M$ has the most singular form allowed by analyticity, i.e.,

$$\beta_{M,M}{}^{\pm}=\gamma_{M,M}{}^{\pm}t^{-\alpha}. \quad (2.13)$$

From Eq. (2.11),

$$\beta_{M,\mu}{}^{\pm}=\gamma_{M,\mu}{}^{\pm}t^{-\alpha}(\sqrt{t})^{M-\mu}. \quad (2.14)$$

Then by factorization, $\gamma_{\mu\mu}{}^{\pm}$ must have a zero of the form $t^{M-\mu}$; therefore,

$$\beta_{\mu,\mu}{}^{\pm}=\gamma_{\mu,\mu}{}^{\pm}t^{-\alpha}t^{M-\mu}. \quad (2.15)$$

Similar arguments can be applied to all possible values of $\bar{\mu}$ and μ. One finds that both analyticity and factorization are satisfied if

$$\beta_{\bar{\mu},\mu}{}^{\pm}=\gamma_{\bar{\mu},\mu}{}^{\pm}t^{-\alpha}(\sqrt{t})^{|M-|\bar{\mu}||}(\sqrt{t})^{|M-|\mu||}; \quad (2.16)$$

here the γ's are analytic and without zeros, by the definition of M. Therefore, the behavior of $\beta_{\mu,\mu}{}^{\pm}$ at $t=0$ is uniquely determined by M. Notice that in addition to $\beta_{MM}{}^{\pm}$, the $\beta_{M\mu}{}^{\pm}$ also have the most singular t factor allowed by the kinematics.

Here we should recall the following point[16]: For given α and M, the $\beta_{\bar{\mu},\mu}{}^{+}$ and $\beta_{\bar{\mu},\mu}{}^{-}$ have the same $t=0$ behavior. But notice that $\beta_{\bar{\mu},\mu}=\beta_{\bar{\mu},\mu}{}^{+}+\beta_{\bar{\mu},\mu}{}^{-}$ has a less singular factor than $\beta_{-\bar{\mu},\mu}=\eta_J(\beta_{\bar{\mu},\mu}{}^{+}-\beta_{\bar{\mu},\mu}{}^{-})$. To be specific, we consider $\beta_{MM}{}^{\pm}\sim t^{-\alpha}$. From Eqs. (2.5) and (2.6), we find

$$\beta_{MM}\sim t^{-\alpha+M}, \quad \text{but} \quad \beta_{-MM}\sim t^{-\alpha}. \quad (2.17)$$

This means that $\beta_{M,M}{}^{+}$ and $\beta_{M,M}{}^{-}$ must be correlated, i.e.,

$$\gamma_{MM}=\gamma_{MM}{}^{+}+\gamma_{MM}{}^{-}\sim t^{M}. \quad (2.18)$$

In general,

$$\gamma_{\bar{\mu},\mu}\equiv\gamma_{\bar{\mu}\mu}{}^{+}+\gamma_{\bar{\mu}\mu}{}^{-}\sim\text{the less singular of} \begin{cases}(\sqrt{t})^{|\bar{\mu}+\mu|-|M-\bar{\mu}|-|M-\mu|}\\ 1\end{cases} \quad (2.19)$$

so that

$$\beta_{\bar{\mu},\mu}=\gamma_{\bar{\mu},\mu}\times\text{the less singular of} \begin{cases}t^{-\alpha}(\sqrt{t})^{|\bar{\mu}+\mu|}\\ t^{-\alpha}(\sqrt{t})^{|M-\bar{\mu}|}(\sqrt{t})^{|M-\mu|}\end{cases}. \quad (2.20)$$

But

$$\beta_{-\bar{\mu},\mu}=\gamma_{-\bar{\mu},\mu}t^{-\alpha}(\sqrt{t})^{|M-\bar{\mu}|}(\sqrt{t})^{|M-\mu|}. \quad (2.21)$$

The main point here is that for any trajectory α with $M>0$, we can find a helicity amplitude (such as f_{MM}) relating the plus amplitude and the minus state. Thus there must exist another trajectory with opposite spin-parity, which is usually called the conspirator of the former. Only an $M=0$ trajectory does not have to have a conspirator.

2. *t Factors of Daughter Trajectory*

For those helicity amplitudes whose residues have the most singular behavior allowed, such as $f_{M,\mu}{}^{t,\pm}$, the $t=0$ behavior of the residues of the daughters is uniquely determined by that of the residues of the parent. A detailed daughter cancellation mechanism is given in Sec. III. Just as in the spinless case, it turns out that all the daughter and conspirator residues $\beta_{M,\mu}{}^{\kappa,\pm}$ should have the same t factor as the parent $\beta_{M,\mu}{}^{\pm}$. Then by factorization, all $\beta_{\bar{\mu},\mu}{}^{\kappa}$ should have the same t factor as the parent, i.e.,

$$\beta_{\bar{\mu},\mu}{}^{\kappa,\pm}\sim t^{-\alpha}(\sqrt{t})^{|M-|\mu||}(\sqrt{t})^{|M-|\bar{\mu}||}. \qquad (2.22)$$

C. *UE* and *EE* Reactions

1. *t Factor of Parent*

We consider first the UE reactions with equal-mass pair $m_b=m_d$. The main difference between the $t=0$ singularity of UE and that of UU is that the full helicity amplitudes $f_{\mu\lambda}{}^t$ do have singularities at $t=0$. The quantities[30] $\sin\frac{1}{2}\theta_t$ and $\cos\frac{1}{2}\theta_t$ are 1 at $t=0$; therefore, the $\bar{f}_{\mu\lambda}{}^t$'s have respectively the same singularities at $t=0$ as the $f_{\mu\lambda}{}^t$'s. But notice that in the UE case, $t=0$ is never inside the physical region. We shall use λ to denote the total helicity at the equal-mass vertex, i.e., $\lambda\equiv D-b$. In accordance with the definition in Sec. II B, μ is used for the unequal-mass vertex. The kinematic singularities of the helicity amplitude are[31]

$$f_{\mu,\lambda}{}^{t,\pm}\equiv\bar{f}_{-\mu,\lambda}{}^t\pm\eta\bar{f}_{-\mu,\lambda}{}^t\sim(\sqrt{t})^{-\zeta_\pm}, \qquad (2.23)$$

where the η is the same as in Eq. (2.4). For boson-boson (BB) reactions,

$$\zeta_\pm=S-\tfrac{1}{2}[1-(\pm)\eta_b\eta_d(-)^{\mu+\lambda+\lambda_m}], \qquad (2.24)$$

where S is even; whereas for fermion-antifermion ($F\bar{F}$) reactions,

$$\zeta_\pm=S-\tfrac{1}{2}[1+(\pm)\eta_b\eta_d(-)^{\mu+\lambda+\lambda_m}], \qquad (2.25)$$

where S is odd. Here $\lambda_m\equiv\max(\mu,\lambda)$, $\mu\geqslant0$, $\lambda\geqslant0$, $S\equiv s_b+s_d$. For definiteness, we put $s_b=s_d$. The singularities of the full helicity amplitudes are

$$\bar{f}_{\mu,\lambda}{}^t,\quad \bar{f}_{-\mu,\lambda}{}^t\sim(\sqrt{t})^{-S}. \qquad (2.26)$$

We consider separately the following cases.

a. Case of $S\geqslant M$. For a boson-boson system with $S\geqslant\lambda\geqslant\mu$, we have

$$\beta_{\mu,\lambda}{}^+=\gamma_{\mu,\lambda}{}^+[(\sqrt{t})^{-S}(\sqrt{t})^{\eta_\mu}](\sqrt{t})^{-\alpha+\lambda}$$
$$=\gamma_{\mu,\lambda}{}^+(\sqrt{t})^{-\alpha}(\sqrt{t})^{-(S-\lambda-\eta_\mu)}, \qquad (2.27)$$

where

$$\eta_\mu=1 \quad \text{if } \mu \text{ is odd}$$
$$=0 \quad \text{if } \mu \text{ is even}$$

and

$$\beta_{\mu,\lambda}{}^-=\gamma_{\mu,\lambda}{}^-(\sqrt{t})^{-\alpha}(\sqrt{t})^{-[S-\lambda-(1-\eta_\mu)]}. \qquad (2.28)$$

The second factor in the singularity shown in Eq. (2.27) is from the kinematic singularity given in Eq. (2.24). The first factor is a result of the partial-wave projection. From Eq. (2.16), the t factor of the UU residue is

$$\beta_{\mu\mu}{}^\pm\equiv\gamma_{\mu\mu}{}^\pm t^{-\alpha}t^{|M-\mu|}. \qquad (2.29)$$

The t factors at $t=0$ for EE reactions are[4]

$$f_{\lambda,\lambda'}{}^t\sim1 \quad \text{if } (-)^{\lambda+\lambda'}=+$$
$$\sim t \quad \text{if } (-)^{\lambda+\lambda'}=-. \qquad (2.30)$$

Therefore,

$$\beta_{\lambda\lambda}{}^\pm\equiv\gamma_{\lambda\lambda}{}^\pm \qquad (2.31)$$

is constant at $t=0$. Factorization requires

$$(\beta_{\mu,\lambda}{}^\pm)^2=\beta_{\mu,\mu}{}^\pm\beta_{\lambda,\lambda}{}^\pm. \qquad (2.32)$$

Notice two things in this equation. First, the left-hand side is always more singular than the right-hand side, so additional zeros must be introduced into the UE residues. Second, the right-hand side has the same t factor for plus and for minus states, but the $\beta_{\mu,\lambda}{}^\pm$ on the left differ by a factor $\sqrt{t}$. Therefore, either for the plus or for the minus states, the two sides of Eq. (2.32) may differ in odd powers of t. In that case, a factor of t must be introduced in $\beta_{\lambda\lambda}$. (It cannot be in $\beta_{\mu,\mu}$, since its t factor is fixed by the convention for the quantum number M.) From Eqs. (2.27)–(2.29) and (2.32), one can reach the conclusion that for the EE reactions

$$\beta_{\lambda\lambda}{}^\pm\sim t^{\frac{1}{2}(1\pm\eta)}, \qquad (2.33)$$

where

$$\eta=-(-)^{S-\lambda+M}. \qquad (2.34)$$

Therefore, the parent trajectory and its conspirator cannot both couple to a given helicity state λ of the equal-mass pair. For the other cases of $S\geq\mu\geq\lambda$ and $\mu\geq S\geq\lambda$, the same result as in Eqs. (2.33) and (2.34) holds. For fermion-antifermion ($F\bar{F}$) system in the t channel, owing to the difference of Eq. (2.25) from Eq. (2.24), we find

$$\beta_{\lambda,\lambda}{}^\pm\sim t^{\frac{1}{2}(1\mp\eta)}. \qquad (2.35)$$

Combining the results of Eqs. (2.33) and (2.35), we have

$$\beta_{\lambda,\lambda}{}^\pm\sim t^{\frac{1}{2}[1\pm(-)^{M+\lambda+1}]}. \qquad (2.36)$$

b. Case of $S\leqslant M$. The same argument as given in Subsec. a will hold. Equations (2.33) and (2.35) are still true for the cases $M\geq S\geq\lambda\geq\mu$, $M\geq S\geq\mu\geq\lambda$, and

[30] If the masses are such that $(m_c^2-m_a^2)(m_d^2-m_b^2)>0$, then $\theta_t=\pi$ at $\theta_s=0$, $\theta_t=0$ at $t=0$, and $t=0$ is outside the s physical region; here, $\bar{f}_{\bar{\mu},\mu}{}^t\sim(\sqrt{t})^{-(\bar{\mu}-\mu)}$ and $\bar{f}_{-\bar{\mu},\mu}{}^t\sim(\sqrt{t})^{-(\bar{\mu}+\mu)}$ for $\bar{\mu}\geqslant\mu\geqslant0$. If $(m_c^2-m_a^2)(m_d^2-m_b^2)<0$, then $\theta_t=0$ at $\theta_s=0$, $\theta_t=\pi$ at $t=0$, and $t=0$ is inside the s physical region; here $\bar{f}_{\bar{\mu},\mu}{}^t\sim(\sqrt{t})^{-(\bar{\mu}+\mu)}$ and $\bar{f}_{-\bar{\mu},\mu}{}^t\sim(\sqrt{t})^{-(\bar{\mu}-\mu)}$. If $m_b=m_d$ and $m_c\neq m_a$, then $\theta_t=\frac{1}{2}\pi$ at $t=0$, $\theta_t=\pi$ at $\theta_s=0$, and $t=0$ is outside all physical regions.

[31] Appendix A of Ref. 6.

$M \geq \mu \geq S \geq \lambda$. But in the case of $\mu \geq M \geq S \geq \lambda$, we need

$$\beta_{\mu,\lambda}{}^{+} = \gamma_{\mu,\lambda}{}^{+}(\sqrt{t})^{-\alpha}(\sqrt{t})^{-\{S-\mu-\frac{1}{2}[1\pm(-)^{\lambda+1}]\}}, \quad (2.37)$$

which will give more zeros in t on the left-hand side in Eq. (2.32) than those on the right. Therefore, more zeros than those in Eqs. (2.33) and (2.35) must be introduced, and the result is

$$\beta_{\lambda,\lambda}{}^{\pm} \sim t^{M-S+\frac{1}{2}[1\pm(-)^{\lambda+1}]} \quad \text{for the } BB \text{ system (i.e., } S \text{ even)} \quad (2.38)$$

and

$$\beta_{\lambda\lambda}{}^{\pm} \sim t^{M-S+\frac{1}{2}[1\pm(-)^{\lambda}]} \quad \text{for the } F\bar{F} \text{ system (i.e., } S \text{ odd).} \quad (2.39)$$

Checking the consistency for all cases, we see that this is the solution for $S \leqslant M$.

From all these results, we see that the most singular factor that a trajectory of quantum number M can give to its residue in UE reactions is for $M=\mu$ and $S>M$:

$$\beta_{\pm M,\lambda} \sim (\sqrt{t})^{-\alpha}.$$

One of the $\beta_{\pm M,\lambda}{}^{\pm}$ will have $(\sqrt{t})^{-\alpha}$, and the other $(\sqrt{t})^{-\alpha+1}$. From Eqs. (2.27) and (2.28), we see that only when $\lambda = S$ does one of $\beta_{\pm M,S}{}^{\pm}$ have the most singular t factor allowed by the kinematics. From the discussion of Sec. I, we foresee that only for this amplitude $f_{\pm M,S}{}^{t}$ can the most singular part of the daughter residue be uniquely determined from that of the parent. This will be further discussed in Sec. III.

2. *t Factor of Daughters at t=0*

The original t factor of the daughter trajectories $\alpha_\kappa \equiv \alpha - \kappa$ is

$$\beta_{\mu,\lambda}{}^{\kappa,+} = \gamma_{\mu,\lambda}{}^{\kappa,+}[(\sqrt{t})^{-S}(\sqrt{t})^{\eta_\mu}](\sqrt{t})^{-\alpha_\kappa+\lambda}, \quad (2.40)$$

where the subscript μ refers to the unequal-mass state, and λ to the equal-mass state. Unlike the parent residue, the daughter $\gamma_{\mu,\lambda}{}^{\kappa,+}$ may still have poles[2] in t at $t=0$. Equation (2.40) says that the even daughters have the same evenness or oddness in the power of $\sqrt{t}$ as the parent, but the odd daughters have the opposite. From factorization,

$$\beta_{\mu,\lambda}{}^{\kappa,\pm} = \beta_{\mu,\mu}{}^{\kappa,\pm}\beta_{\lambda,\lambda}{}^{\kappa,\pm}. \quad (2.41)$$

The t factor on the right-hand side is the same for all κ. Using the argument we used in Sec. II C *1*, and also checking with the discussion in Sec. III, it turns out that the even daughter residues $\beta_{\mu,\lambda}{}^{\kappa,+}$ (or $\beta_{\mu,\lambda}{}^{\kappa,-}$) should have the same t factor as the parent residue $\beta_{\mu,\lambda}{}^{+}$ (or $\beta_{\mu,\lambda}{}^{-}$). But the odd daughter residues should have the same t factor as the conspirator residue. Therefore, we obtain the following t factors of the daughter residues in the EE reaction:

$$\begin{aligned} &\text{for } M \leqslant S, \quad \beta_{\lambda,\lambda}{}^{\kappa,\pm} = \gamma_{\lambda,\lambda}{}^{\kappa,\pm} t^{\frac{1}{2}(1\pm\eta_\kappa)}, \\ &\qquad\qquad \eta_\kappa = (-)^{M+\lambda+1+\kappa}; \\ &\text{for } M \geqslant S, \quad \beta_{\lambda,\lambda}{}^{\kappa,\pm} = \beta_{\lambda,\lambda}{}^{\kappa,\pm} t^{M-S+\frac{1}{2}[1\pm\eta_\kappa{}']}, \\ &\qquad\qquad \eta_\kappa{}' = (-)^{S+\lambda+1+\kappa}. \end{aligned} \quad (2.42)$$

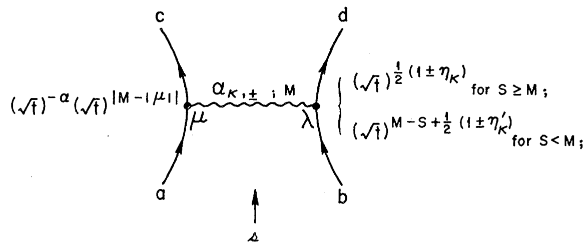

FIG. 1. t factors of Regge residues $\beta_{\mu,\lambda}(t)$ and $\hat{\beta}_{\mu,s\lambda}(t)$: Here $m_b=m_d$, $m_a \neq m_c$; for $\beta_{\mu,\lambda}(t)$, $s=S=s_b+s_d$; for $\hat{\beta}_{\mu,s\lambda}(t)$, which is defined in Eq. (3.38), s is the total spin defined in Eq. (3.33); κ is the daughter number, $\eta_\kappa \equiv (-)^{M+\lambda+\kappa+1}$; $\eta_\kappa{}' \equiv (-)^{s+\lambda+\kappa+1}$; notice that the t factors at the equal-mass vertex are dependent on κ as well as on M, s, and λ.

For a fixed helicity state λ, either the even or the odd trajectories can couple to the equal-mass vertex at $t=0$. As discussed in Sec. I A, for equal-mass vertex of identical particles or particle-antiparticle, this fact is consistent with the Pauli principle and G-parity conservation, because the even and the odd trajectories have opposite parity. We summarize all the results in Fig. 1.

D. Comments on the Situation When $\alpha(t=0)$ Is an Integer

For integer α, the sense and the nonsense channels must decouple,[32] i.e.,

$$\beta_{\bar{\mu},\mu}{}^{\pm} \sim (\alpha - n)^{1/2} \quad \text{for} \quad \bar{\mu} > n > \mu > 0. \quad (2.43)$$

This behavior should be displayed also at $t=0$, if $\alpha(t=0)=n$. Namely, there should be an additional $\sqrt{t}$ factor for those sense-nonsense amplitudes. This additional $\sqrt{t}$ factor might complicate the already complicated $t=0$ behavior we have just discussed. However, we argue that in the UU reaction the $t=0$ behavior of $\beta_{MM}{}^{\pm}$ should not be changed by the fact that the α of quantum number M also passes through an integer at $t=0$. Namely, $\beta_{MM}{}^{\pm}$ should still have the most singular behavior allowed by the kinematics, i.e., $\beta_{MM}{}^{\pm} \sim t^{-\alpha}$. In this way, the quantum number M is still a physically well-defined quantity. The trajectory α of quantum number M still contributes its full strength in the forward direction. Once the behavior of the $\beta_{MM}{}^{\pm}$ is fixed, in addition to the behavior given by Eq. (2.43) the $t=0$ behavior for all other residues is fixed, including those of EE and UE reactions. If $M \leqslant \alpha(0)$, the $\beta_{MM}{}^{\pm}$ are sense-sense. Therefore, all the trajectories α with $M \leqslant \alpha(0)$, and $\alpha(0)=n$, will choose sense in all channels, i.e., all the nonsense-nonsense residues should have an additional factor of t to that given in Fig. 1. But if $M > \alpha(0)$, and $\alpha(0)=n$, the trajectory α

[32] The possibility of having a multiplicative fixed pole at a nonsense wrong-signature point of α can be ruled out at $t=0$, upon the following basis: The daughter residues become singular if the parent residue has a multiplicative fixed pole. For references and a detailed discussion of the α factors and fixed pole in the parent residue, see C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. **161**, 1563 (1967); S. Mandelstam and L. L. Wang, *ibid.* **160**, 1490 (1967).

will choose nonsense in all channels. In this case, the full amplitude can never have a pole at $t=0$ corresponding to this particle. Therefore, this particle decouples from all physical reactions.

The one-Lorentz-pole result for the EE reaction says precisely this.[10] Assuming that the pion has a mass equal to zero and has M equal to 1, Mandelstam[33] first used this group-theoretical result to derive the Adler self-consistency relation for the soft pions.[34] But the difficulty associated with this theory, as was realized earlier,[35] is just that mentioned in the last paragraph. Once the fact is established that the $M=1$ pion couples only to the nonsense channels, it is clear that the soft pion decouples totally from all physical reactions. Thus, no physical consequences can be deduced from this type of soft-pion theory.

E. Discussion

(1) From Fig. 1, the t factor of the residues in the EE reactions is $\beta_{\lambda',\lambda}{}^{\kappa,\pm}\sim\sqrt{t}$ if $(-)^{\lambda+\lambda'}=-1$, for any $M\leq S$ ($\beta_{\lambda',\lambda}{}^{\kappa,\pm}\sim t^{M-S}\sqrt{t}$ for $M>S$). The kinematic factor of the full t-channel helicity amplitudes is also $f_{\lambda',\lambda'}{}^{t}\sim\sqrt{t}$. Therefore, our result is consistent with the analyticity of $f_{\lambda',\lambda}{}^{t}\sim\sqrt{t}$, though we did not impose it in advance.

(2) We apply the results of Fig. 1 to the much-studied reaction $NN\rightarrow NN$. The t-channel reaction is $N\bar{N}\rightarrow N\bar{N}$. The total intrinsic spin is $S=\frac{1}{2}+\frac{1}{2}=1$. The behavior of the five amplitudes for $M\leq S$, i.e., $M=0$ or 1, is

$$\begin{aligned}\beta_{11}{}^{\kappa,+}&\sim(\sqrt{t})^2 && \text{if } (-)^{M+\kappa}=+1\\ &\sim 1 && \text{if } (-)^{M+\kappa}=-1\,,\end{aligned} \tag{2.44}$$

$$\beta_{10}{}^{\kappa,+}\sim\sqrt{t}\,,\quad \text{independent of } M \text{ and } \kappa\,, \tag{2.45}$$

$$\begin{aligned}\beta_{00}{}^{\kappa,+}&\sim 1 && \text{if } (-)^{M+\kappa}=+1\\ &\sim(\sqrt{t})^2 && \text{if } (-)^{M+\kappa}=-1\,,\end{aligned} \tag{2.46}$$

$$\begin{aligned}\beta_{00}{}^{\kappa,-}&\sim(\sqrt{t})^2 && \text{if } (-)^{M+\kappa}=+1\\ &\sim 1 && \text{if } (-)^{M+\kappa}=-1\,,\end{aligned} \tag{2.47}$$

$$\begin{aligned}\beta_{11}{}^{\kappa,-}&\sim 1 && \text{if } (-)^{M+\kappa}=+1\\ &\sim(\sqrt{t})^2 && \text{if } (-)^{M+\kappa}=-1\,.\end{aligned} \tag{2.48}$$

The first four amplitudes have $G=(-)^{I+J}$, and the last one has $G=(-)^{I+J+1}$. However, $\beta_{10}{}^{-}\equiv 0$ due to the total decoupling of the singlet and the triplet state. Notice that the coupling at $t=0$ depends not only on M but also on the evenness or the oddness of κ for a trajectory.

Consider now the plus parent trajectories, such as P, P', ρ, ω, and A_2. If they have $M=0$, they couple to $F_{00}{}^{J,+}$; if $M=1$, they couple to $F_{11}{}^{J,+}$ at $t=0$, and their even conspirators can couple to $F_{00}{}^{J,-}$. But their odd daughters totally decouple from the $N\bar{N}$ system in either case, owing to internal symmetry. Notice that the nondependence of $\beta_{10}{}^{+}\sim\sqrt{t}$ on M and κ is consistent with the kinematic behavior of $F_{10}{}^{J+}\sim\sqrt{t}$. Trajectories such as A_1 can only couple to $F_{1,1}{}^{J,-}$. Being a parent trajectory, A_1 can couple to $F_{1,1}{}^{J,-}$ at $t=0$ if it has $M=0$. Then its odd daughters would couple to $F_{0,0}{}^{J,-}$. However if A_1 were the first daughter of some trajectory, it can couple to $F_{1,1}{}^{J,-}$ only when it had $M=1$. The pion trajectory couples to $F_{0,0}{}^{J,-}$. If it is a parent trajectory, it can couple at $t=0$ only if it has $M=1$. Its odd daughters can couple to $F_{1,1}{}^{J,-}$. Its even conspirators can couple to $F_{1,1}{}^{J,+}$; its odd conspirators totally decouple from the $N\bar{N}$ system. However, if the pion happened to be a first daughter trajectory, with $M=0$, it could couple to the $N\bar{N}$ system in $F_{0,0}{}^{J,-}$ at $t=0$. But then its parent $\alpha(0)=\alpha_\pi(0)+1$ would couple to $F_{1,1}{}^{J,-}$, and its importance for high s would be second only to that of the Pomeranchon α_P. This is not observed in experiments. Therefore either this possibility is out of the question or the parent trajectory just happens to decouple from the $N\bar{N}$ system.

(3) In the case of $M>S$, the trajectory totally decouples from the equal-mass system. This is a well-known Lorentz-pole result,[10,11] but the analyticity approach also gives the vanishing power.

(4) We see that the quantum number M cannot be uniquely introduced in EE reactions by the analyticity approach. Hence, we may appreciate the powerful restrictions due to analyticity and factorization in the unequal-mass reactions. But the identification of the quantum number M with the $O(4)$ M can only be made after considering its role in the EE reactions, which we shall discuss in Sec. III. The exact symmetry origin of M in the UU reactions is still unclear.

III. STRUCTURE OF REGGE TRAJECTORIES AND RESIDUES AT $t=0$: GENERAL SPIN

A. Residues of Conspirators and Daughters at $t=0$

From the discussion in Sec. I, we see that the way to generalize the calculation to general spin is quite obvious. Instead of $Q_{-\alpha-1}(z_t)$, we use $E_{m,m'}{}^{\alpha}(z_t)$ in Eqs. (1.5) and (1.9), where $E_{0,0}{}^{\alpha}(z_t)=Q_{-\alpha-1}(z_t)$. The only complication is the additional requirement of a conspirator and its daughters. We shall show how the analyticity is achieved by the collaboration of the daughter sequence and its conspirator daughter sequence.

1. *UU Reactions*

As we have discussed in Sec. II B for a trajectory of quantum number M, the residues $\beta_{\mu,M}{}^{\pm}$ with arbitrary μ have the most singular form allowed. Thus, the daughter coefficients can be uniquely determined from the analyticity of $f_{\mu,M}{}^{\pm}(s,t)$ like in Eq. (1.5). To be specific, we consider $M\geqslant\mu\geqslant 0$. The parent contribution

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[33] S. Mandelstam, Phys. Rev. **168**, 1884 (1968).
[34] S. Adler, Phys. Rev. **139**, B1638 (1968).
[35] R. F. Sawyer, Phys. Rev. Letters **21**, 764 (1968); S. Mandelstam (private communication).

is

$$f_{\mu,M}{}^{t,\pm}(s,t)\approx\beta_{\mu,M}{}^{\pm}(t)E_{M,\mu}{}^{\alpha+,+}(z_t)+\beta_{\mu,M}{}^{\mp}(t)E_{M,\mu}{}^{\alpha-,-}(z_t)\,, \quad (3.1)$$

where $\alpha_+(0)=\alpha_-(0)$. But we know that the singular parts of $\beta_{M,\mu}{}^+$ and $\beta_{M,\mu}{}^-$ are related, i.e., $\gamma_{\mu,M}{}^+(t)=-\gamma_{\mu,M}{}^-(t)$ up to the μth derivative, and

$$\beta_{\mu,M}(t)=\beta_{\mu,M}{}^+(t)+\beta_{\mu,M}{}^-(t)\sim t^{-\alpha}(\sqrt{t})^{M+\mu}$$

instead of $t^{-\alpha}(\sqrt{t})^{M-\mu}$. As far as the most singular part is concerned, $\beta_{\mu,M}{}^+=-\beta_{\mu,M}{}^-$. At $t=0$, Eq. (3.1) becomes

$$f_{\mu,M}{}^{t,\pm}(s,t)\approx\beta_{\mu,M}{}^+(t)E_{M,-\mu}{}^{\alpha}(z_t)\,, \quad (3.2)$$

where $E_{M,-\mu}{}^{\alpha}=E_{M,\mu}{}^{\alpha,+}-E_{M,\mu}{}^{\alpha,-}$. Using the argument of Sec. I, the daughter coefficient for a trajectory of quantum number M is such that

$$\sum_{\text{all }\kappa=0}^{\infty} a_{\mu,M}{}^{M,\kappa}(t=0)E_{M,-\mu}{}^{\alpha}(z_t)=[\tfrac{1}{2}(z_t-1)]^{\alpha(t)-M}. \quad (3.3)$$

The detailed calculation of $a_{M,\mu}{}^{M,\kappa}(t=0)$ is given in Appendix B, Eq. (B14) and Eq. (B16). For $M\geqslant\mu\geqslant 0$,

$$\begin{aligned}a_{\mu,M}{}^{M,\kappa}&=g_{-\mu,M}{}^{\kappa,\alpha}, \quad \text{with} \quad \alpha-\alpha_\kappa=\kappa,\\ &=(-)^{M+\mu}\tan\pi(\alpha-M)\,\Gamma(\alpha-M+1)\Gamma(\alpha+\mu+1)\\ &\quad\times(-)^{\kappa}\frac{2\alpha_\kappa+1}{\Gamma(\kappa+1)}\left[\frac{\Gamma(\alpha_\kappa+M+1)\Gamma(\alpha_\kappa-\mu+1)}{\Gamma(\alpha_\kappa-M+1)\Gamma(\alpha_\kappa+\mu+1)}\right]^{1/2}\\ &\quad\times[\Gamma(\alpha+\alpha_\kappa+2)]^{-1}. \quad (3.4)\end{aligned}$$

For other values of μ, the value of $a_{M,\mu}{}^{M,\kappa}$ can be obtained from the symmetry property of the $E_{\mu,M}{}^{\alpha}$ given by Eqs. (B2) and (B3).

Here it is seen that the analyticity property of $f_{\mu,M}{}^{t,\pm}(s,t)$ is achieved by collaboration of $\alpha_{+,\kappa}$ and $\alpha_{-,\kappa}$ for all positive integer values of κ. Also, $a_{\bar{\mu},\mu}{}^{M,\kappa}$ can be obtained uniquely by factorization, i.e.,

$$a_{\bar{\mu},\mu}{}^{M,\kappa}=a_{M,\mu}{}^{M,\kappa}a_{M,\bar{\mu}}{}^{M,\kappa}/a_{M,M}{}^{M,\kappa}, \quad (3.5)$$

even though we cannot start with $f_{\bar{\mu},\mu}{}^{t,\pm}$ to calculate the daughter coefficient. Without proving them explicitly, we expect the following results for $f_{-\bar{\mu},\mu}{}^{t}$ with $\bar{\mu}\geqslant\mu\geqslant 0$:

$$\sum_{\text{all }\kappa=0}^{\infty} a_{\bar{\mu},\mu}{}^{M,\kappa}E_{-\bar{\mu},\mu}{}^{\alpha}(z_t)=d_0[\tfrac{1}{2}(z_t-1)]^{\alpha-\bar{\mu}}+\cdots+d_n[\tfrac{1}{2}(z_t-1)]^{\alpha-\bar{\mu}-n}, \quad (3.6)$$

where $n=\frac{1}{2}[|M-\bar{\mu}|+|M-\mu|-(\bar{\mu}-\mu)]$. The d's are explicitly known, since all the $a_{\bar{\mu},\mu}{}^{M,\kappa}$ are known from Eq. (3.5). For $f_{\bar{\mu},\mu}{}^{t}$,

$$\sum_{\text{all }\kappa=0}^{\infty} a_{\bar{\mu},\mu}{}^{M,\kappa}E_{\bar{\mu},\mu}{}^{\alpha}(z_t)=d_0'[\tfrac{1}{2}(z_t-1)]^{\alpha-\bar{\mu}}+\cdots+d_{n'}'[\tfrac{1}{2}(z_t-1)]^{\alpha-\bar{\mu}-n'}, \quad (3.7)$$

where

$$n'=\max(\tfrac{1}{2}(\bar{\mu}+\mu-|M-\bar{\mu}|-|M-\mu|),0)\,,$$

and the d' are determined. The additional terms on the right are just allowed by the additional zeros in the residues.

2. UE and EE Reactions

In the UU reactions the analyticity requirement is solely from the t-channel helicity amplitudes f^t which are analytic at $t=0$. Once the analyticity properties of the f^t's are satisfied, the f^s's are automatically analytic at $t=0$, since the crossing matrix[36] is analytic[4] at $t=0$. But the analyticity requirement in the UE reaction is much more complicated. First, the f^t's have a definite singularity structure at $t=0$ that has to be satisfied. Second, the f^s's are analytic at $t=0$. But this is not automatically given, since both f^t's and the crossing matrix are singular at $t=0$. This gives the well-known constraints[8] on the f^t's. We shall show how these constraints in the UE reactions help to determine the daughter residues. First, let us discuss the analyticity requirement due to the f^t's.

a. Analyticity requirement due to $f_{cA,Db}{}^{t}$. As discussed in Sec. II C, in UE reactions only the residue with $\lambda=S\equiv s_b+s_d$ and $\mu=M$, and $S\geqslant M$ have the most singular t factor allowed by the kinematics. Also as given in Sec. II C for $\lambda=S$ and $\mu=M$, one of $\beta_{M,S}{}^+$ and $\beta_{M,S}{}^-$ has $(\sqrt{t})^{-\alpha}$ and the other has $(\sqrt{t})^{-\alpha+1}$. To be specific, we discuss the case

$$\beta_{M,S}{}^+\sim(\sqrt{t})^{-\alpha} \quad (3.8)$$

and

$$\beta_{M,S}{}^-\sim(\sqrt{t})^{-\alpha+1}. \quad (3.9)$$

As also discussed in Sec. II C, the daughter residues must be

$$\beta_{M,S}{}^{\kappa,+}\sim(\sqrt{t})^{-\alpha} \quad \text{for} \quad \kappa=\text{positive even integer} \quad (3.10)$$

and

$$\beta_{M,S}{}^{\kappa,+}\sim(\sqrt{t})^{-\alpha+1} \quad \text{or} \quad \beta_{M,S}{}^{\kappa,+}\equiv 0 \quad \text{for} \quad \kappa=\text{positive odd integer}. \quad (3.11)$$

The odd daughters of the minus trajectory should have the same t factors as the even plus trajectories, i.e.,

$$\beta_{M,S}{}^{\kappa,-}\sim(\sqrt{t})^{-\alpha} \quad \text{for} \quad \kappa=\text{positive odd integers} \quad (3.12)$$

and

$$\beta_{M,S}{}^{\kappa,-}(t)\sim(\sqrt{t})^{-\alpha+1} \quad \text{or} \quad \beta_{M,S}{}^{\kappa,-}\equiv 0 \quad \text{for} \quad \kappa=\text{positive even integers}.^{37} \quad (3.13)$$

[36] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 332 (1964); I. J. Muzinich, J. Math. Phys. **5**, 1481 (1964); G. Cohen-Tannoudji, A. Morel, and H. Navelet, *ibid.* **46**, 239 (1968).

[37] In the other case, $\beta_{M,s}{}^{\kappa,+}\sim(\sqrt{t})^{-\alpha}$ for $\kappa=$positive odd integers, and $\beta_{M,s}{}^{\kappa,-}\sim(\sqrt{t})^{-\alpha}$ for $\kappa=$positive even integers; $\alpha_\kappa=\alpha_{\kappa,+}$ for $\kappa=$positive odd integers, and $\alpha_\kappa=\alpha_{\kappa,-}$ for $\kappa=$negative even integers. See Refs. 4 and 5.

We now look at the contributions by the parent and the first conspirator to the $f^{t,+}$,

$$f_{M,S}{}^{t,+}(s,t)\approx\beta_{M,S}{}^{+}(t)E_{S,M}{}^{\alpha+,+}(z_t)+\beta_{M,S}{}^{\kappa=1,-}(t)E_{S,M}{}^{(\alpha-1),-}(z_t)\,,\quad(3.14)$$

where $\alpha_+(0)=\alpha_-(0)$. Therefore, for large s and $t=0$,

$$f_{M,S}{}^{t,+}(s,t)\sim\gamma_{M,S}{}^{+}(\sqrt{t})^{-\alpha}(s\sqrt{t})^{\alpha-S}+\gamma_{M,S}{}^{1,-}(\sqrt{t})^{-\alpha}(s\sqrt{t})^{\alpha-S-2}.\quad(3.15)$$

The leading $s^{\alpha-S}$ term has a $(\sqrt{t})^{-S}$ singularity, which is just the singularity of $f_{M,S}{}^{+}$. But the second term has $1/t$ additional singularity, so it must cancel with the $(s\sqrt{t})^{\alpha-S-2}$ term from $E_{S,M}{}^{\alpha,+}(z_t)$ and $E_{S,M}{}^{\alpha_2,+}(z_t)$. The net result is

$$\sum_{\text{even }\kappa=0}^{\infty} b_{M,S}{}^{\kappa,+}E_{S,M}{}^{\alpha_\kappa,+}(z_t)+\sum_{\text{odd }\kappa=1}^{\infty} b_{M,S}{}^{\kappa,-}E_{S,M}{}^{\alpha_\kappa,-}(z_t)=(\tfrac{1}{2}z_t)^{\alpha-S},\quad(3.16)$$

where all the β's and α's are their values at $t=0$.

We do the same thing for $f_{M,S}{}^{-}$,

$$f_{M,S}{}^{t,-}(s,t)=\beta_{M,S}{}^{\kappa=1,-}(t)E_{S,M}{}^{(\alpha-1),+}(z_t)+\beta_{M,S}{}^{+}(t)E_{S,M}{}^{\alpha+,-}(z_t)\quad(3.17)$$

$$\approx\gamma_{M,S}{}^{\kappa=1,-}(\sqrt{t})^{-\alpha}(s\sqrt{t})^{\alpha-1-S}+\gamma_{M,S}{}^{+}(\sqrt{t})^{-\alpha}(s\sqrt{t})^{\alpha-1-S}.\quad(3.18)$$

From this equation we observe two things. First, both terms in Eq. (3.18) have a $(\sqrt{t})^{-(S+1)}$ factor, which exceeds the singular form of the full amplitude $f_{M,S}{}^{t,-}\sim(\sqrt{t})^{-(S-1)}$ by a factor of t^{-1}. Second, as already mentioned in Sec. II C 2, the $\beta_{M,S}{}^{\kappa=1,-}(t)$ must have the same t factor as $\beta_{M,S}{}^{+}(t)$; otherwise the singularity in the second term cannot be cancelled. Therefore, there must be a total cancellation between the two terms. All these facts require that

$$\sum_{\text{odd }\kappa=1}^{\infty} b_{M,S}{}^{\kappa,-}(t)E_{S,M}{}^{\alpha_\kappa,+}(z_t)+\sum_{\text{even }\kappa=0}^{\infty} b_{M,S}{}^{\kappa,+}(t)E_{S,M}{}^{\alpha_\kappa,-}(z_t)=0\,.\quad(3.19)$$

Now, adding Eqs. (3.16) and Eq. (3.19), we obtain

$$\sum_{\text{even }\kappa=0}^{\infty} b_{M,S}{}^{\kappa,+}E_{S,M}{}^{\alpha_\kappa}+\sum_{\text{odd }\kappa=1}^{\infty} b_{M,S}{}^{\kappa,-}E_{S,M}{}^{\alpha_\kappa}=(\tfrac{1}{2}z_t)^{\alpha-S}\quad(3.20)$$

and

$$\sum_{\text{even }\kappa=0}^{\infty} b_{M,S}{}^{\kappa,+}E_{S,-M}{}^{\alpha_\kappa}-\sum_{\text{odd }\kappa=1}^{\infty} b_{M,S}{}^{\kappa,-}\times E_{S,-M}{}^{\alpha_\kappa}=(\tfrac{1}{2}z_t)^{\alpha-S}.\quad(3.21)$$

Equation (3.20) can also be rewritten in the form

$$\sum_{\text{all }\kappa=0}^{\infty} b_{M,S}{}^{M,\kappa}E_{S,M}{}^{\alpha_\kappa}=(\tfrac{1}{2}z)^{\alpha-S},\quad(3.22)$$

where

$b_{M,S}{}^{M,\kappa}=b_{M,S}{}^{\kappa,+}$ for $\kappa=$positive even integers

and

$b_{M,S}{}^{M,\kappa}=b_{M,S}{}^{\kappa,-}$ for $\kappa=$positive odd integers.

So we see that the analyticity property of $f_{\pm M,S}{}^{t}$ is achieved by the collaboration of the even trajectories of one spin-parity (α_+ in our case) and the odd trajectories of the conspirator (α_- in our case). From the symmetry property of $E_{S,M}{}^{\alpha}$ in Eqs. (B2) and (B3) in Appendix B, we easily show that the b's obtained from Eq. (3.22) atuomatically satisfy Eq. (3.21). The solution to Eq. (3.22) is calculated in Appendix B [see Eq. (B26)]:

$$\begin{aligned}b_{M,S}{}^{M,\kappa}&=h_{M,S}{}^{\kappa,\alpha},\quad\text{with}\quad\alpha-\alpha_\kappa=\kappa,\\&=(-)^{M+S}\pi^{-1}\tan\pi(\alpha+S)\,\Gamma(\alpha-S+1)\\&\quad\times(\tfrac{1}{2})^{\kappa}(2\alpha_\kappa+1)\times[\Gamma(\kappa+1)\Gamma(2\alpha_\kappa+2)]^{-1}\\&\quad\times\left[\frac{\Gamma(\alpha_\kappa+S+1)\Gamma(\alpha_\kappa+M+1)\Gamma(\alpha_\kappa-M+1)}{\Gamma(\alpha_\kappa-S+1)}\right]^{1/2}\\&\quad\times F(-\kappa,\,\alpha_\kappa-M+1;\,2\alpha_\kappa+2;\,2).\quad(3.23)\end{aligned}$$

By factorization we can find the daughter residues of the EE reactions at $t=0$, using Eqs. (3.4) and (3.23):

$$\begin{aligned}c_{S,S'}{}^{M,\kappa}&=b_{S,M}{}^{M,\kappa}b_{S',M}{}^{M,\kappa}/a_{M,M}{}^{M,\kappa}\\&=(-)^{S+S'}\pi^{-1}\frac{\tan\pi(\alpha+S)\,\tan\pi(\alpha+S')}{\tan\pi(\alpha-M)}\,\frac{\Gamma(\alpha-S+1)\Gamma(\alpha-S'+1)}{\Gamma(\alpha-M+1)\Gamma(\alpha+M+1)}(-\tfrac{1}{4})^{\kappa}(2\alpha_\kappa+1)\\&\quad\times\frac{\Gamma(\alpha_\kappa+M+1)\Gamma(\alpha_\kappa-M+1)\Gamma(\alpha+\alpha_\kappa+2)}{\Gamma(\kappa+1)[\Gamma(2\alpha_\kappa+2)]^2}\left[\frac{\Gamma(\alpha_\kappa+S+1)\Gamma(\alpha_\kappa+S'+1)}{\Gamma(\alpha_\kappa-S+1)\Gamma(\alpha_\kappa-S'+1)}\right]^{1/2}\\&\quad\times[F(-\kappa,\,\alpha_\kappa-M+1;\,2\alpha_\kappa+2;\,2)]^2.\quad(3.24)\end{aligned}$$

Comparing with Eqs. (A9) and (A10), we see again that this is proportional to

$$d_{S,\alpha_\kappa,S}{}^{M,\alpha+1}(i\pi/2)d_{S',\alpha_\kappa,S'}{}^{M,\alpha+1}(i\pi/2).$$

Therefore, the sum of all daughters just corresponds to a one-Lorentz-pole expansion

$$\sum_{\text{all }\kappa=0}^{\infty} c_{S,S'}{}^{M,\kappa}E_{S',S}{}^{\alpha_\kappa}(z_t)\sim \sum_{\text{all }\kappa=0}^{\infty} d_{S,\alpha_\kappa,S}{}^{M,\alpha+1}(i\pi/2) \times d_{S',\alpha_\kappa S'}{}^{M,\alpha+1}(i\pi/2)E_{S',S}{}^{\alpha_\kappa}(z_t). \quad (3.25)$$

Notice that if κ is a positive even integer, $c_{S,S'}{}^{M,\kappa}$ are the residues of even trajectories with the same spin-parity, but if κ is a positive odd integer, $c_{S,S'}{}^{M,\kappa}$ are the residues of odd trajectories with spin-parity opposite to that with κ positive and even.

For other UE amplitudes $f_{\mu,\lambda}{}^t$, the daughter residues can be determined in terms of the first $n=S-\lambda_m+M-\mu$ arbitrary daughter residues, where $\lambda_m\equiv\max(|\mu|,|\lambda|)$, as already defined in Eq. (2.25). The equation corresponding to Eq. (3.22) becomes

$$\sum_{\text{all }\kappa=0}^{\infty} b_{\mu,\lambda}{}^{M,\kappa}E_{\lambda,\mu}{}^{\alpha}(z_t)=d_0(\tfrac{1}{2}z_t)^{\alpha-\lambda_m}+d_1(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-1} +\cdots+d_n(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-n}, \quad (3.26)$$

where $n=S-\lambda_m+M-\mu$, and the d's are undetermined. We foresee that the analyticity requirement from f^s can eliminate some of the arbitrariness. We shall discuss it in the next subsection.

b. Analyticity requirement due to $f_{cd;ab}{}^s$. The s-channel helicity amplitudes are related to the t-channel helicity amplitudes by crossing[36]:

$$f_{cd;ab}{}^s=\sum_{a'b'c'd'} d_{a'a}{}^{s_a}(\chi_a)d_{b'b}{}^{s_b}(\chi_b)d_{c'c}{}^{s_c}(\chi_c) \times d_{d'd}{}^{s_d}(\chi_d)f_{c'a';d'b'}{}^t. \quad (3.27)$$

For $m_b=m_d=m$, the crossing angles are given by

$$\cos\chi_a=[-(s+m_a{}^2-m^2)(t+m_a{}^2-m_c{}^2) -2m_a{}^2(m_c{}^2-m_a{}^2)]/\mathcal{S}_{ab}\mathcal{T}_{ac},$$

$$\cos\chi_c=[(s+m_c{}^2-m^2)(t+m_c{}^2-m_a{}^2) -2m_c{}^2(m_c{}^2-m_a{}^2)]/\mathcal{S}_{cd}\mathcal{T}_{ac},$$

$$\cos\chi_b=[t(s+m^2-m_a{}^2)-2m^2(m_c{}^2-m_a{}^2)]/ \mathcal{S}_{ab}[t(t-4m^2)]^{1/2},$$

and

$$\cos\chi_d=[t(s+m^2-m_c{}^2)-2m^2(m_c{}^2-m_a{}^2)]/ \mathcal{S}_{cd}[t(t-4m^2)]^{1/2},$$

where

$$(\mathcal{S}_{ab})^2=[s-(m_a+m_b)^2][s-(m_a-m_b)^2].$$

Notice that $\cos\chi_b$ and $\cos\chi_d$ are singular at $t=0$. The f^t's are also singular at $t=0$. But the f^s's are analytic at $t=0$. Therefore, the f^t's are constrained at $t=0$ through Eq. (3.27). Let us make the following observations:

(1) For the crossing angles, we need only keep the $(\sqrt{t})s$ terms at $t=0$ and $s\to\infty$. All the other terms, such as ts, can be dropped, since they will correspond to more regular terms of the Regge pole. So we find

$$\cos\chi_b\approx -2m^2(m_c{}^2-m_a{}^2)/(s\sqrt{t})\times 4m = -m(m_c{}^2-m_a{}^2)/2s\sqrt{t}=(-\cos\theta_t)^{-1}\equiv\cos\chi, \quad (3.28)$$

$$\cos\chi_d\approx -2m(m_c{}^2-m_a{}^2)/s\sqrt{t}=(-\cos\theta_t)^{-1} \equiv\cos\chi, \quad (3.29)$$

$$\cos\chi_a\approx -s(m_a{}^2-m_c{}^2)/s|m_a{}^2-m_c{}^2|=-1, \quad (3.30)$$

and

$$\cos\chi_c\approx -1. \quad (3.31)$$

Therefore,

$$\chi_a=\chi_c\approx\pi.$$

Notice that Eqs. (3.28) and (3.29) are also true as far as the singular part at $t=0$ is concerned. Substituting Eqs. (3.28)–(3.31) into Eq. (3.27), we obtain

$$f_{-cd;-ab}{}^s\approx(-)^{s_c-c+s_a-a}\sum_{b',d'} d_{b'b}{}^{s_b}(\chi) \times d_{d'd}{}^{s_d}(\chi)f_{ca;d'b'}{}^t. \quad (3.32)$$

(2) In Eq. (3.32), the functions on the right-hand side are singular at $t=0$, but the $f_{-cd;-ab}{}^s$ are regular. As we vary b and d, we shall obtain all the constraint relations on the f^t's for fixed c and a. It turns out that it is much nicer to use the irreducible form of the crossing matrix $d_{b',b}{}^{s_b}(\chi)d_{d',d}{}^{s_d}(\chi)$, which will be dependent on $b-d$ and $b'-d'$, since the daughter coefficients depend only on $b-d$, not individually on b or d. So we shall transform Eq. (3.32) to its irreducible form.[38]

$$d_{b',b}{}^{s_b}(\chi)d_{d',d}{}^{s_d}(\chi)=(-)^{b'-b}d_{-b',-b}{}^{s_b}(\chi)d_{d',d}{}^{s_d}(\chi) =(-)^{(s_b-b)-(s_b-b')}\sum_{\mathrm{s}} c(s_d,s_b,\mathrm{s};d',-b') \times c(s_d,s_b,\mathrm{s};d,-b)d_{\lambda',\lambda}{}^{\mathrm{s}}(\chi), \quad (3.33)$$

where $\lambda'=d'-b'$, $\lambda=b-d$. The introduction of s_b into the phase in Eq. (3.33) is just to conform with the convention in Ref. 11. Substituting Eq. (3.33) into Eq. (3.32), we obtain

$$f_{-cd;-ab}{}^s=\sum_{\mathrm{s}}\sum_{\lambda'} d_{\lambda',\lambda}{}^{\mathrm{s}}(\chi)(-)^{s_b-b} \times c(s_d,s_b,\mathrm{s};d,-b)\sum_{b',d';d'-b'=\lambda'}(-)^{-(s_b-b')} \times c(s_d,s_b,\mathrm{s};d',-b')\times f_{ca;d'b'}{}^t. \quad (3.34)$$

Using the orthogonality property of the Clebsch-Gordan coefficients, we have

$$\sum_{b,d;d-b=\lambda}(-)^{-(s_b-b)}c(s_d,s_b,\mathrm{s};d,-b)f_{-c,d;-a,b}{}^s\equiv\hat{f}_{-\mu;\mathrm{s}\lambda}{}^s =\sum_{\lambda'} d_{\lambda',\lambda}{}^{\mathrm{s}}(\chi)\sum_{b',d';d'-b'=\lambda'}(-)^{-(s_b-b')}c(s_d,s_b,\mathrm{s};d',-b') \times f_{ca;d'b'}{}^t\equiv\sum_{\lambda'} d_{\lambda',\lambda}{}^{\mathrm{s}}(\chi)\hat{f}_{\mu;\mathrm{s}\lambda'}{}^t \quad (3.35)$$

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[38] This performance is hinted at the $O(4)$ expansion in Ref. 11, or the Lorentz expansion in Ref. 10.

or

$$\hat{f}_{-\mu;s\lambda}{}^{s}=\sum_{\lambda'} d_{\lambda',\lambda}{}^{s}(\chi)\hat{f}_{\mu;s\lambda'}{}^{t}=\sum_{\lambda'} d_{\lambda',\lambda}{}^{s}(\chi)\times\sum_{J}(2J+1)\hat{F}_{\mu;s\lambda'}{}^{J}d_{\lambda',\mu}{}^{J}(z_t). \quad (3.36)$$

The implications of these equations are interesting. The left-hand side is still analytic at $t=0$. As with the original helicity amplitude, we can use this crossing relation to discuss the analyticity properties[4,31] of the new set of amplitudes $\hat{f}_{\mu;s\lambda'}{}^{t}$ by studying the inverted crossing relation corresponding to Eq. (3.36):

$$f_{\mu,s\lambda'}{}^{t}=\sum_{\lambda} d_{\lambda',\lambda}{}^{s}(\chi)\hat{f}_{-\mu,s\lambda}{}^{s}. \quad (3.37)$$

The result is precisely the same as the original helicity amplitudes as given in Eqs. (2.23)–(2.25) of Sec. II C with the replacement of S by s. Therefore all the arguments for the original helicity amplitudes and residues discussed in Sec. II C and III A 2 a go through. The new residues are related to the old ones by

$$\hat{\beta}_{\mu;s\lambda}{}^{\pm}=\sum_{d',b';d'-b'=\lambda'}(-)^{-(s_b-b')}c(s_d,s_b,\mathrm{s};d',-b')\times\beta_{ca;d'b'}{}^{\pm}. \quad (3.38)$$

Notice that $\hat{\beta}_{\mu;s\lambda}{}^{\pm}$ still satisfy factorization. Therefore the results given in Fig. 1 apply to the $\hat{\beta}_{\mu;s\lambda}$. Again we see that for $\mathrm{s}<M$, the equal-mass channel totally decouples from the trajectory α. For $M\leqslant \mathrm{s}$, the daughter residues are uniquely determined in terms of the parent residue only for $\hat{f}_{M;ss}{}^{t}$. For fixed $\mathrm{s}=M$, the daughter residues are also uniquely determined for all $\hat{f}_{M;s\lambda}{}^{t}$ with arbitrary λ.

From the discussion in Sec. III A *2 a*, and Eq. (3.26), the daughter cancellation will mean

$$\hat{f}_{M;s\lambda}{}^{t}=\hat{\beta}_{M,s\lambda}[\tfrac{1}{2}(z_t-1)]^{\frac{1}{2}|M-\lambda|}[\tfrac{1}{2}(z_t+1)]^{\frac{1}{2}|M+\lambda|}\times[d_0(\tfrac{1}{2}z_t)^{\alpha-\lambda_m}+d_1(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-1}+\cdots+d_n(\tfrac{1}{2}z_t)^{\alpha-s}], \quad (3.39)$$

where $n=\mathrm{s}-\lambda_m$, and $\hat{\beta}_{M,s\lambda}\sim(\sqrt{t})^{-\alpha}$. The d's are arbitrary constants at $t=0$ and are related to $\hat{\beta}_{M,s\kappa}{}^{\kappa}$ for $\kappa=0, 1, 2,\cdots, n$. Notice that for all λ in Eq. (3.39), there is a leading asymptotic term of

$$\hat{f}_{M;s\lambda}{}^{t}\sim\beta_{M,s\lambda}z_t{}^{\alpha}\sim s^{\alpha}, \quad s\to\infty. \quad (3.40)$$

But for $\mu\neq M$, $\hat{\beta}_{\mu,s\lambda}\sim(\sqrt{t})^{-\alpha+|M-\mu|}$, the leading asymptotic behavior is

$$\hat{f}_{\mu,s\lambda}{}^{t}\sim s^{\alpha-|M-\mu|}. \quad (3.41)$$

Now we want to see if these arbitrary parameters d can be determined by the analyticity requirement of $\hat{f}_{-\mu;s\lambda}{}^{s}$ from Eq. (3.36).

(3) In addition to the f^s's being analytic at $t=0$, because of total helicity conservation in the forward direction the asymptotic behavior of f^s changes at $t=0$; we obtain from the kinematics

$$f_{cd;ab}{}^{s}=(\sqrt{2}\sin\tfrac{1}{2}\theta_s)^{|(a-b)-(c-d)|}\times(\sqrt{2}\cos\tfrac{1}{2}\theta_s)^{|(a-b)+(c-d)|}\bar{f}_{cd;ab}{}^{s}=(\sqrt{2}\sin\tfrac{1}{2}\theta_s)^{|\mu-\lambda|}(\sqrt{2}\cos\tfrac{1}{2}\theta_s)^{|(a-b)+(c-d)|}\bar{f}_{cd;ab}{}^{s}, \quad (3.42)$$

where $\bar{f}^s$ are analytic and zero-free wherever $\sin\frac{1}{2}\theta_s=0$ or $\cos\frac{1}{2}\theta_s=0$. In Eq. (3.40), both $\sin\frac{1}{2}\theta_s\sim1$, $\cos\frac{1}{2}\theta_s\sim1$ as $s\to\infty$ for $t\neq0$, but at $t=0$ $\sin\frac{1}{2}\theta_s\sim(\sqrt{s})^{-1}$. It follows that for $\mu-\lambda=0$ only, we have

$$f_{cd;ab}{}^{s}\sim s^{\alpha}\sim(\sqrt{t})^{-\alpha}z_t{}^{\alpha} \quad \text{at} \quad t=0. \quad (3.43)$$

The same is true for the $\hat{f}_{\mu,s\lambda}$, i.e.,

$$\hat{f}_{\mu;s\lambda}{}^{s}\sim\delta_{\mu\lambda}s^{\alpha}+O(s^{\alpha-1}) \quad \text{at} \quad t=0. \quad (3.44)$$

But from Eqs. (3.36) and (3.41), μ must be equal to M to make the s^{α} term survive. In conclusion, basically there are two requirements due to $\hat{f}^s$: First, the analyticity requires that

$$\hat{f}_{-M;s\lambda}{}^{s}\sim(\sqrt{t})^{-\alpha}(\tfrac{1}{2}z_t)^{\alpha}. \quad (3.45)$$

Second, the additional symmetry at $t=0$ requires that

$$\hat{f}_{-M;s\lambda}{}^{s}\sim\delta_{\lambda,-M}(\sqrt{t})^{-\alpha}(\tfrac{1}{2}z_t)^{\alpha}. \quad (3.46)$$

Substituting these into Eq. (3.37), we obtain

$$\hat{f}_{M;s\lambda'}{}^{t}\sim d_{\lambda',-M}{}^{s}(\chi)(\sqrt{t})^{-\alpha}(\tfrac{1}{2}z_t)^{\alpha}. \quad (3.47)$$

Using Eq. (3.39), we obtain from Eq. (3.47)

$$\hat{\beta}_{M,s\lambda'}[\tfrac{1}{2}(z_t-1)]^{\frac{1}{2}|M-\lambda|}[\tfrac{1}{2}(z_t+1)]^{\frac{1}{2}|M+\lambda|}\times[d_0(\tfrac{1}{2}z_t)^{\alpha-\lambda_m}+d_1(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-1}+\cdots+d_n(\tfrac{1}{2}z_t)^{\alpha-s}]=c(\mathrm{s},M)d_{\lambda'-M}{}^{s}(\chi)(\sqrt{t})^{-\alpha}(\tfrac{1}{2}z_t)^{\alpha}, \quad (3.48)$$

where $c(\mathrm{s},M)$ is the proportionality constant. Notice that $c(\mathrm{s},M)$ depends only on s and M, not on λ'; so the λ' dependence of $\hat{\beta}_{M,s\lambda'}$ is explicitly taken care of by the $d(\chi)$'s. Thus, $\hat{\beta}_{M,s\lambda'}$ is λ'-independent. We then use the representation

$$d_{\lambda',-M}{}^{s}(\chi)=[\tfrac{1}{2}(1+\cos\chi)]^{\frac{1}{2}|\lambda'-M|}[\tfrac{1}{2}(1-\cos\chi)]^{\frac{1}{2}|\lambda'+M|}\times\left(\frac{\Gamma(\mathrm{s}+\lambda'+1)\Gamma(\mathrm{s}+M+1)}{\Gamma(\mathrm{s}-\lambda'+1)\Gamma(\mathrm{s}-M+1)}\right)^{1/2}[\Gamma(1+\lambda'+M)]^{-1}\times F(-\mathrm{s}+\lambda_m,\mathrm{s}+\lambda_m+1;1+\lambda_m-\lambda_n;\tfrac{1}{2}-\tfrac{1}{2}\cos\chi), \quad (3.49)$$

where $\lambda_m=\max(M,|\lambda'|)$, $\lambda_n=\min(M,|\lambda'|)$, and λ_m is the same as defined in Eq. (3.25). From the fact that

$$\cos\chi=(-z_t)^{-1},$$

it then follows that

$$d_{\lambda',-M}{}^{s}(\chi)=[\tfrac{1}{2}(1-z_t)]^{\frac{1}{2}|\lambda'-M|}[\tfrac{1}{2}(1+z_t)]^{\frac{1}{2}|\lambda'+M|}\times(\tfrac{1}{2})^{\lambda_m}(\tfrac{1}{2}z_t)^{-\lambda_m}\left(\frac{\Gamma(\mathrm{s}+\lambda'+1)\Gamma(\mathrm{s}+M+1)}{\Gamma(\mathrm{s}-\lambda'+1)\Gamma(\mathrm{s}-M+1)}\right)^{1/2}\times[\Gamma(1+\lambda'+M)]^{-1}F(-s+\lambda_m,\mathrm{s}+\lambda_m+1;1+\lambda_m-\lambda_n;\tfrac{1}{2}(1+1/z_t)). \quad (3.50)$$

Since s, λ_m, and λ_n are all integers, F is a finite power series in $\frac{1}{2}(1+1/z_t)$,

$$F(-s+\lambda_m,\ s+\lambda_m+1;\ 1+\lambda_m-\lambda_n;\ \tfrac{1}{2}(1+1/z_t))=1+\frac{(-s+\lambda_m)(s+\lambda_m+1)}{1+\lambda_m-\lambda_n}\tfrac{1}{2}(1+1/z_t)+\cdots$$
$$+\frac{\Gamma(-s+\lambda_m+n)\Gamma(s+\lambda_m+1+n)\Gamma(1+\lambda_m-\lambda_n)}{\Gamma(-s+\lambda_m)\Gamma(s+\lambda_m+1)\Gamma(1+\lambda_m-\lambda_n+n)(s-\lambda_m)!}[\tfrac{1}{2}(1+1/z_t)]^{s-\lambda_m}, \quad (3.51)$$

where $n=s-\lambda_m$. From Eq. (3.51), we can calculate explicitly the asymptotic expansion in z_t of

$$(z_t)^{-\lambda_m}\bar{d}_{\lambda',-M}{}^{s}(\chi)\equiv b_0(\tfrac{1}{2}z_t)^{-\lambda_m}+b_1(\tfrac{1}{2}z_t)^{-\lambda_m-1}+\cdots+b_n(\tfrac{1}{2}z_t)^{-s}. \quad (3.52)$$

So Eq. (3.48) is

$$\hat{\beta}_{M,s}[d_0(\tfrac{1}{2}z_t)^{\alpha-\lambda_m}+d_1(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-1}+\cdots+d_n(\tfrac{1}{2}z_t)^{\alpha-s}]$$
$$=c'(M,s)[b_0(\tfrac{1}{2}z_t)^{\alpha-\lambda_m}+b_1(\tfrac{1}{2}z_t)^{\alpha-\lambda_m-1}+\cdots+b_n(\tfrac{1}{2}z_t)^{\alpha-s}], \quad (3.53)$$

where the b's can be explicitly calculated from Eqs. (3.50)–(3.52). Then from Eqs. (3.26), (3.53), and (B28), the daughter residues are uniquely determined by the parent residue $\hat{\beta}_{M,s\lambda}$:

$$\hat{b}_{M,s\lambda}{}^{M,\kappa}=d_0h_{M,\lambda}{}^{\kappa,\alpha}+d_1h_{M,\lambda}{}^{\kappa,\alpha-1}+\cdots+d_nh_{M,\lambda}{}^{\kappa,\alpha-n}, \quad (3.54)$$

where the h's are given in Eq. (B26), and the d's are explicitly known from Eq. (3.53). However, notice that the daughter residues $\beta_{M,\lambda}{}^{\kappa}$ of the original helicity amplitudes $f_{M,\lambda}{}^{t}$ are not uniquely determined, since the s dependence of $\hat{\beta}_{M,s}$ is not known. Only in the reactions of total spin 1 do the $\beta_{M,\lambda}{}^{\kappa}$ happen also to be uniquely determined.[21]

Using factorization again as in Eq. (3.24), one can calculate the daughter coefficients $\beta_{s\lambda,s'\lambda'}c_{s\lambda;s'\lambda'}{}^{\kappa,M}$ for equal-mass reactions. They should correspond to the expansion of one Lorentz pole. We shall not show this explicitly. (The identification is trivial for the cases of $\lambda=s$ and $\lambda'=s'$.) We quote the $O(4)$ result and then make two remarks to complete the discussions:

$$c_{s\lambda;s'\lambda'}{}^{\kappa,M}=d_{\alpha_\kappa s'\lambda'}{}^{M,\alpha+1}(i\pi/2)\times d_{\alpha_\kappa s'\lambda'}{}^{M,\alpha+1}(i\pi/2), \quad (3.55)$$

$$\hat{f}_{s'\lambda';s\lambda}{}^{t,s}(t=0)=\beta_{s\lambda,s'\lambda'}\sum_{\kappa}d_{\alpha_\kappa s'\lambda'}{}^{M,\alpha+1}(i\pi/2)\times d_{\alpha_\kappa s\lambda}{}^{M,\alpha+1}(i\pi/2)e_{\lambda,\lambda'}{}^{\alpha_\kappa}(z_t), \quad (3.56)$$

where the e's are the second-kind functions[26] corresponding to $d_{\lambda,\lambda'}{}^{\alpha_\kappa}(z_t)$. The explicit definition of the e's is given in Appendix B.

(1) $\hat{f}^s$ is related to $\hat{f}^t$ by crossing:

$$\hat{f}_{s'\nu';s\nu}{}^{s}(s,\ t=0)=\sum_{\lambda',\lambda}d_{\lambda'\nu'}{}^{s'}(\pi/2)d_{\lambda\nu}{}^{s}(\pi/2)\times\hat{f}_{s'\lambda';s\lambda}{}^{t}(s,\ t=0). \quad (3.57)$$

Inasmuch as

$$\hat{f}_{s'\nu';s\nu}{}^{s}\sim(\sin\tfrac{1}{2}\theta_s)^{|\nu-\nu'|}\sim(\sqrt{t})^{|\nu-\nu'|}, \quad (3.58)$$

the $\hat{f}^t$'s are constrained at $t=0$, i.e.,

$$\sum_{\lambda',\lambda}d_{\lambda'\nu'}{}^{s'}(\pi/2)d_{\lambda\nu}{}^{s}(\pi/2)\hat{f}_{s'\lambda';s\lambda}{}^{t}(s,\ t=0)\sim\delta_{\nu\nu'}. \quad (3.59)$$

The fact is mentioned by Bitar and Tindle[15] that using their addition theorem, one can show that Eq. (3.56) does satisfy the constraint equation (3.59). The Bitar-Tindle addition theorem[15] says that

$$\sum_{\kappa}d_{\alpha_\kappa s'\lambda'}{}^{M,\alpha+1}(i\pi/2)d_{\alpha_\kappa s\lambda}{}^{M,\alpha+1}(i\pi/2)e_{\lambda,\lambda'}{}^{\alpha_\kappa}(z_t)=\sum_{\mu}d_{\lambda,\mu}{}^{s}(-\pi/2)d_{\mu,\lambda'}{}^{s'}(\pi/2)D_{s's\mu}{}^{M,\alpha+1}(\gamma^s), \quad (3.60)$$

where

$$\cosh\gamma^s=-(s-2m^2)/m^2$$

and

$$D_{s's\mu}{}^{M,\alpha+1}(\gamma^s)\sim s^{\alpha-|M-|\mu||}\quad \text{for}\quad s\to\infty\,.$$

Substituting Eqs. (3.60) and (3.56) into Eq. (3.57), we obtain

$$\hat{f}_{s'\nu',s\nu}{}^{s}(s,\ t=0)=\beta_{s\nu,s'\nu'}\delta_{\nu,\nu'}D_{s',s\mu}{}^{M,\alpha+1}(\gamma^s). \quad (3.61)$$

Therefore the conspiracy equation (3.59) is satisfied.

(2) Notice that Eq. (3.61) also says that

$$\hat{f}_{s'\mu,s\mu}{}^{s}\sim s^{\alpha-|M-|\mu||}.$$

From Eq. (3.35), the original helicity amplitudes

$$f_{cd;ab}{}^{s}\sim\delta_{0,[(a-b)-(c-d)]}s^{\alpha-|M-|a-c||}. \quad (3.62)$$

This says that[22,39] $f_{cd;ab}{}^{s}\sim s^{\alpha}$ at $t=0$ only when $a-c=b-d=M$.

Therefore in this subsection we have shown that the constraint relations on the f^t's of the UE reactions are determined uniquely by the daughter residues $\hat{b}_{M,s\lambda}{}^{\kappa}$, but not the original helicity residue $b_{M,\lambda}{}^{\kappa}$. It has also been demonstrated that the conspiracy relations in the EE reactions are also satisfied by the calculated daughter residues (presumably the one-Lorentz-pole solution). From our discussion here and in Ref. 39, one can easily show that around $t=0$, the amplitude

[39] Notice that this result is also true for the UU reaction at $t=0$. We give a brief derivation here. Using the relation given by Eqs. (B1) and (B2) of Appendix B of Ref. 4, at $t=0$, $\cos\chi_a\cos\chi_c=\cos\chi_b\cos\chi_d=\cos\theta_s=+1$, $\cos\chi_a\cos\chi_b=\cos\chi_c\cos\chi_d=\cos\theta_t=-1$. Therefore, the crossing matrix gives $f_{cd;ab}{}^{s}=f_{-c-a;db}{}^{t}+f_{-\bar{\mu},\mu}{}^{t}$, where $\bar{\mu}=c-a$, $\mu=d-b$. But from Eq. (3.6), since $\sin\frac{1}{2}\theta_t\sim(ts)^{1/2}$, $f_{cd;ab}{}^{s}=f_{-c-a;db}{}^{t}=(\sin\frac{1}{2}\theta_t)^{|\bar{\mu}+\mu|}(\cos\frac{1}{2}\theta_t)^{|\bar{\mu}-\mu|}\hat{f}_{-\bar{\mu},\mu}{}^{t}$, $\sim\delta_{-\bar{\mu},\mu}s^{\alpha-|M-|\bar{\mu}||}$, $=\delta_{0,[(a-b)-(c-d)]}s^{\alpha-|M-|a-b||}$. For the UE reactions, this is also true according to Eq. (3.46). This shows that the behavior of the f^s's at $t=0$ is independent of the external masses. The behavior of the f^t's is dependent on the external masses, since their singularities at $t=0$ depends on the external masses.

$f_{cd;ab}{}^s \sim s^{\alpha-\frac{1}{2}|M-|a-c||-\frac{1}{2}|M-|b-d||}$, independent of external masses, as it should, even though the large-s behavior of the f^t's around $t=0$ are strongly dependent on the external masses.

B. Derivatives of Trajectories at $t=0$

1. UU Reactions

In Sec. III A, we used the analyticity properties of the parity-conserving helicity amplitudes $f_{M,M}{}^{t,\pm}$ to find the daughter residues at $t=0$. To find the restrictions on the derivatives of the trajectories, we shall consider the analyticity property of the original helicity amplitude $f_{MM}{}^t$ and $f_{-MM}{}^t$. The parent doublet contributions to the $f_{MM}{}^{t,\pm}$ are

$$f_{M,M}{}^{t,\pm}=\beta^{\pm}E_{M,M}{}^{\alpha+,+}+\beta^{\mp}E_{M,M}{}^{\alpha-,-}, \quad (3.63)$$

where we omit the subscripts on the β's.

$$\bar{f}_{M,M}{}^t=f_{M,M}{}^{t,+}+f_{M,M}{}^{t,-}$$
$$=\beta^+E_{M,M}{}^{\alpha+}+\beta^-E_{M,M}{}^{\alpha-} \quad (3.64)$$

and

$$\bar{f}_{-M,M}{}^t=\beta^+E_{M,-M}{}^{\alpha+}-\beta^-E_{M,-M}{}^{\alpha-}. \quad (3.65)$$

Notice that $\bar{f}_{M,M}{}^t$ is analytic, but from the fact $\beta^{\pm}(t)=t^{-\alpha}\gamma^{\pm}(t)$ and $E_{M,M}{}^{\alpha}(z_t)\sim(st)^{\alpha-M}$, it follows that each term of the right-hand side behaves like t^{-M} (actually there are more singular terms, but they are cancelled by daughters). So there are constraints on both $\beta^{\pm}$ and $\alpha^{\pm}$. We expand $E_{M,M}{}^{\alpha\pm(t)}(z_t)$ in t; then Eq. (3.64) becomes

$$\begin{aligned}\bar{f}_{M,M}{}^t=t^{-\alpha}\{&[\gamma^+(0)+\gamma^-(0)]E_{M,M}{}^{\alpha(0)}(z_t)\\&+t[\gamma^{+,(1)}(0)+\gamma^{-,(1)}(0)+\alpha_+{}^{(1)}(0)\gamma^+(0)\\&+\alpha_-{}^{(1)}(0)\gamma^-(0)](\partial/\partial\alpha)E_{M,M}{}^{\alpha}(z_t)\\&+t^2[\gamma^{+,(2)}(0)+\gamma^{-,(2)}(0)+\alpha_+{}^{(1)}\gamma^{+,(1)}(0)\\&+\alpha_-{}^{(1)}(0)\gamma^{-,(1)}(0)+\alpha_+{}^{(2)}(0)\gamma^+(0)\\&+\alpha_-{}^{(2)}(0)\gamma^-(0)](\partial^2/\partial\alpha^2)E_{M,M}{}^{\alpha}(z_t)+\cdots\\&+t^M[\gamma^{+,(M)}(0)+\gamma^{-,(M)}(0)+\cdots\\&+\alpha_+{}^{(M)}(0)\gamma^+(0)+\alpha_-{}^{(M)}(0)\gamma^-(0)]\\&\times(\partial/\partial\alpha)^ME_{M,M}{}^{\alpha}(z_t)+\cdots\}. \quad (3.66)\end{aligned}$$

The first term says that $\gamma^+(t)=-\gamma^-(t)$ up to the $(M-1)$th derivative, so that $\beta^+(t)+\beta^-(t)\sim t^{-(\alpha-M)}$, though individually $\beta^{\pm}(t)\sim t^{-\alpha}$. This is just the result of Sec. II B *1*. The further terms of Eq. (3.66) imply that $\alpha_+(t)=\alpha_-(t)$ up to the $(M-1)$th derivative, just like the residues $\gamma^+(t)$ and $\gamma^-(t)$. Similar argument ought to be true for the daughter residues and trajectories. Therefore,

$$\left.\begin{aligned}\alpha_{\kappa,+}(t)&=\alpha_{\kappa,-}(t)\\ \gamma^{\kappa,+}(t)&=-\gamma^{\kappa,-}(t)\end{aligned}\right\}\text{up to the }(M-1)\text{th derivative.} \quad (3.67)$$

However, in Eq. (3.65) the singularity on the right-hand side just matches that on the left-hand side, so there is no restriction like that just discussed.

We are now going to calculate the restriction on the slope of the daughter trajectories with respect to the parent, using the same method as in Sec. I B. Corresponding to Eqs. (1.24) and (3.3), the analyticity requirement of the $\ln z_t$ term in the expansion of $f_{M,M}{}^t$ Eq. (3.64) would imply

$$\sum_\kappa a_{MM}{}^{\kappa,M}[\alpha_{\kappa,+}{}^{(1)}(0)-\alpha_{\kappa,-}{}^{(1)}(0)]E_{MM}{}^{\alpha_\kappa}(z_t)$$
$$=[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)][\tfrac{1}{2}(z_t-1)]^{\alpha-M}. \quad (3.68)$$

Here no d terms like that of Eq. (3.2) are allowed, because the right-hand side is already too singular. Similarly, from the $\ln z_t$ term in Eq. (3.65), we obtain

$$\sum_\kappa a_{MM}{}^{\kappa,M}[\alpha_{\kappa,+}{}^{(1)}(0)+\alpha_{\kappa,-}{}^{(1)}(0)]E_{M,-M}{}^{\alpha_\kappa}(z_t)$$
$$=[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)][\tfrac{1}{2}(z_t-1)]^{\alpha-M}$$
$$+d_1[\tfrac{1}{2}(z_t-1)]^{\alpha-M-1}. \quad (3.69)$$

Notice there here the d_1 term is allowed. The use of Eq. (B14) in Eq. (3.68) gives

$$a_{M,M}{}^{\kappa,M}[\alpha_{\kappa,+}{}^{(1)}(0)-\alpha_{\kappa,-}{}^{(1)}(0)]/$$
$$[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)]=g_{M,M}{}^{\kappa,\alpha}. \quad (3.70)$$

From Eq. (3.4), we know that $a_{M,M}{}^{\kappa,M}=g_{-M,M}{}^{\kappa,\alpha}$. Therefore,

$$\begin{aligned}\alpha_{\kappa,+}{}^{(1)}(0)-\alpha_{\kappa,-}{}^{(1)}(0)&=(g_{M,M}{}^{\kappa,\alpha}/g_{-M,M}{}^{\kappa,\alpha})\\&\times[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)]=[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)]\\&\times[\Gamma(\alpha-M+1)\Gamma(\alpha_\kappa+M+1)/\Gamma(\alpha+M+1)\\&\times\Gamma(\alpha_\kappa-M+1)]_{t=0}. \quad (3.71)\end{aligned}$$

But we know that for $M>1$,

$$\alpha_+{}^{(1)}(0)=\alpha_-{}^{(1)}(0). \quad (3.72)$$

So Eq. (3.71) has a nontrivial solution only for $M=1$, and

$$\alpha_{\kappa,+}{}^{(1)}(0)-\alpha_{\kappa,-}{}^{(1)}(0)=[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)]$$
$$\times\alpha_\kappa(0)[\alpha_\kappa(0)+1]/\alpha(0)[\alpha(0)+1]. \quad (3.73)$$

Using Eq. (B14), Eq. (3.69) gives

$$a_{M,M}{}^{\kappa,M}[\alpha_{\kappa,+}{}^{(1)}(0)+\alpha_{\kappa,-}{}^{(1)}(0)]=[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)]$$
$$\times g_{M,-M}{}^{\kappa,\alpha}+d_1g_{M,-M}{}^{\kappa,\alpha-1}, \quad (3.74)$$

where

$$d_1=[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)]a_{M,M}{}^{0,M}P_1{}^{\alpha}$$
$$+[\alpha_{1,+}{}^{(1)}(0)+\alpha_{1,-}{}^{(1)}(0)]a_{M,M}{}^{1,M}P_0{}^{\alpha_1}, \quad (3.75)$$

where $P_1{}^\alpha$ is the coefficient of $(z_t)^{\alpha-M-1}$ in $E_{M,-M}{}^\alpha$, and $P_0{}^{\alpha_1}$ is the coefficient of $z_t{}^{\alpha_1-M}$ in $E_{M,-M}{}^{\alpha_1}$. Using the fact that

$$a_{M,M}{}^{0,M}P_1{}^{\alpha}=-a_{M,M}{}^{1,M}P_0{}^{\alpha_1}, \quad (3.76)$$

$$a_{M,M}{}^{0,M}P_1{}^{\alpha}=-(\alpha-M)^2/2\alpha, \quad (3.77)$$

$$a_{M,M}{}^{\kappa,M}=g_{M,-M}{}^{\kappa,\alpha}, \quad (3.78)$$

and

$$g_{M,-M}{}^{\kappa,\alpha-1}/g_{M,-M}{}^{\kappa,\alpha}=\kappa(2\alpha-\kappa+1)/(\alpha-M)^2, \quad (3.79)$$

one obtains

$$[\alpha_{\kappa,+}{}^{(1)}(0)+\alpha_{\kappa,-}{}^{(1)}(0)]-[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)] =\frac{\kappa(2\alpha-\kappa+1)}{2\alpha}\{[\alpha_{1,+}{}^{(1)}(0)+\alpha_{1,-}{}^{(1)}(0)] -[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)]\}, \quad (3.80)$$

for all M. Notice that Eq. (3.80) is independent of M. Combining the results of Eqs. (3.72), (3.73), and (3.80), we obtain all the constraints on the slope of the daughter and conspirator trajectories:

For $M>1$,

$$\alpha_{\kappa,+}{}^{(1)}(0)=\alpha_{\kappa,-}{}^{(1)}(0)$$

and

$$\alpha_\kappa{}^{(1)}(0)-\alpha^{(1)}(0)=\frac{\kappa[2\alpha(0)-\kappa+1]}{2\alpha(0)}\times[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]; \quad (3.81)$$

for $M=1$,

$$\alpha_{\kappa,+}{}^{(1)}(0)-\alpha_{\kappa,-}{}^{(1)}(0)=\frac{\alpha_\kappa(0)[\alpha_\kappa(0)+1]}{\alpha(0)[\alpha(0)+1]}\times[\alpha_+{}^{(1)}(0)-\alpha_-{}^{(1)}(0)]$$

and

$$[\alpha_{\kappa,+}{}^{(1)}(0)+\alpha_{\kappa,-}{}^{(1)}(0)]-[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)] =\frac{\kappa[\alpha(0)-\kappa+1]}{2\alpha(0)}\{[\alpha_{1,+}{}^{(1)}(0)+\alpha_{1,-}{}^{(1)}(0)] -[\alpha_+{}^{(1)}(0)+\alpha_-{}^{(1)}(0)]\}; \quad (3.82)$$

for $M=0$, there is no doublet, and

$$\alpha_\kappa{}^{(1)}(0)-\alpha^{(1)}(0)=\frac{\kappa[2\alpha(0)-\kappa+1]}{2\alpha(0)}\times[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]. \quad (3.83)$$

As we have shown, all these constraints are independent of the external spins.

2. *UE Reactions*

As in Sec. I B, we again would like to check whether the restrictions on $\alpha_{\kappa,\pm}{}^{(1)}(0)$ in the UE reaction are consistent with those in the UU reactions. Again we find they are consistent, and the conditions from the UE reactions are less restrictive. Therefore, Eqs. (3.81)–(3.83) are the restrictions on the $\alpha_{\kappa,\pm}{}^{(1)}(0)$.

The restrictions on the slopes of the α's from $f_{M,s}{}^t$ and $\hat{f}_{M,ss}{}^t$ in the UE reactions are all of a form similar to Eq. (3.74):

$$b_{M,s}{}^{\kappa,M}\alpha_\kappa{}^{(1)}(0)=\alpha^{(1)}(0)h_{M,s}{}^{\kappa,\alpha}+\bar{d}_1h_{M,s}{}^{\kappa,\alpha-1}+\bar{d}_2h_{M,s}{}^{\kappa,\alpha-2}, \quad (3.84)$$

where the h's are given explicitly in Eq. (B26). The d's can be calculated:

$$\bar{d}_1=b_{M,s}{}^{0,M}P_1{}^{\alpha}\alpha^{(1)}+b_{M,s}{}^{1,M}P_0{}^{\alpha_1}\alpha_1{}^{(1)},$$

$$\bar{d}_2=b_{M,s}{}^{0,M}P_2{}^{\alpha}\alpha^{(1)}+b_{M,s}{}^{1,M}P_1{}^{\alpha_1}\alpha_1{}^{(1)}+b_{M,s}{}^{2,M}P_0{}^{\alpha_2}\alpha_2{}^{(1)},$$

where $P_j{}^{\alpha_i}$ is the coefficient of $(\frac{1}{2}z_t)^{\alpha_i-S-j}$ in $E_{M,s}{}^{\alpha_i}(z_t)$. As we mentioned in Sec. II C, the analyticity of the amplitude is achieved by the collaboration of the even trajectories and the odd trajectories of the conspirator. The odd daughters of the *same* spin-parity are either totally decoupled or unrelated. By the specification of Eqs. (3.11)–(3.14) and (3.22), in Eq. (3.84),

$$\alpha_\kappa=\alpha_{\kappa,+} \quad \text{if} \quad \kappa \text{ is a positive even integer},$$

$$\alpha_\kappa=\alpha_{\kappa,-} \quad \text{if} \quad \kappa \text{ is a positive odd integer}.$$

From the restriction of Eq. (3.22), it follows that

$$b_{M,s}{}^{0,M}P_1{}^{\alpha}=-b_{M,s}{}^{1,M}P_0{}^{\alpha_1}, \quad (3.85)$$

$$b_{M,s}{}^{0,M}P_2{}^{\alpha}=-b_{M,s}{}^{1,M}P_1{}^{\alpha_1}-b_{M,s}{}^{2,M}P_0{}^{\alpha_2}, \quad (3.86)$$

and

$$b_{M,s}{}^{\kappa,M}=h_{M,s}{}^{\kappa,\alpha}. \quad (3.87)$$

Thus Eq. (3.84) becomes

$$\alpha_\kappa{}^{(1)}(0)-\alpha^{(1)}(0) =[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]P_0{}^{\alpha_1}h_{M,s}{}^{1,\alpha}h_{M,s}{}^{\kappa,\alpha-1}/h_{M,s}{}^{\kappa,\alpha} +[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]P_1{}^{\alpha_1}h_{M,s}{}^{1,\alpha}h_{M,s}{}^{\kappa,\alpha-2}/h_{M,s}{}^{\kappa,\alpha} +[\alpha_2{}^{(1)}(0)-\alpha^{(1)}(0)]P_0{}^{\alpha_2}h_{M,s}{}^{2,\alpha}h_{M,s}{}^{\kappa,\alpha-2}/h_{M,s}{}^{\kappa,\alpha}. \quad (3.88)$$

Comparing Eq. (3.88) with Eq. (3.81), we see that the restriction given by Eq. (3.88) on $\alpha_\kappa{}^{(1)}$ depends on one more free parameter $\alpha_2{}^{(1)}(0)$, which is determined in Eq. (3.81). Therefore, their consistency is not very obvious. We show explicilty in Appendix C that Eq. (3.88) and Eq. (3.81) are consistent. So, the constraint on the slope of the trajectory is indeed independent of the external masses and spins, and the daughter trajectories are not forced to be parallel to the parent for any M.

IV. CONCLUSION

From this analysis, we see the implications of analyticity and factorization. The positions of the daughter and conspirators and the most singular parts of the daughter and the conspirator residues at $t=0$ can be *uniquely* determined with respect to those of the parent trajectory. The solution in the equal-mass reaction corresponds to that of a one-Lorentz-pole expansion. In addition, the t factors of the residues and the constraints with free parameters on the derivatives of the trajectories and the less singular parts of the residues can also be calculated. It is also shown how all the conspiracy relations are satisfied. However, notice that analyticity and factorization *cannot* imply anything about the regular parts of the daughter and conspirator residues in unequal-mass reactions, just as in equal-mass reactions they can never specify the vanishing parts of the daughter-conspirator residues. Any specifications in a model beyond those mentioned above must be justified by dynamics. Therefore, we do not expect an $O(4)$ expansion at $t=0$ for the unequal-mass reactions to have a fundamental meaning, as it had for the equal-mass reactions.

Note added in manuscript. After this manuscript was written, we received a report by J. B. Bronzan [Phys. Rev. **181**, 2111 (1969)], who also derived Eq. (3.67).

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APPENDIX A: USEFUL FORMULAS

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \tag{A1}$$

where $(a)_n\equiv\Gamma(a+n)/\Gamma(a)$.

$$F(a,b,c,1)=\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b). \tag{A2}$$

$$(d^n/dz^n)F(a,b,c,z)=[(a)_n(b)_n/(c)_n]\times F(a+n,\ b+n;\ c+n;\ z). \tag{A3}$$

$$[c-2a-(b-a)z]F(a,b;\ c;\ z)+a(1-z)F(a+1,\ b,\ c,\ z)-(c-a)F(a-1,\ b;\ c;\ z)=0. \tag{A4}$$

$$F(a,b;\ 2b;\ z)=(1-z)^{-\frac{1}{2}a}F(\tfrac{1}{2}a,\ b-\tfrac{1}{2}a,\ b+\tfrac{1}{2};\ z^2/4(z-1)). \tag{A5}$$

$$\Gamma(z)\Gamma(1-z)=-z\Gamma(-z)\Gamma(z)=\pi/\sin\pi z. \tag{A6}$$

$$\Gamma(2z)=2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \tag{A7}$$

All formulas from (A1) to (A7) are from *Higher Transcendental Functions*, Vol. 1.[40]

$$\frac{d^n}{dz^n}[f(z)g(z)]=\sum_{\kappa}\frac{n!}{\kappa!(n-\kappa)!}f^{(\kappa)}(z)g^{(n-\kappa)}(z). \tag{A8}$$

The contribution of one $O(4)$ pole to the helicity amplitude is given by Ref. 11 as

$$\hat{f}_{s'\lambda',s\lambda}{}^t\equiv\sum_{A,b,c,D;\lambda'=c-A;\lambda=D-b}(-)^{-(s_b-b)-(s_a-A)}c(s_d,s_b,\mathrm{s};\ d,\ -b)c(s_c,s_a,\mathrm{s};\ c,\ -A)f_{cA,Db}{}^t$$
$$\sim T_{s',s}{}^{\alpha,M}\sum_{\text{all }\kappa=0}^{\infty}d_{s'\alpha_\kappa\lambda'}{}^{M,\alpha+1}(i\pi/2)d_{s\alpha_\kappa\lambda}{}^{M,\alpha+1}(i\pi/2)e_{\lambda\lambda'}{}^{\alpha_\kappa}(z_t), \tag{A9}$$

where the e's are defined in Eq. (B1) of Appendix B. From Ref. 41 and Ref. 15,

$$d_{s\alpha_\kappa s}{}^{M,\alpha+1}(i\pi/2)=(-)^{\alpha+M-s}\left[\frac{(2s+1)\Gamma(\alpha-s+1)}{\Gamma(s+M+1)\Gamma(s-M+1)\Gamma(s+\alpha+2)}\times\frac{(2\alpha_\kappa+1)\Gamma(\alpha_\kappa+s+1)\Gamma(\alpha_\kappa+M+1)\Gamma(\alpha_\kappa-M+1)\Gamma(2\alpha+2-\kappa)}{\Gamma(\alpha_\kappa-s+1)\Gamma(1+\kappa)}\right]^{1/2}$$
$$\times 2^{\alpha_\kappa-s}[\Gamma(2\alpha_\kappa+2)]^{-1}F(-\kappa,\ \alpha_\kappa+1-M;\ 2\alpha_\kappa+2;\ 2). \tag{A10}$$

APPENDIX B: EXPANSION COEFFICIENTS IN E FUNCTIONS

Andrews and Gunson[26] have a method of finding the expansion coefficients in $e_{m,m'}{}^j(z)$.[42] We shall adapt their method to calculate the coefficients which we need in this paper. Their e functions are defined to be, for $m\geqslant m'$,

$$e_{m,m'}{}^j\equiv\tfrac{1}{2}\{\Gamma(j+m+1)\Gamma(j-m+1)\Gamma(j+m'+1)\times\Gamma(j-m'+1)\}^{1/2}[\Gamma(2j+2)]^{-1}\times[\tfrac{1}{2}(1+z)]^{\frac{1}{2}(m+m')}[\tfrac{1}{2}(1-z)]^{\frac{1}{2}(m-m')}\times[\tfrac{1}{2}(z-1)]^{-j-1-m}\times F(j+m+1,\ j+m'+1;\ 2j+2;\ 2/1-z). \tag{B1}$$

The symmetry properties of the e's are

$$e_{m,m'}{}^j(z)=(-)^{m-m'}e_{m',m}{}^j(z)=(-)^{m-m'}e_{-m,-m'}{}^j(z) \tag{B2}$$

[40] *Higher Transcedental Functions,* edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953).
[41] S. Störm, Arkiv Fysik **29**, 467 (1965).
[42] We benefited greatly from discussions with Dr. C. G. Itzykson on this subject.

and

$$e_{m,-m'}{}^{j}(-z)=-e^{\pm i\pi(j-m)}e_{m,m'}{}^{j}(z) \quad (\pm \text{ for } \mathrm{Im}z\gtrless 0). \quad \text{(B3)}$$

Their theorem says: If $z^{j+1}f_j(z)$ is analytic for all z (including $z=\infty$) outside some ellipse E with foci at ± 1, then for z outside E,

$$f_j(z)=\sum_{\mu=j}^{\infty} g_{m,m'}{}^{\mu}e_{m,m'}{}^{\mu}(z)\,, \quad \text{(B4)}$$

where

$$g_{m,m'}{}^{\mu}=-(2\mu+1)\pi^{-1}\tan\pi(j-m)(2\pi i)^{-1} \times\int_c e_{-m,-m'}{}^{-\mu-1}(t)f_j(t)dt\,, \quad \text{(B5)}$$

the contour c enclosing ± 1 and all the singularities of $z^{j+1}f_j(z)$. This means that the contour integration just picks out the residue of the singularity of the integrand at $z=\infty$. The E functions used in the paper are

$$E_{m,m'}{}^{\alpha_\kappa}(z)\equiv e_{m,m'}{}^{-\alpha_\kappa-1}\times[\tfrac{1}{2}(1+z)]^{-\frac{1}{2}|m+m'|} \times[\tfrac{1}{2}(1-z)]^{-\frac{1}{2}|m-m'|}. \quad \text{(B6)}$$

To be specific, we discuss the case of $m\geqslant m'\geqslant 0$. With the change of variable $\mu=-\alpha_\kappa-1$, $j=-\alpha-1$, Eqs. (B4) and (B5) become

$$\bar{f}_\alpha(z)\equiv f_\alpha(z)[\tfrac{1}{2}(1+z)]^{-\frac{1}{2}|m+m'|}[\tfrac{1}{2}(1-z)]^{-\frac{1}{2}|m-m'|} =\sum_{\text{all }\kappa=0}^{\infty} g_{m,m'}{}^{\kappa}E_{m,m'}{}^{\alpha_\kappa}(z)\,, \quad \text{(B7)}$$

$$g_{m,m'}{}^{\kappa}=(2\alpha_\kappa+1)\pi^{-1}\tan\pi(-\alpha-1-m)(2\pi i)^{-1} \times\int_c E_{-m,-m'}{}^{-\alpha_\kappa-1}(t)\bar{f}_\alpha(t)dt \quad \text{(B8)}$$

$$=\eta_\kappa\int_c \bar{f}_\alpha(t)[\tfrac{1}{2}(t-1)]^{-\alpha_\kappa-1+m} \times F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1,\,2\alpha_\kappa+2;\,2/(1-t))dt\,, \quad \text{(B9)}$$

where

$$\eta_\kappa=-(-)^{m-m'}\pi^{-1}\tan\pi(\alpha+m)(2\pi i)^{-1}(2\alpha_\kappa+1) \times\tfrac{1}{2}[\Gamma(\alpha_\kappa-m+1)\Gamma(\alpha_\kappa+m+1)\Gamma(\alpha_\kappa-m'+1) \times\Gamma(\alpha_\kappa+m'+1)]^{1/2}[\Gamma(2\alpha_\kappa+2)]^{-1}. \quad \text{(B10)}$$

1. Applications to UU Reaction

From Eq. (3.3),

$$\bar{f}_\alpha(t)=[\tfrac{1}{2}(z_t-1)]^{\alpha-m}. \quad \text{(B11)}$$

Substituting Eq. (B11) into Eq. (B9), we obtain

$$g_{m,m'}{}^{\kappa,\alpha}\equiv\eta_\kappa\int_c [\tfrac{1}{2}(z_t-1)]^{\alpha-m}[\tfrac{1}{2}(z_t-1)]^{-\alpha_\kappa-1+m}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,2/(1-z_t))dz_t$$

$$=\eta_\kappa\int_c [\tfrac{1}{2}(z_t-1)]^{\alpha-\alpha_\kappa-1}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,2/(1-z_t))dz_t. \quad \text{(B12)}$$

As we mentioned before, the contour integral is just to bring out the residue of the singularity at $z_t=\infty$. Changing variable $u\equiv 2/(1-z_t)$,

$$g_{m,m'}{}^{\kappa,\alpha}=\eta_\kappa 2(-)^{\alpha-\alpha_\kappa-1}\int_c\left(\frac{1}{u}\right)^{\alpha-\alpha_\kappa+1}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,u)du\,, \quad \text{(B13)}$$

where the contour is around $u=0$. Using Eq. (A3), we obtain

$$g_{m,m'}{}^{\kappa,\alpha}=\eta_\kappa 2(-)^{\alpha-\alpha_\kappa-1}\times\frac{2\pi i}{(\alpha-\alpha_\kappa)!}F^{(\alpha-\alpha_\kappa)}(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1:\,2\alpha_\kappa+2;\,0)$$

$$=\eta_\kappa 2(-)^{\alpha-\alpha_\kappa-1}\frac{2\pi i}{(\alpha-\alpha_\kappa)!}\,\frac{\Gamma(\alpha-m+1)}{\Gamma(\alpha_\kappa-m+1)}\,\frac{\Gamma(\alpha-m'+1)}{\Gamma(\alpha_\kappa-m'+1)}\,\frac{\Gamma(2\alpha_\kappa+2)}{\Gamma(\alpha+\alpha_\kappa+2)}$$

$$=(-)^{m-m'}\pi^{-1}\tan\pi(\alpha+m)\,\Gamma(\alpha-m+1)\Gamma(\alpha-m'+1) \times(-)^{\alpha-\alpha_\kappa}\frac{(2\alpha_\kappa+1)}{(\alpha-\alpha_\kappa)!}\left[\frac{\Gamma(\alpha_\kappa+m+1)\Gamma(\alpha_\kappa+m'+1)}{\Gamma(\alpha_\kappa-m+1)\Gamma(\alpha_\kappa-m'+1)}\right]^{1/2}[\Gamma(\alpha+\alpha_\kappa+2)]^{-1}. \quad \text{(B14)}$$

For spinless case, this reduces to a simpler result

$$g_{0,0}{}^{\kappa,\alpha}=\pi^{-1}\tan\pi\alpha\,[\Gamma(\alpha+1)]^2\times(-)^{\alpha-\alpha_\kappa}(2\alpha_\kappa+1)[\Gamma(\alpha-\alpha_\kappa+1)\Gamma(\alpha+\alpha_\kappa+2)]^{-1}. \quad \text{(B15)}$$

Therefore, the uniquely determined daughter coefficients for Eq. (3.3) are

$$a_{M,\mu}{}^{M,\kappa}=g_{M,-\mu}{}^{\kappa,\alpha}$$
$$=(-)^{M+\mu}\pi^{-1}\tan\pi(\alpha-M)\,\Gamma(\alpha-M+1)\Gamma(\alpha+\mu+1)$$
$$\times(-)^{\alpha-\alpha_\kappa}\frac{2\alpha_\kappa+1}{(\alpha-\alpha_\kappa)!}\left[\frac{\Gamma(\alpha_\kappa+M+1)\Gamma(\alpha_\kappa-\mu+1)}{\Gamma(\alpha_\kappa-M+1)\Gamma(\alpha_\kappa+\mu+1)}\right]^{1/2}[\Gamma(\alpha+\alpha_\kappa+2)]^{-1}. \quad \text{(B16)}$$

To calculate the restrictions of analyticity in higher orders or restrictions on the derivatives of the trajectories the expansion of the following function is useful:

$$\hat{f}_\alpha(z_t)=[\tfrac{1}{2}(z_t-1)]^{\alpha-m}+d_1[\tfrac{1}{2}(z_t-1)]^{\alpha-m-1}+\cdots+d_n[\tfrac{1}{2}(z_t-1)]^{\alpha-m-n},$$
$$g_{m,m'}{}^{\kappa}=g_{m,m'}{}^{\kappa,\alpha}+d_1g_{m,m'}{}^{\kappa,\alpha-1}+\cdots+d_ng_{m,m'}{}^{\kappa,\alpha-n}. \quad \text{(B17)}$$

2. Applications to *UE* Reactions

As indicated in Eq. (3.22), here

$$\hat{f}_\alpha(z_t)=(\tfrac{1}{2}z_t)^{\alpha-m}.$$

Equation (B9) becomes

$$h_{m,m'}{}^{\kappa,\alpha}=\eta_\kappa\int_c(\tfrac{1}{2}z_t)^{\alpha-m}[\tfrac{1}{2}(z_t-1)]^{-\alpha_\kappa-1+m}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1,\,2\alpha_\kappa+2;\,2/(1-z_t))dz_t. \quad \text{(B18)}$$

After a change of variable, we obtain

$$h_{m,m'}{}^{\kappa}=\eta_\kappa(\tfrac{1}{2})^{\alpha-m-1}(-)^{-\alpha_\kappa-1+m}\int_c du\left(\frac{1}{u}\right)^{\alpha-\alpha_\kappa+1}(u-2)^{\alpha-m}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1,\,2\alpha_\kappa+2;\,u), \quad \text{(B19)}$$

where the contour is around $u=0$. The contour integration gives

$$I\equiv\frac{2\pi i}{(\alpha-\alpha_\kappa)!}\left(\frac{d}{du}\right)^{(\alpha-\alpha_\kappa)}[(u-2)^{\alpha-m}F(\alpha_\kappa-m+1,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,u)]_{u=0}. \quad \text{(B20)}$$

Using the formula

$$\left(\frac{d}{du}\right)^{\alpha-\alpha_\kappa}[f(u)g(u)]=\sum_{n=0}^{\alpha-\alpha_\kappa}\binom{\alpha-\alpha_\kappa}{n}f^{(\alpha-\alpha_\kappa-n)}g^{(n)}, \quad \text{(B21)}$$

one obtains

$$I=\frac{2\pi i}{(\alpha-\alpha_\kappa)!}\sum_n\frac{(\alpha+\alpha_\kappa)!}{n!(\alpha-\alpha_\kappa-n)!}\frac{\Gamma(\alpha-m+1)}{\Gamma[\alpha-m+1-(\alpha-\alpha_\kappa)+n]}(-2)^{\alpha-m-(\alpha-\alpha_\kappa)+n}$$
$$\times\frac{\Gamma(\alpha_\kappa-m+1+n)}{\Gamma(\alpha_\kappa-m+1)}\frac{\Gamma(\alpha_\kappa-m'+1+n)}{\Gamma(\alpha_\kappa-m'+1)}\frac{\Gamma(2\alpha_\kappa+2)}{\Gamma(2\alpha_\kappa+2+n)}$$
$$=2\pi i\Gamma(\alpha-m+1)(-2)^{\alpha_\kappa-m}\frac{\Gamma(2\alpha_\kappa+2)}{\Gamma(\alpha_\kappa-m+1)\Gamma(\alpha_\kappa-m'+1)}\sum_{n=0}^{\alpha-\alpha_\kappa}\frac{(-2)^n}{n!(\alpha-\alpha_\kappa-n)!}\frac{\Gamma(\alpha_\kappa-m'+1+n)}{\Gamma(2\alpha_\kappa+2+n)}. \quad \text{(B22)}$$

Substituting Eq. (B22) into (B19), we obtain

$$h_{m,m'}{}^{\kappa,\alpha}=(-)^{m-m'}\pi^{-1}\tan\pi(\alpha+m)\,\Gamma(\alpha-m+1)(\tfrac{1}{2})^{\alpha-\alpha_\kappa}(2\alpha_\kappa+1)\left[\frac{\Gamma(\alpha_\kappa+m+1)\Gamma(\alpha_\kappa+m'+1)}{\Gamma(\alpha_\kappa-m+1)\Gamma(\alpha_\kappa-m'+1)}\right]^{1/2}$$
$$\times\sum_{n=0}^{\alpha-\alpha_\kappa}\frac{(-2)^n}{n!(\alpha-\alpha_\kappa-n)!}\frac{\Gamma(\alpha_\kappa-m'+1+n)}{\Gamma(2\alpha_\kappa+2+n)}. \quad \text{(B23)}$$

Let us reexpress the summation as

$$\sum_{n=0}^{\alpha-\alpha_\kappa}\frac{(-2)^n}{n!(\alpha-\alpha_\kappa-n)!}\frac{\Gamma(\alpha_\kappa-m'+1+n)}{\Gamma(2\alpha_\kappa+2+n)}$$
$$=-[\pi^{-1}\sin\pi(\alpha-\alpha_\kappa)]\sum_n\frac{2^n}{n!}\frac{\Gamma(\alpha_\kappa-m'+1+n)\Gamma(n-\alpha+\alpha_\kappa)}{\Gamma(2\alpha_\kappa+2+n)}$$
$$=-[\pi^{-1}\sin\pi(\alpha-\alpha_\kappa)]\frac{\Gamma(-\alpha+\alpha_\kappa)\Gamma(\alpha_{\kappa-m+1})}{\Gamma(2\alpha_\kappa+2)}F(-\alpha+\alpha_\kappa,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,2)\,. \quad \text{(B24)}$$

Then substituting Eq. (B24) into Eq. (B23), and using Eq. (A6), in the form

$$\sin\pi(\alpha-\alpha_\kappa)\,\Gamma(-\alpha+\alpha_\kappa)=-\pi[(\alpha-\alpha_\kappa)\Gamma(\alpha-\alpha_\kappa)]^{-1}, \quad \text{(B25)}$$

we obtain

$$h_{m,m'}{}^{\kappa,\alpha}=(-)^{m-m'}\pi^{-1}\tan\pi(\alpha+m)\,\Gamma(\alpha-m+1)(\tfrac{1}{2})^{\alpha-\alpha_\kappa}(2\alpha_\kappa+1)[\Gamma(\alpha-\alpha_\kappa+1)\Gamma(2\alpha_\kappa+2)]^{-1}$$
$$\times\left[\frac{\Gamma(\alpha_\kappa+m+1)\Gamma(\alpha_\kappa+m'+1)\Gamma(\alpha_\kappa-m'+1)}{\Gamma(\alpha_\kappa-m+1)}\right]^{1/2}F(-\alpha+\alpha_\kappa,\,\alpha_\kappa-m'+1;\,2\alpha_\kappa+2;\,2)\,. \quad \text{(B26)}$$

For higher-order restrictions due to analyticity, such as Eq. (B17) for the UU reaction

$$\bar{f}_\alpha(z_t)=(\tfrac{1}{2}z_t)^{\alpha-m}+d_1(\tfrac{1}{2}z_t)^{\alpha-m-1}+\cdots+d_{2n}(\tfrac{1}{2}z_t)^{\alpha-m-2n}\,. \quad \text{(B27)}$$

The expansion coefficients are

$$h_{m,m'}{}^{\kappa}=h_{m,m'}{}^{\kappa,\alpha}+d_1h_{m,m'}{}^{\kappa,\alpha-1}+\cdots+d_{2n}h_{m,m'}{}^{\kappa,\alpha-2n}\,. \quad \text{(B28)}$$

In the spinless case, $h_{0,0}{}^{\kappa,\alpha}$ in Eq. (B26) can be reduced to a simpler form by using Eqs. (A5) and (A2):

$$F(-\alpha+\alpha_\kappa,\,\alpha_\kappa+1;\,2\alpha_\kappa+2;\,2)$$
$$=(-)^{\frac{1}{2}(\alpha-\alpha_\kappa)}F(-\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa,\,\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+1;\,\alpha_\kappa+\tfrac{3}{2};\,1)$$
$$=(-)^{\frac{1}{2}(\alpha-\alpha_\kappa)}\Gamma(\alpha_\kappa+\tfrac{3}{2})\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{3}{2})+\Gamma(-\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{1}{2})\,. \quad \text{(B29)}$$

Substituting Eq. (B29) in Eq. (B26), we obtain

$$h_{0,0}{}^{\kappa,\alpha}=\tan\pi\alpha\Gamma(\alpha+\tfrac{1}{2})(-)^{\frac{1}{2}(\alpha-\alpha_\kappa)}(\tfrac{1}{2})^{\alpha+\alpha_\kappa+1}(2\alpha_\kappa+1)$$
$$\times[\Gamma(\alpha-\alpha_\kappa+1)\Gamma(\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{3}{2})\Gamma(-\tfrac{1}{2}\alpha+\tfrac{1}{2}\alpha_\kappa+\tfrac{1}{2})]^{-1}. \quad \text{(B30)}$$

APPENDIX C: CONSISTENCY OF CONDITIONS ON SLOPE OF DAUGHTER TRAJECTORIES IN UU AND UE REACTIONS

Here we want to check explicitly that the restrictions on the slope of the daughter trajectories from UU reactions and UE reactions are consistent. The restriction from the UU reactions, for $M>1$, is

$$\alpha_{\kappa,+}{}^{(1)}(0)=\alpha_{\kappa,-}{}^{(1)}(0)\,,$$
$$\alpha_\kappa{}^{(1)}(0)-\alpha^{(1)}(0)=\tfrac{1}{2}\kappa[2\alpha(0)-\kappa+1][\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]\,. \quad \text{(3.81)}$$

The restriction from the UE reaction is

$$\alpha_\kappa{}^{(1)}(0)-\alpha^{(1)}(0)$$
$$=[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]P_0{}^{\alpha_1}h_{M,s}{}^{1,\alpha}h_{M,s}{}^{\kappa,\alpha-1}/h_{M,s}{}^{\kappa,\alpha}$$
$$+[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]P_1{}^{\alpha_1}h_{M,s}{}^{1,\alpha}h_{M,s}{}^{\kappa,\alpha-2}/h_{M,s}{}^{\kappa,\alpha}$$
$$+[\alpha_2{}^{(1)}(0)-\alpha^{(1)}(0)]P_0{}^{\alpha_2}h_{M,s}{}^{2,\alpha}h_{M,s}{}^{\kappa,\alpha-2}/h_{M,s}{}^{\kappa,\alpha}\,, \quad \text{(3.88)}$$

where

$$\alpha_\kappa=\alpha_{\kappa,+}\quad\text{for}\quad\kappa=\text{positive even integer},$$
$$\alpha_\kappa=\alpha_{\kappa,-}\quad\text{for}\quad\kappa=\text{positive odd integer},$$

and $P_j{}^{\alpha_i}$ is the coefficient of $(\frac{1}{2}z_t)^{\alpha_i-s-j}$ in $E_{s,M}{}^{\alpha_i}(z_t)$. We shall show that Eq. (3.81) and Eq. (3.88) are consistent. Notice that Eq. (3.88) is less restrictive than Eq. (3.81). From Eq. (3.81),

$$\alpha_2{}^{(1)}(0)-\alpha^{(1)}(0)=[\alpha_1{}^{(1)}(0)-\alpha^{(1)}(0)]\times[2\alpha(0)-1]/\alpha(0)\,. \quad \text{(C1)}$$

Substituting Eq. (C1) into Eq. (3.88), we obtain

$$\alpha_\kappa{}^{(1)}-\alpha^{(1)}=[\alpha_1{}^{(1)}-\alpha^{(1)}]\left\{P_0{}^{\alpha_1}h_{M,s}{}^{1,\alpha}h_{M,s}{}^{\kappa,\alpha-1}+\left[P_1{}^{\alpha_1}h_{M,s}{}^{1,\alpha}+\frac{2\alpha-1}{\alpha}P_0{}^{\alpha_2}h_{M,s}{}^{2,\alpha}\right]h_{M,s}{}^{\kappa,\alpha-2}\right\}\Big/h_{M,s}{}^{\kappa,\alpha}. \quad \text{(C2)}$$

We shall show that Eq. (C2) is just Eq. (3.81). By our normalization,

$$P_0{}^{\alpha}h_{M,s}{}^{0,\alpha}=1\,, \quad \text{(C3)}$$

so we shall always calculate the P's and the h's with respect to P_0^{α} and $h_{M,s}^{0,\alpha}$. From Eq. (B23), we can calculate all the h's:

$$\frac{h_{M,s}^{1,\alpha}}{h_{M,s}^{0,\alpha}}=\frac{2\alpha_1+1}{2\alpha+1}\left[\frac{\Gamma(\alpha_1+s+1)\Gamma(\alpha_1+M+1)\Gamma(\alpha-s+1)\Gamma(\alpha-M+1)}{\Gamma(\alpha_1-s+1)\Gamma(\alpha_1-M+1)\Gamma(\alpha+s+1)\Gamma(\alpha+M+1)}\right]^{1/2}$$
$$\times(-2)^{-1}\left[\frac{\Gamma(\alpha_1-M+1)}{(2\alpha_1+2)}+\frac{\Gamma(\alpha_1-M+2)}{\Gamma(2\alpha_1+2+1)}(-2)\right]\left[\frac{\Gamma(\alpha-M+1)}{\Gamma(2\alpha+2)}\right]^{-1}. \quad \text{(C4)}$$

After some manipulations, this becomes

$$h_{M,s}^{1,\alpha}/h_{M,s}^{0,\alpha}=(2\alpha-1)M[(\alpha-s)/(\alpha-M)(\alpha+M)(\alpha+s)]^{1/2}, \quad \text{(C5)}$$

$$h_{M,s}^{2,\alpha}/h_{M,s}^{0,\alpha}=\frac{2\alpha_2+1}{2\alpha+1}\left[\frac{\Gamma(\alpha_2+s+1)\Gamma(\alpha_2+M+1)\Gamma(\alpha-s+1)\Gamma(\alpha-M+1)}{\Gamma(\alpha_2-s+1)\Gamma(\alpha_2-M+1)\Gamma(\alpha+s+1)\Gamma(\alpha+M+1)}\right]^{1/2}(-2)^{-2}\left[\frac{\frac{1}{2}\Gamma(\alpha_2-M+1)}{\Gamma(2\alpha_2+2)}\right.$$
$$\left.+\frac{\Gamma(\alpha_2-M+2)}{\Gamma(2\alpha_2+3)}(-2)+\frac{\frac{1}{2}\Gamma(\alpha_2-M+3)}{\Gamma(2\alpha_2+4)}(-2)^2\right]\left[\frac{\Gamma(\alpha+1-M)}{\Gamma(2\alpha+2)}\right]^{-1}$$
$$=\alpha(2\alpha-3)[M^2+\tfrac{1}{2}(\alpha-1)][(\alpha-s)(\alpha-s-1)/$$
$$(\alpha+M)(\alpha+M-1)(\alpha+s)(\alpha+s-1)(\alpha-M)(\alpha-M-1)]^{1/2}. \quad \text{(C6)}$$

$$h_{M,s}^{\kappa,\alpha-1}/h_{M,s}^{\kappa,\alpha}=2\kappa(\alpha-s)^{-1}F(-\kappa+1,\ \alpha_\kappa-M+1,\ 2\alpha_\kappa+2,\ 2)F^{-1}(-\kappa;\ \alpha_\kappa-M+1;\ 2\alpha_\kappa+2;\ 2), \quad \text{(C7)}$$

$$h_{M,s}^{\kappa,\alpha-2}/h_{M,s}^{\kappa,\alpha}=4\kappa(\kappa-1)[(\alpha-s)(\alpha-s-1)]^{-1}F(-\kappa+2,\ \alpha_\kappa-M+1;\ 2\alpha_\kappa+2;\ 2)$$
$$\times F^{-1}(-\kappa;\ \alpha_\kappa-M+1;\ 2\alpha_\kappa+2;\ 2). \quad \text{(C8)}$$

From Eq. (B1) and Eq. (B6), we can calculate the P's:

$$P_0^{\alpha_1}/P_0^{\alpha}=[(\alpha-s)(\alpha+s)(\alpha-M)(\alpha+M)]^{1/2}/2\alpha(2\alpha-1), \quad \text{(C9)}$$

$$P_0^{\alpha_2}/P_0^{\alpha}=[(\alpha-s-1)(\alpha-s)(\alpha+s-1)(\alpha+s)\times(\alpha-M-1)(\alpha-M)(\alpha+M-1)(\alpha+M)]^{1/2}\times[(2\alpha-3)(2\alpha-2)(2\alpha-1)(2\alpha)]^{-1}, \quad \text{(C10)}$$

$$P_1^{\alpha_1}/P_0^{\alpha}=(P_0^{\alpha_1}/P_0^{\alpha})\{\tfrac{1}{2}(-\alpha_1+s)+(\alpha_1-M)\times(\alpha_1-s)/2\alpha_1\}$$
$$=(P_0^{\alpha_1}/P_0^{\alpha})M(-\alpha_1+s)/2\alpha_1. \quad \text{(C11)}$$

Combining Eqs. (C3), (C5), and (C6), we obtain

$$h_{M,s}^{1,\alpha}P_0^{\alpha_1}=M(\alpha-s)/2\alpha. \quad \text{(C12)}$$

Combining Eqs. (C3), (C5), (C11), and (C12), we obtain

$$h_{M,s}^{1,\alpha}P_1^{\alpha_1}=\frac{M(\alpha-s)}{2\alpha}\frac{(-\alpha+1+s)M}{2\alpha-2}. \quad \text{(C13)}$$

Combining Eqs. (C3), (C6), and (C10), we obtain

$$h_{M,s}^{2,\alpha}P_0^{\alpha_2}=\tfrac{1}{2}(\alpha-s)(\alpha-s-1)[M^2+\tfrac{1}{2}(\alpha-1)]/(2\alpha-2)(2\alpha-1). \quad \text{(C14)}$$

So from Eqs. (C13) and (C14)

$$P_1^{\alpha_1}h_{M,s}^{1,\alpha}+[(2\alpha-1)/\alpha]P_0^{\alpha_2}h_{M,s}^{2,\alpha}=(\alpha-s)(\alpha-s-1)/8\alpha. \quad \text{(C15)}$$

Now substituting Eqs. (C7), (C8), (C13), and (C15) into Eq. (C2), we obtain

$$\alpha_\kappa^{(1)}-\alpha^{(1)}=[\alpha_1^{(1)}-\alpha^{(1)}](\kappa/2\alpha)[2MF(-\kappa+1)+(\kappa-1)F(-\kappa+2)]/F(-\kappa), \quad \text{(C16)}$$

where

$$F(-n)\equiv F(-n;\alpha_\kappa-M+1;2\alpha_\kappa+2;2).$$

Using the identity Eq. (A4),

$$2MF(-\kappa+1)+(\kappa-1)F(-\kappa+2)=(2\alpha-\kappa+1)F(-\kappa), \quad \text{(C17)}$$

we obtain finally

$$\alpha_\kappa^{(1)}-\alpha^{(1)}=(\alpha_1^{(1)}-\alpha^{(1)})\kappa(2\alpha-\kappa+1)/2\alpha. \quad \text{(C18)}$$

This is just Eq. (3.81) and Eq. (1.34). Notice that there is no s dependence in the equation.