TABLE II. Self-consistent ρ parameters for Veneziano and Veneziano-Regge models, with the indicated values of N, as a function of the width of the ϵ included in the input force. 100% ϵ means an ϵ width $4\frac{1}{2}$ times that of the ρ, corresponding to the retention of only a single Veneziano term in Eq. (3). Units and notation are the same as in Table I. Errors, where shown, indicate an appreciable range of values yielding equally satisfactory self-consistency. For 100% ϵ, there are two somewhat different solutions, as explained in the text; the first is preferred to some extent by the criteria of Refs. 7 and 9.

	Veneziano-Regge $N=4$			Veneziano-Regge $N=20$			Veneziano $N=20$		
% ϵ	ν_F	m_ρ	Δm_ρ	ν_F	m_ρ	Δm_ρ	ν_F	m_ρ	Δm_ρ
00	−5.40	711	29	−5.30	711	100±6	−5.37	711	72
10	−5.40	711	32	−5.35	711	89±6	−5.40	711	89
20	−5.40	711	36				−5.40	711	98
30	−5.40	711	42				−5.45	711	93
50	−5.50	711	42	−5.45	711	83±6	−5.50	711	89±6
70	−5.60	711	52				−5.60	711	83±8
76	−5.65	711	64						
80	−5.65	710	77	−5.65	710	61±6	−5.70	710	77±8
83	−5.67	706	91				−5.70	706	77±8
100	−6.50	753	100±25	−6.51	753	90±40	−6.51	753	80±30
100	−5.80	712	86±9	−5.80	712	72±36	−5.80	712	72±36

100% ϵ, i.e., with $\gamma_2=0$ and only a single Veneziano term retained in Eq. (3), yields results in excellent agreement with the experimental parameters of the ρ, although, as mentioned above, there is some uncertainty in the theoretical value for the ρ width. With $N=4$, if γ_2 is chosen so as to significantly reduce the ϵ width from the value for a single Veneziano term, then the results in Table II indicate that the bootstrap value for the mass is slightly reduced, and the value for the width becomes appreciably too narrow. For $N=20$, the results are in reasonable agreement with experiment throughout the range of ϵ widths investigated for both the Veneziano and Veneziano-Regge cases. They are still perfectly consistent with keeping only one Veneziano term, and, in fact, doing so gives a theoretical mass in somewhat better agreement with experiment.

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General Solution for Regge Residues and Trajectories

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A parametrization for Regge vertices is presented. These vertices have the most general t dependence consistent with constraints at $t=0$ and pseudothresholds, and are valid for general spins and general masses and for nonparallel trajectories. The assumptions upon which this work is based are analyticity, crossing symmetry, factorization (unitarity), and Regge asymptotic behavior. In the unequal-mass case, we find that the general Regge vertex has a particularly simple expansion around $t=0$.

I. INTRODUCTION

THE problem of constructing a Regge expansion that has the proper kinematic singularities (the conspiracy problem) has received much attention during the last two years.[1] One reason why so much work has been expended by so many people is that different cases have been treated separately. The equal-mass case[2–4] was thought to be entirely separate from the unequal-mass case,[5–7] daughters separate from conspirators. Some authors consider only low value of spin and Lorentz number M, others only consider residues for the parent and first daughter, or only the most singular parts of the residue. The approaches range from elegant group theory,[2,5,8] which makes use of special symmetries at $t=0$, through techniques using Feynman diagrams[9] or Bethe-Salpeter models,[5,10] and finally brute-force

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[9] R. F. Sawyer, Phys. Rev. **167**, 1372 (1968).
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techniques in which analyticity is enforced term by term.[11–14]

We have previously presented a formalism[15] that is able to treat the Regge vertex $\beta_{kS\lambda}(t)$ for the parallel-trajectory case in a manner that is independent of external masses and spin, and that handles the nonleading pieces of the vertex in a general way. In the present work we shall discuss this formalism in some detail and, in addition, shall extend our solution to the nonparallel case.

Our basic assumptions are the same as those of the "brute-force" school[11–14]: Regge expansion, factorization of residues, analyticity, and crossing symmetry. We shall, however, make important use of certain group-theory identities, in order to demonstrate that our general expansion satisfies the analyticity constraints.

In Sec. II we set up the problem and discuss our notation. The consequences of parity and charge-conjugation symmetry will be discussed. We also point out the connections of the spin basis S, λ with the Breit-frame multipole expansion. We finally present some rough arguments to justify the introduction of daughter trajectories and the Lorentz number M.

In Sec. III we discuss the basic building block of our formalism—a nongeneral form for the residue, which satisfies the constraints at $t=0$ and pseudothreshold, but which does not have the most general nonleading behavior. The main group-theory identity is introduced in this section.

In Sec. IV we generalize the expansion of Sec. III, so that away from $t=0$, the vertices for daughter trajectories are no longer determined once the parent vertex is given. The formalism of this section applies to general spins and masses, including the often-neglected case in which $S<M$. It is quite possible that the results of this section are the same as the recent results of Cosenza, Sciarrino, and Toller,[16] though expressed through a different type of expansion. As will be shown in Sec. V, our expansion has the advantage that it leads to a simple Taylor expansion for the vertex. That is, the residue will be shown to have a simple expansion in powers of t once the solution is known at $t=0$.

In Sec. VI we show that the previous results can be easily generalized to the case of nonparallel trajectories.[17] A straightforward proof is given for the trajectory formula that has been given by Bronzan.[18,19]

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[13] J. B. Bronzan, C. E. Jones, and P. K. Kuo, Phys. Rev. **175**, 2200 (1968).
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[18] G. Domokos, S. Kovesi-Domokos, and P. Suranyi, Nuovo Cimento **56**, 233 (1968).

In the Appendix we outline the brute-force method for the unequal-mass case. The brute-force method verifies that our expansion is indeed the most general expansion consistent with analyticity and factorization.

The main result of our approach is the formula given in (121) for the most general Regge vertex and in (116) for the most general trajectory. The expansion around $t=0$, as given in (89) for the unequal-mass case, extends Bronzan's residue formula to general spins.

We shall extend our formalism to cover thresholds, nonsense factors, and simplified equal-mass residues in a future paper.

II. GENERAL FORMALISM

The basic object of our concern is the t-channel c.m. helicity amplitude $T_{\lambda_b\lambda_d\lambda_a\lambda_c}(\theta_t)$ as defined by Jacob and Wick.[20] It will be convenient for us to combine the helicity indices with Clebsch-Gordan coefficients as follows:

$$T_{S'\lambda',S\lambda}\equiv\sum_{\lambda_a\lambda_b\lambda_c\lambda_d}T_{\lambda_b\lambda_d,\lambda_a\lambda_c}C(S_aS_cS;\lambda_a-\lambda_c\lambda)\times C(S_bS_dS';\lambda_b-\lambda_d\lambda')(-)^{S_c-\lambda_c+S_d-\lambda_d}. \quad (1)$$

The phase factor is needed because of our use of the Jacob-Wick "backwards-particle" convention. The spins S and S' should not be confused with the t-channel spins [defined without the phase $(-)^{S_c-\lambda_c+S_d-\lambda_d}$] which are convenient for describing the t-channel threshold behavior. Rather, the spins S and S' are convenient for describing the behavior near $t=0$ and pseudothresholds.

Let us now *assume* that the amplitude has a Regge asymptotic expansion:

$$T_{S'\lambda',S\lambda}\sim\beta_{S'\lambda'}{}^{\sigma\tau\xi}(t)\beta_{S\lambda}{}^{\sigma\tau\xi}(t)D_{\lambda\lambda'}{}^{\alpha}(\theta_t)\times(1+\tau e^{-i\pi\alpha})/\sin\pi\alpha. \quad (2)$$

The function $D_{\lambda\lambda'}{}^{\alpha}(\theta)$ is the analytic continuation of the rotation matrix $d_{\lambda\lambda'}{}^{J}(\theta)$ that has the asymptotic behavior $D_{\lambda\lambda'}{}^{\alpha}(\theta)\sim(\cos\theta)^{\alpha}$ even for $\alpha<-\frac{1}{2}$.[21]

We shall not attempt to justify (2) except to say that it is largely motivated by the unitarity conditions. A Regge trajectory is characterized by its spin α, signature τ, normality σ, and charge conjugation ξ.

Normality is defined for external particles as $\sigma_a=\eta_a(-)^{S_a-v_a}$, where η is the intrinsic parity, and where $v=\frac{1}{2}$ for fermions and $v=0$ for bosons. For a Regge trajectory this definition must be altered, since the "spin" becomes complex. The normality of a trajectory is given by $\sigma=\eta\tau$, where η is the intrinsic parity of the particles on the trajectory. The main significance of σ for our purposes is shown in the following relation, which summarizes the consequence of the invariance of strong in-

[19] J. B. Bronzan, Phys. Rev. **178**, 2302 (1969); **180**, 1423 (1969); **181**, 2111 (1969).
[20] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[21] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Co., New York, 1953).

teractions under a mirror reflection[2,20]:

$$\beta_{S\lambda}{}^{\sigma}(t)=\sigma\eta_a\bar{\eta}_c(-)^{S-v}\beta_{S-\lambda}{}^{\sigma}(t)\,, \tag{3}$$

where $\bar{\eta}_i\equiv\eta_i(-)^{2S_i}$, so that $\bar{\eta}=1$ for an antiproton. We are led to use this unusual convention for parity because the backwards-particle convention in (3) leads to an extra minus sign if the backwards particle is a fermion.

The quantum number ξ is relevant when the Regge trajectory is coupled to like particles. We define like particles to be either a particle-antiparticle pair or two identical particles.

We shall follow Toller[2] in defining ξ:

$$\begin{aligned}\xi&=G(-)^I \quad \text{if} \quad B=Y=0\\ &=(-)^{I-Y/2} \quad \text{if} \quad B\neq 0 \quad \text{or} \quad Y\neq 0\,,\end{aligned} \tag{4}$$

where B is the baryon number, Y is the hypercharge, and G is the G parity of a particle ($\xi=C$ for a neutral particle).

If we use invariance under charge conjugation in the case $B=Y=0$, or invariance under exchange of identical particles in the case $B\neq 0$ or $Y\neq 0$, then we can derive the identity

$$\beta_{S\lambda}{}^{\sigma\tau\xi}(t)=\sigma\tau\xi(-)^{S-\lambda}\beta_{S\lambda}{}^{\sigma\tau\xi}(t)\,. \tag{5}$$

This relation implies that a Regge trajectory can couple to two like particles only if $(-)^{S-\lambda}=\sigma\tau\xi$.

It is interesting to note that the parameters S and σ have a special significance. The residue $\beta_{S\lambda}{}^{\sigma}$ given by

$$\begin{aligned}\beta_{S\lambda}{}^{\sigma}\equiv\sum_{\lambda_a\lambda_c}\beta_{\lambda_a\lambda_c}(t)(-)^{S_c-\lambda_c}\{C(S_aS_cS;\,\lambda_a-\lambda_c\lambda)\\ +\eta_a\bar{\eta}_c\sigma(-)^{S-v}C(S_aS_cS;\,\lambda_a-\lambda_c-\lambda)\}\end{aligned} \tag{6}$$

is exactly the Breit-frame multipole vertex defined by Durand, De Celles, and Marr.[22,23] The t-channel c.m. frame of particles a and c becomes the Breit frame when one replaces the incoming particle c with an outgoing antiparticle. The phase factor $(-)^{S_c-\lambda_c}$ in (6) is canceled by this replacement rule.

To show the correspondence with multipole vertices, let us consider the $\Delta\bar{N}\rho$ couplings. As the N has two possible helicities, and the ρ has three possible helicities, the total number of couplings must be six (if we allow the ρ to have both parities). The multipole moments for ρ exchange are given by

$\beta_{1,1}{}^{+}$ magnetic dipole, $\beta_{2,1}{}^{+}$ electric quadrupole,
$\beta_{1,1}{}^{-}$ electric dipole, $\beta_{2,1}{}^{-}$ magnetic quadrupole,
$\beta_{1,0}{}^{-}$ longitudinal dipole, $\beta_{2,0}{}^{+}$ longitudinal quadrupole.

The subscripts S,λ refer to the $\Delta\bar{N}$ system, and σ is the parity of the ρ. The ρ found in nature has $\sigma=1$. We have an (electric, magnetic) multipole for $\eta_a\bar{\eta}_c\sigma(-)^{S-v}=(+1,\,-1)$.

[22] L. Durand, P. C. De Celles, and R. B. Marr, Phys. Rev. **126**, 1882 (1962).
[23] L. Jones, Phys. Rev. **163**, 1530 (1967).

Freedman and Wang,[6] following a suggestion by Mandelstam, were the first to show that if $M_a\neq M_c$ and/or $M_b\neq M_d$ [either the UU (unequal-mass–unequal-mass) or the EU (equal-mass–unequal-mass) case], then each leading Regge pole must be associated with an infinite number of daughter poles in order to guarantee the proper analyticity at $t=0$. Freedman and Wang showed that at $t=0$ the trajectories must be integrally spaced. In order to have the daughters contribute to the same processes, we shall require all daughters to have the same isotopic spin, hypercharge, and baryon number. The dependence of charge conjugation ξ on the daughter number k is more subtle. This is because the charge-conjugation selection rule involves the helicity of the state and the signature of the trajectory. In order to satisfy the constraints the odd daughters must decouple at $t=0$. This condition is only compatible with $\xi_k=\xi_0$. Thus if the parent trajectory has a normal charge conjugation $(-)^J$, then the odd daughters must have an abnormal charge conjugation $(-)^{J+1}$.

We shall find that in order to have the proper analyticity at $t=0$, it may be necessary for the parent trajectory to be doubled, with the two trajectories having opposite parities [labeled by $\alpha_k{}^{\sigma}(t)$]. In the case of a single parent (which only occurs for certain boson trajectories), all daughters must have the same normality σ in order for them all to contribute to a process involving the scattering of scalar unequal-mass particles.

According to our preceding arguments, the helicity amplitude should have an expansion of the form

$$T_{S'\lambda',S\lambda}=\sum_k\left[\beta_{kS'\lambda'}{}^{+}\beta_{kS\lambda}{}^{+}D_{\lambda\lambda'}{}^{\alpha_k+}(\theta_t)\times\frac{1+\tau_ke^{-i\pi(\alpha_k{}^{+}-v)}}{\sin\pi(\alpha_k{}^{+}-v)}-(\text{term with }\sigma=-1)\right], \tag{7}$$

where k labels the kth daughter. We have placed a minus sign in front of $\beta^-\beta^-$ merely for future convenience. We could have used a plus sign and multiplied β^- by $\sqrt{(-1)}$. In addition, (7) does not commit us to have trajectories of both parities, since we could have $\beta^-=0$.

From (3) we can determine the signature of the daughters by the following considerations. If the first daughter is to help cancel unwanted singularities arising from the parent trajectory, then the term with $k=1$ must have the same phase near $t=0$ as does the parent. The unitarity relation can be used to show that the ratio of residues

$$\beta_{kS'\lambda'}\beta^{\sigma}{}_{kS\lambda}{}^{\sigma}/\beta_{0S'\lambda'}{}^{\sigma}\beta_{0S\lambda}{}^{\sigma}$$

is a real analytic function. Since these residues do not allow for any relative phase factors, we find that we must have $\tau_1=-\tau_0$ in order for the phase factor $1+\tau_ke^{-i\pi(\alpha_k-v)}$ to be unchanged. The same argument can be extended to higher daughters, giving the results

$$\tau_k=\tau(-)^k.$$

In Secs. III–V we make the drastic assumption that all trajectories are integrally spaced. This assumption seems to disagree with the real world, where particles of a given mass seem to have a unique spin, but we are forsaking the real world because the assumption $\alpha_k^\sigma(t)=\alpha_0(t)-k$ allows a considerable simplification in our treatment of the problem. In Sec. VI we discuss how our formalism can be modified to give expansions for residues and trajectories in the *nonparallel* case.

The assumption of parallel trajectories allows (7) to be written as

$$T_{S'\lambda' S\lambda}=\frac{1+\tau e^{-i\pi(\alpha-v)}}{\sin\pi(\alpha-v)}\sum_k [\eta_b\bar{\eta}_d(-)^{S'-v}\beta_{kS'-\lambda'}\beta_{kS\lambda} +\eta_a\bar{\eta}_c(-)^{S-v}\beta_{kS'\lambda'}\beta_{kS-\lambda}]D_{\lambda\lambda'}{}^{\alpha k}(\theta_t)(-)^k, \quad (8)$$

where

$$\beta_{kS\lambda}{}^\sigma\equiv\beta_{kS\lambda}+\sigma\eta_a\bar{\eta}_c(-)^{S-v}\beta_{kS-\lambda}.$$

The question to which we are addressing ourselves in this paper can now be asked: What is the most general expansion of $\beta_{kS\lambda}(t)$ that does not violate the analyticity constraints at $t=0$ and pseudothresholds?

These constraints are easily expressed in terms of the s-channel c.m. helicity amplitude for the process $ab\to cd$. We shall now state our analyticity assumption. We *assume* that the s-channel amplitude has the following dominant behavior for small t and large s[24]:

$$S_{\mu_c\mu_d,\mu_a\mu_b}\sim(t-t_{\min})^{|\mu-\mu'|/2}s^{\alpha}, \quad (9)$$

where $\mu\equiv\mu_a-\mu_c$, $\mu'\equiv\mu_b-\mu_d$. Note that the combination of helicities in the definition of μ is the combination relevant to the t-channel reaction. The factor $t-t_{\min}$ vanishes on the boundary of the physical region and is necessary because of angular momentum conservation. This factor is also motivated by the helicity crossing matrices.[25,26] We find that $t_{\min}$ is given by

$$\begin{aligned}t_{\min}&=(E_a{}^s-E_c{}^s)^2-(p_a{}^s-p_c{}^s)^2\\&=\{M_a{}^2-M_c{}^2-M_b{}^2+M_d{}^2-[\Delta(s,a,b)-\Delta(s,c,d)]^2\}/4s\\&=-(M_a{}^2-M_c{}^2)(M_b{}^2-M_d{}^2)/s-(M_a{}^2+M_b{}^2-M_c{}^2-M_d{}^2)\\&\quad\times(M_a{}^2M_b{}^2-M_c{}^2M_d{}^2)/s^2+O(s^{-3}),\end{aligned} \quad (10)$$

where

$$p_a{}^s\equiv\Delta(s,a,b)/2s^{1/2}$$

and

$$\Delta(s,a,b)\equiv\{[s-(M_a+M_b)^2][s-(M_a-M_b)^2]\}^{1/2}. \quad (11)$$

Thus the analyticity constraint for the case $M_a\neq M_c$ and/or $M_b\neq M_d$ is simply the requirement that the s-channel c.m. helicity amplitude should be analytic at $t=0$, thresholds, and pseudothresholds. In the case $M_a=M_c$, $M_b=M_d$, we note that $t_{\min}=0$ and the analyticity constraint is still given by (9).

In order to guarantee this analyticity, we must introduce an extra quantum number M. The need for this quantum number can be crudely seen as follows. To highest order in s, Eq. (9) becomes

$$S_{\mu_c\mu_d,\mu_a\mu_b}\sim t^{|\mu-\mu'|/2}s^{\alpha_0}. \quad (12)$$

However, by crossing (8) to the s channel and invoking parity conservation, it becomes apparent that the s-channel amplitude must be of the form[27]

$$S_{\mu_c\mu_d,\mu_a\mu_b}\sim[\gamma_{\mu_a\mu_c}(t)\gamma_{\mu_b\mu_d}(t)+\gamma_{-\mu_a-\mu_c}(t)\gamma_{-\mu_b-\mu_d}(t)]s^{\alpha_0}. \quad (13)$$

In order to get (13), we have made use of our limitation that not more than two trajectories have the same α_k. If (12) and (13) are to be compatible, then $\gamma(t)$ must have the following behavior:

$$\begin{aligned}\gamma_{\mu_a\mu_c}(t)&\sim\bar{\gamma}_{\mu_a\mu_c}(t)t^{|\mu-M|/2},\\ \gamma_{\mu_b\mu_d}(t)&\sim\bar{\gamma}_{\mu_b\mu_d}(t)t^{|\mu'-M|/2},\end{aligned} \quad (14)$$

where $\bar{\gamma}$ is an arbitrary function that is analytic at $t=0$, and $M-\mu$ is an integer (in order to have analyticity at $t=0$).

We see that a Regge trajectory gives a nonzero contribution to the s-channel c.m. amplitude at $t=0$ only when the spin flip μ is equal to M.[15,9]

We have found so far that the *trajectory* is characterized by the parameters $\alpha_k(t)$, M, σ, τ, and ξ. The Regge *residue* involves the coupling of the trajectory to two external particles. Thus the residue depends not only on all the parameters of the trajectory, but also on the parameters of the external particles. Whenever possible, we shall omit most of the parameters and simply write the residue as $\beta_{kS\lambda}$.

III. LEADING BEHAVIOR OF RESIDUE

In this section we shall demonstrate our methods by investigating the following nongeneral form for the residue, which was also discussed by Bitar and Tindle[28]:

$$\begin{aligned}\beta_{kS\lambda}&=d_{\alpha_kS\lambda}{}^{nM}(\phi_{ta}),\\ \beta_{kS'\lambda'}&=d_{\alpha_kS'\lambda'}{}^{nM}(\phi_{tb}),\end{aligned} \quad (15)$$

where $n\equiv\alpha_0(t)$ and

$$\begin{aligned}\sinh\phi_{ta}&\equiv p_a{}^t/M_a=\Delta(t,a,c)/2t^{1/2}M_a,\\ \sinh\phi_{tb}&\equiv P_b{}^t/M_b=\Delta(t,b,d)/2t^{1/2}M_b.\end{aligned} \quad (16)$$

The function $d_{\alpha_kS\lambda}{}^{nM}(\phi)$ has been widely discussed by many people.[2,3,8] It is defined as the representation of a boost that takes particle a from rest to momentum $p_a{}^t$ in the z direction:

$$\delta_{\lambda\lambda'}d_{SS'\lambda}{}^{nM}(\phi)\equiv\langle nMS'\lambda'|e^{-K_z\phi}|nMS\lambda\rangle. \quad (17)$$

[24] H. Stapp, Phys. Rev. **160**, 1251 (1967).
[25] L. L. Wang, Phys. Rev. **142**, 1187 (1966).
[26] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968); G. Fox, Ph.D. thesis, Cambridge University, 1967 (unpublished).
[27] G. Fox and E. Leader, Phys. Rev. Letters **18**, 628 (1967); **18**, 766(E) (1967).
[28] K. Bitar and G. L. Tindle, Phys. Rev. **175**, 1835 (1968).

The states $|nMS\lambda\rangle$ are basis states of the homogeneous Lorentz group. If n is an integer, then we have

$$d_{SS'\lambda}{}^{nM}(\phi)=\sum_{\mu_\pm} C(J_+J_-S;\,\mu_+\mu_-\lambda)\times C(J_+J_-S';\,\mu_+\mu_-\lambda)e^{(\mu_+-\mu_-)\phi},\quad(18)$$

with $J_++J_-=n$, $J_+-J_-=M$.

We have obtained (18) by making use of the following properties of the homogeneous Lorentz group:

$$|nMS\lambda\rangle\equiv\sum_{\mu_\pm}|J_+\mu_+,J_-\mu_-\rangle C(J_+J_-S;\,\mu_+\mu_-\lambda)\,,$$

$$\langle J_+'\mu_+'|e^{K_z\phi}|J_+\mu_+\rangle=e^{\mu_+\phi}\delta_{J_+J_+'}\delta_{\mu_+\mu_+'}\,,$$

$$\langle J_-'\mu_-'|e^{K_z\phi}|J_-\mu_-\rangle=e^{-\mu_-\phi}\delta_{J_-J_-'}\delta_{\mu_-\mu_-'}\,.$$

We shall need to know $d_{\alpha S\lambda}{}^{nM}$ for the case in which $n-M$ is an integer, but of much greater interest is the case in which we allow n (but not M) to become complex. There are three cases for which we would like to have expansions for $d_{\alpha S\lambda}{}^{nM}(\phi)$: (1) α is an integer; (2) $n-\alpha$ is an integer and $\cosh\phi\to 0$; and (3) $n-\alpha$ is an integer and $\cosh\phi\to\infty$. These expansions have appeared various times in the literature,[8,28] and there is no need to repeat them here. In Sec. V we shall, however, examine the expansion for case (3) in some detail.

The reason we have written (15) is *not* because of any group-theoretic arguments about the symmetry of the scattering amplitude near $t=0$. Rather, we are led to investigate (15) because of the following sum rule:

$$\sum_{k=0} d_{\alpha_k S'\lambda'}{}^{nM}(-\phi_{tb})d_{\lambda'\lambda}{}^{\alpha_k}(\theta_t)d_{\alpha_k S\lambda}{}^{nM}(\phi_{ta})=\sum_{\tau} d_{\lambda'\tau}{}^{S'}(-\chi_b)d_{S'S\tau}{}^{nM}(\phi_{ba})d_{\tau\lambda}{}^{S}(\chi_a)\,,\quad(19)$$

where χ_a and χ_b turn out to be the Trueman-Wick crossing angles for particles a and b from the t-channel c.m. system to the s-channel c.m. system, and ϕ_{ba} is the boost angle from the frame in which particle a is at rest to the frame in which b is at rest:

$$\cosh\phi_{ba}=(-s+M_a{}^2+M_b{}^2)/2M_aM_b\,.\quad(20)$$

The derivation of (19) for integral n is straightforward. We simply sandwich the Lorentz-transformation identity[29]

$$e^{+K_z\phi_{tb}}e^{-iJ_y\theta_t}e^{-K_z\phi_{ta}}=e^{iJ_y\chi_c}e^{-K_z\phi_{ba}}e^{-iJ_y\chi_a}\quad(21)$$

between the basis states $|nMS\lambda\rangle$ and $\langle nMS'\lambda'|$.

Upon inserting a complete set of basis states between the boost and rotation operators, we obtain (19). The summation over k stops at $k=n-M$.

If n becomes complex, then (19) is more difficult to justify. The main problem is the question whether the states $|nMn-k\lambda\rangle$ with $k=0,\cdots,\infty$ form a complete basis. A second question is how to continue analytically $d_{\lambda\lambda'}{}^{\alpha k}$, $d_{n-kS\lambda}{}^{nM}$, and $d_{SS'\lambda}{}^{nM}$.

[29] S. Klein, Ph.D. thesis, Brandeis University, 1967 (unpublished).

Bitar and Tindle claim to have derived (19) by using Carlson's theorem and the analytic continuations of Salam. An alternative justification of (19) is given by the brute-force expansion discussed in the Appendix.

In order to check whether the choice of residue given by (15) has the proper behavior at $t=0$ and pseudothresholds, we must examine the amplitude when crossed to the s-channel c.m. system. The crossing formula has been given by Trueman and Wick[30] and by Muzinich[31]:

$$S_{c'd',a'b'}=\pm\sum_{abcd} T_{bd,ac}d_{aa'}{}^{S_a}(\chi_a)d_{bb'}{}^{S_b}\times(\chi_b)d_{cc'}{}^{S_c}(\chi_c)d_{dd'}{}^{S_d}(\chi_d)\,.\quad(22)$$

The $(\pm)$ phase ambiguity is of no importance to us, and we shall simplify our formulas by ignoring such over-all phases. The crossing angles are given by

$$\cos\chi_a=-\left[\frac{(s+M_a{}^2-M_b{}^2)(t+M_a{}^2-M_c{}^2)-2M_a{}^2\Delta}{\Delta(s,a,b)\Delta(t,a,c)}\right],$$

$$\cos\chi_b=+[a\leftrightarrow b,\,c\leftrightarrow d]\,,\quad(23)$$

$$\cos\chi_c=+[a\leftrightarrow c,\,b\leftrightarrow d]\,,$$

$$\cos\chi_d=-[a\leftrightarrow d,\,b\leftrightarrow c]\,,$$

where $\Delta=M_a{}^2-M_b{}^2-M_c{}^2+M_d{}^2$. Our choice of phases for the crossing angles is motivated by (21).

When we combine (15), (22), (1), and (8), we get

$$S_{c'd',a'b'}=\frac{1+\tau e^{-i\pi(n-v)}}{\sin\pi(n-v)}\sum_{k\lambda\lambda',abcd}[\eta_b\bar\eta_d(-)^{S'-v}\times d_{n-kS'-\lambda'}{}^{nM}(+\phi_{tb})d_{n-kS\lambda}{}^{nM}(\phi_{ta})+\eta_a\bar\eta_c(-)^{S-v}d_{n-kS'-\lambda'}{}^{n-M}(+\phi_{tb})d_{n-kS\lambda}{}^{n-M}(\phi_{ta})]\times D_{\lambda'\lambda}{}^{n-k}(\theta_t)(-)^kC(S_aS_cS;\,a-c\lambda)\times C(S_bS_dS;\,b-d\lambda')d_{aa'}{}^{S_a}(\chi_a)d_{bb'}{}^{S_b}(\chi_b)\times d_{cc'}{}^{S_c}(\chi_c)d_{dd'}{}^{S_d}(\chi_d)\,,\quad(24)$$

where we have use the symmetry property[28]

$$d_{n-kS\lambda}{}^{nM}(\phi_{ta})=d_{n-kS-\lambda}{}^{n-M}(\phi_{ta})\,.\quad(25)$$

By using another symmetry property of the boost functions

$$d_{\alpha S\lambda}{}^{nM}(\phi)=(-)^{\alpha-S}d_{\alpha S,-\lambda}{}^{nM}(-\phi)\quad(26)$$

and the analytic continuation of (19), we can immediately carry out the sum over k:

$$\sum_k d_{n-k,S',-\lambda'}{}^{nM}(\phi_{tb})D_{\lambda'\lambda}{}^{n-k}(\theta_t)d_{n-k,S\lambda}{}^{nM}(\phi_{ta})(-)^k=e^{i\pi(n+S')}\sum_\tau d_{\lambda'\tau}{}^{S'}(-\chi_b)D_{S'S\tau}{}^{nM}(\phi_{ab})d_{\tau\lambda}{}^{S}(\chi_a)\,,\quad(27)$$

[30] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 332 (1964).
[31] I. J. Muzinich, J. Math. Phys. **5**, 1481 (1964).

where $D_{S'S\lambda}{}^{nM}$ is the analytic continuation of $d_{S'S\lambda}{}^{nM}$ that has the same asymptotic behavior as $D_{\lambda'\lambda}{}^{n}(\theta)$. The sum over the helicity indices is also straightforward, with the following Clebsch-Gordan identity[32]:

$$\sum_{\lambda ab} C(S_aS_cS;\, a-c\lambda)d_{aa'}{}^{S_a}(\chi_a)d_{cc'}{}^{S_c}(\chi_c)d_{\tau\lambda}{}^{S}(-\chi_a)$$
$$=C(S_aS_cS;\, a'-c''\tau)d_{c''c'}{}^{S_c}(\chi_a+\chi_c). \quad (28)$$

The final result is

$$S_{cd,ab}=\frac{e^{i\pi(n-v)}+\tau}{\sin\pi(n-v)}\sum_{\mu} C(S_aS_cS;\, a-c'\mu)$$
$$\times C(S_bS_dS';\, b-d'\mu)d_{c'c}{}^{S_c}(\chi_a+\chi_c)$$
$$\times d_{d'd}{}^{S_d}(\chi_d+\chi_b)[\eta_b\bar{\eta}_d(-)^{S'-v}D_{SS'\mu}{}^{nM}(\phi_{ab})$$
$$+\eta_a\bar{\eta}_c(-)^{S-v}D_{SS'\mu}{}^{n-M}(\phi_{ab})]. \quad (29)$$

One can show that $\sin\frac{1}{2}(\chi_a+\chi_c)$ has the behavior

$$\sin\tfrac{1}{2}(\chi_a+\chi_c)\sim\{(t-t_{\min})/[t-(M_a+M_c)^2]\}^{1/2} \quad (30)$$

by using (23) and the identity $\sin(\theta+\theta')=\cos\theta\sin\theta'+\sin\theta\cos\theta'$. Since $\cos\frac{1}{2}(\chi_a+\chi_c)$ and $\sin\frac{1}{2}(\chi_a+\chi_c)$ are well behaved at $t=0$ and pseudothreshold (they do not blow up), so too must be the function $d_{cc'}{}^{S_c}(\chi_a+\chi_c)$. It is now easy to check that $S_{cd,ab}$ does not violate the analytic behavior that was assumed in (9).

The asymptotic behavior of (29) as $s\to\infty$ can be determined by using (30) and

$$D_{SS'\mu}{}^{nM}(\phi_{ab})\sim s^{n-|M-\mu|}, \quad (31)$$

where $t_{\min}$ is given by (10). Thus we get

$$S_{cd,ab}\sim\frac{e^{i\pi(n-v)}+\tau}{\sin\pi(n-v)}\sum_{\mu''} C(S_aS_cS;\, a-c'\mu'')$$
$$\times C(S_bS_dS';\, b-d'\mu'')(t-t_{\min})^{(|\mu''-\mu|+|\mu''-\mu'|)/2}$$
$$\times(s^{n-|M-\mu''|}+\eta s^{n-|M+\mu''|}), \quad (32)$$

where $\mu=a-c$, $\mu=b=d$, and $\eta=\eta_a\eta_b\bar{\eta}_c\bar{\eta}_d(-)^{S-S'}$. The s^n behavior is given by

$$S_{cd,ab}\sim\frac{e^{i\pi(n-v)}+\tau}{\sin\pi(n-v)}s^n$$
$$\times[t^{(|M-\mu|+|M-\mu'|)/2}+\eta t^{(|M+\mu|+|M+\mu'|)/2}]. \quad (33)$$

The singularity structure given by (32) is consistent with the expected kinematic singularities of the s-channel amplitude, shown in (9). The discussion following (12) shows that (32) and (33) have exactly the desired asymptotic behavior.

[32] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

IV. GENERALIZED k DEPENDENCE

In the previous section we demonstrated that if we choose

$$\beta_{kS\lambda}(t)=d_{n-kS\lambda}{}^{nM}(\phi_{ta})a_S(t), \quad (34)$$

then the scattering amplitude satisfies the analyticity constraints. However, (34) has an important deficiency: It is not the most general function that satisfies the constraints. That is, the k and λ dependence of $\beta_{kS\lambda}(t)$ is severely constrained. Once the residue for $k=\lambda=0$ is known, (34) gives the residue for all other values of k and λ. In this section we shall modify (34) to allow for the most general k dependence and λ dependence.

The most general expansion of $\beta_{kS\lambda}(t)$ is given by

$$\beta_{kS\lambda}(t)=\sum_{k'=0}^{k}\sum_{M'=-S}^{S} d_{n-k,S\lambda}{}^{n-k',M'}(\phi_a)$$
$$\times\begin{Bmatrix} J_+' & J_-' & n-k\\ J_- & J_+ & J\end{Bmatrix}a_{Sk'M'}(t), \quad (35)$$

where

$$J_\pm=\tfrac{1}{2}(n\pm M),\quad J_\pm'=\tfrac{1}{2}(n-k'\pm M'),$$
$$J=\tfrac{1}{2}(k'+|M-M'|),$$

and where $\{\ \}$ is a 6-j symbol defined by Edmonds.[32]

We can show that (35) is the most general expansion of $\beta_{kS\lambda}(t)$ for arbitrary t, by expressing $a_{Sk'M'}(t)$ in terms of $\beta_{kS\lambda}$. In order to invert (35), it is convenient to use the following expression for $d_{n-k,S\lambda}{}^{n-k,M'}(\phi_a)$:

$$d_{\alpha_kS\lambda}{}^{\alpha_kM'}(\phi_a)=C_{\alpha_k\lambda}C'_{\alpha_kSM'}(t)d_{M'\lambda}{}^{S}(\chi_a{}^\infty), \quad (36)$$

where

$$\sinh\phi_a=p_a/M_a=(\sinh\chi_a{}^\infty)^{-1},$$
$$C_{\alpha_k\lambda}=[(-)^\lambda\Gamma(\alpha_k+\lambda+1)\Gamma(\alpha_k-\lambda+1)]^{1/2},$$
$$C_{\alpha_kSM'}{}'(t)=\left(\frac{t^{1/2}M_a}{\Delta(t,a,b)}\right)^{-\alpha}$$
$$\times\left[\frac{(2S+1)\Gamma(\alpha_k+M+1)\Gamma(\alpha_k-M+1)}{\Gamma(2\alpha_k+1)\Gamma(\alpha_k+S+1)\Gamma(\alpha_k-S+1)}\right]^{1/2}.$$

We have obtained (36) by comparing the s^n pieces on both sides of (19). The angle $\chi_a{}^\infty$ is the limit of the crossing angle χ_a as $s\to\infty$.

Since the summation $n-k'$ in (35) is bounded by k, we expect that we can solve for $a_{Sk'M'}$ in terms of $\beta_{kS\lambda}$ with $k<k'$. Let us assume, for example, that $a_{Sk'M}$ is known for $k'<k$. We must show that we can find $a_{SkM}(t)$. We can write (35) as

$$\beta_{kS\lambda}(t)=\sum_{k'=0}^{k-1}\sum_{M'=-S}^{S} d_{n-k,S\lambda}{}^{n-k',M'}(\phi)$$
$$\times\begin{Bmatrix} J_+' & J_-' & n-k\\ J_- & J_+ & J\end{Bmatrix}a_{Sk'M'}(t)+C_{\alpha_k\lambda}\sum_{M'=-S}^{S} C_{\alpha_kSM'}(t)$$
$$\times d_{M'\lambda}{}^{S}(\chi)\begin{Bmatrix} J_+' & J_-' & n-k\\ J_- & J_+ & J\end{Bmatrix}a_{SkM'}(t).$$

We can now solve for $a_{SkM}(t)$:

$$a_{SkM'}(t)=\left[C_{\alpha kSM'}\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J\end{Bmatrix}\right]^{-1}$$
$$\times\sum_{\lambda=-S}^{S}C_{\alpha k\lambda}{}^{-1}d_{\lambda M'}{}^{S}(-\chi)$$
$$\times\left[\beta_{kS\lambda}(t)-\sum_{k'=0}^{k-1}\sum_{M''=-S}^{S}d_{n-k,S\lambda}{}^{n-k',M''}\right.$$
$$\left.\times\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J\end{Bmatrix}a_{Sk'M''}(t)\right].$$

It is apparent that the expansion (35) completely spans the k, λ space, since the inversion is nonsingular, except possibly at points where $n-M$ is an integer. We shall examine the behavior of the residue at these nonsense points in a future paper.

It will be useful for us to express the residue in an *overcomplete* expansion:

$$\beta_{kS\lambda}(t)=\sum_{n''M''J''}d_{n-k,S\lambda}{}^{n''M''}(\phi_{ta})\times\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J''\end{Bmatrix}a_{k''M''J''S}(t),$$
$$\beta_{kS'\lambda'}(t)=\sum_{n'M'J'}d_{n-k,S'\lambda'}{}^{n'M'}(\phi_{tb})\times\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J'\end{Bmatrix}a_{n'M'J'S'}(t),\qquad(37)$$

where $n''=n-k''$, $n'=n-k'$, $J_\pm''=\frac{1}{2}(n''\pm M'')$, and $J_\pm'=\frac{1}{2}(n'\pm M')$. We should point out that k', k'', $|M''-M|$, and $|M'-M|$ are integers. The summation over J' and J'' is *not* needed for completeness, but we shall find it a useful extension. The statement that (37) is overcomplete means that the functions can be expanded in terms of the coefficients $a_{n''M''J''S}$ but that these coefficients are not unique.

We shall now demonstrate that the analyticity constraint at $t=0$ is equivalent to the following condition on the expansion coefficient:

$$a_{n''M''J''S}(t)=\bar{a}_{n''M''J''S}(t)t^{J''},\qquad(38)$$

where $\bar{a}_{n''M''J''S}(t)$ is a dynamical coefficient that is analytic at $t=0$.

Our procedure is to examine the s-channel amplitude just as was done in (24). From the discussion following (24), we see that our main task will be to carry out the following summation:

$$S_{\mu'\mu''}=\sum_{\lambda'\lambda''k}d_{\mu'\lambda'}{}^{S'}(-\chi_b)d_{n-k,S'\lambda'}{}^{n'M'}(-\phi_{tb})$$
$$\times D_{\lambda'\lambda''}{}^{n-k}(\theta_t)d_{n-k,S''\lambda''}{}^{n''M''}(\phi_{ta})d_{\lambda''\mu''}{}^{S''}(\chi_a)$$
$$\times\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J'\end{Bmatrix}\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J''\end{Bmatrix}.\qquad(39)$$

We can use a permuted form of the main sum rule (19) to carry out the sum over λ'':

$$\sum_{\lambda''}D_{\lambda'\lambda''}{}^{n-k}(\theta_t)d_{n-kS\lambda''}{}^{n''M''}(\phi_{ta})d_{\lambda''\mu''}{}^{S}(\chi_a)$$
$$=\sum_{L'}d_{n-k,L'\lambda'}{}^{n''M''}(+\phi_{tb})\times d_{\lambda'\mu'}{}^{L'}(\chi_b)D_{L'S\mu''}{}^{n''M''}(\phi_{ab}).\qquad(40)$$

The summation over L' extends from $L'=M''$ to $L'=+\infty$. However, the k summation will restrict L' to a finite range, as we shall soon see.

The summation over k is

$$G_{\lambda'}=\sum_k d_{n-kS'\lambda'}{}^{n'M'}(-\phi_{tb})d_{n-kL'\lambda'}{}^{n''M''}(\phi_{tb})\times\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J'\end{Bmatrix}\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J''\end{Bmatrix}.\qquad(41)$$

This summation can be carried out if we use (18) and (26) to express $d_{n-kS\lambda}{}^{n''M''}$. The k-dependent pieces of the summation (41) are

$$\sum C(J_+'J_-'\,n-k;\,\mu_+'\mu_-'\lambda')C(J_+''J_-''\,n-k;\,\mu_+''\mu_-''\lambda')\times\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J'\end{Bmatrix}\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J''\end{Bmatrix}.\qquad(42)$$

We now use the identity[32]

$$C(J_+'J_-'\,n-k;\,\mu_+'\mu_-'\lambda')\begin{Bmatrix}J_+' & J_-' & n-k\\ J_- & J_+ & J'\end{Bmatrix}$$
$$=(-)^k c\sum C(J_+'J'J_+;\,\mu_+'m'\tau_+')$$
$$\times C(J_-J'J_-',\tau_-'m'\mu_-')C(J_+J_-\,n-k,\,\mu_+'\mu'\lambda),\qquad(43)$$

where

$$c=(-)^{J'+(n'-n+M'-M)/2}[(2J_++1)(2J_-'+1)]^{-1/2}.\qquad(44)$$

We shall henceforth drop the constant c, since it can be absorbed into the dynamical coefficient $a_{n'M'J'S'}$ of (37).

The sum over k in (42), which is one of the main steps of our analysis, can now be done using the identity[32]

$$\sum_k C(J_+J_-\,n-k;\,\tau_+'\tau_-'\lambda')C(J_+J_-\,n-k;\,\tau_+''\tau_-''\lambda')=\delta_{\tau_+''\tau_+'}\delta_{\tau_-''\tau_-'}.\qquad(45)$$

Now that the summation over k has been done, we can proceed to evaluate (41) and (18):

$$G_{\lambda'}=\sum_{\mu_\pm'\mu_\pm''m'm''}C(J_+'J_-'S';\,\mu_+'\mu_-'\lambda)$$
$$\times C(J_+''J_-''L'';\,\mu_+''\mu_-''\lambda')$$
$$\times e^{(\mu_+'-\mu_-'-\mu_+''+\mu_-'')\phi_{tb}}C(J_+'J'J_+;\,\mu_+'m'\tau_+')$$
$$\times C(J_-J'J_-'';\,\tau_-'m'\mu_-'')C(J_-''J'J_-;\,\mu_-''m'\tau_-')$$
$$\times C(J_+J''J_+'';\,\tau_+'m''\mu_+'').\qquad(46)$$

The summation over $\mu_\pm'$ and $\mu_\pm''$ can be done, since

$$\begin{aligned}\mu_+'-\mu_-'-\mu_+''+\mu_-'' &= \tau_+'-m'-\tau_-'-m'-\tau_+'-m''+\tau_-'-m'' \\ &= -2(m''+m'). \end{aligned} \qquad (47)$$

In order to simplify (46), we use the following relations[32]:

$$C(J_+'J'J_+;\mu_+'m'\tau_+')C(J_+J''J_+'';\tau_+m''\mu_+'')$$
$$=\sum_J C(J'J''J;m'm''m)C(J_+'JJ_+'';\mu_+m\mu_+')$$
$$\times\begin{Bmatrix}J_+' & J_+'' & J\\ J'' & J' & J_+\end{Bmatrix}(-)^{J_+'+J'+J_+''+J''}$$
$$\times[(2J+1)(2J_++1)]^{1/2}, \qquad (48)$$

$$\sum_{m'm''} C(J'J''J;m'm''m)\times C(J'J''J''';m'm''m)=\delta_{JJ'''}, \qquad (49)$$

and

$$\sum_{\mu_\pm'} C(J_+'J_-'S';\mu_+'\mu_-'\lambda')C(J_+''J_-''L';\mu_+''\mu_-''\lambda')$$
$$\times C(J_+'JJ_+'';\mu_+'m\mu_+'')C(J_-''JJ_-';\mu_-''-m\mu_-')$$
$$=\sum_{\bar J} C(J\bar J J;m0m)C(S'\bar J L';\lambda'0\lambda')\begin{Bmatrix}J_+' & J_-' & S'\\ J_+'' & J_-'' & L'\\ J & J & \bar J\end{Bmatrix}$$
$$\times[(2J_-'+1)(2J_+''+1)]^{1/2}, \qquad (50)$$

where the bracketed quantity is a 9-J coefficient.

Upon neglecting constant factors, which can be absorbed into the coefficient in (37), we arrive at

$$G_{\lambda'}=\sum_{J\bar J m} b_{J\bar J S'L'}C(J\bar J J;m0m)\times C(S\bar J L';\lambda'0\lambda')e^{-2m\phi tb}, \qquad (51)$$

where

$$b_{JJS'L'}=\begin{Bmatrix}J_+' & J_+'' & J\\ J'' & J' & J_+\end{Bmatrix}\begin{Bmatrix}J_-' & J_-'' & J\\ J'' & J & J_-\end{Bmatrix}\times\begin{Bmatrix}J_+' & J_-' & S'\\ J_+'' & J_-'' & L'\\ J & J & \bar J\end{Bmatrix}. \qquad (52)$$

All the manipulations since (41) have been made in an effort to extract the λ' dependence in a clean manner. We have accomplished this in (51), so we are now able to carry out the summation on λ' in (39):

$$\sum_{\lambda'} d_{\mu'\lambda'}{}^{S'}(-\chi_b)d_{\lambda'\mu''}{}^{L'}(\chi_b)C(S'\bar J L';\lambda'0\lambda')$$
$$=C(S'\bar J L';\mu',\mu''-\mu',\mu'')d_{\mu''-\mu',0}{}^{\bar J}(\chi_b). \qquad (53)$$

We can finally write (39) as

$$S_{\mu'\mu''}=\sum_{L'J\bar J m} b_{\bar J J L'S}C(J\bar J J;m0m)$$
$$\times e^{-2m\phi tb}d_{\mu''-\mu',0}{}^{\bar J}(\chi_b)C(S'\bar J L';\mu',\mu''-\mu',\mu'')$$
$$\times D_{L'S\mu''}{}^{n''M''}(\phi_{ab}). \qquad (54)$$

We are now able to examine the behavior of the s-channel amplitude in the vicinity of $t=0$. The expected t dependence is given by (9):

$$S_{\mu',\mu''}\sim(t-t_{\min})^{|\mu'-\mu''|/2}, \qquad (55)$$

while the dominant behavior of (54) is

$$S_{\mu',\mu''}\sim e^{2J_{\max}\phi tb}[\sin^{|\mu'-\mu''|}(\chi_b)\cos^{2J_{\max}-|\mu'-\mu''|}(\chi_b)]\times(\cosh\phi_{ab})^{n''-|M''-\mu''|}, \qquad (56)$$

where $J_{\max}=J'+J''$ is the maximum value of J allowed by (48). We need not worry about the last factor, since it is independent of t:

$$\cosh\phi_{ab}=(-s+M_a{}^2+M_b{}^2)/2M_aM_b.$$

There are three possible mass cases to be considered:

(a) $M_b\neq M_d$. In this case $\cos\chi_b$ is analytic at $t=0$, while $e^{\phi tb}\sim t^{-1/2}$. Thus

$$S_{\mu'\mu''}\sim t^{-(J'+J'')}. \qquad (57)$$

(b) $M_b=M_d$, $M_a\neq M_b$. In this case $e^{\phi tb}$ is analytic at $t=0$, but $\cos\chi_b\sim(M_a{}^2-M_c{}^2)t^{-1/2}s^{-1}$. Thus

$$S_{\mu'\mu''}\sim t^{-(J'+J'')}. \qquad (58)$$

(c) $M_b=M_d$, $M_a=M_c$. In this case both $e^{\phi tb}$ and $\cos\chi_b$ are analytic at $t=0$, and we find

$$S_{\mu'\mu''}\sim d_{\mu'-\mu'',0}{}^{J'+J''}(\tfrac{1}{2}\pi)\sim\text{const},$$

whereas we expected $t^{|\mu'-\mu''|/2}$.

In all three cases we find that the amplitude is more singular than it is allowed to be. Thus the residues (35) require extra factors of t to restore the proper analyticity. If $J'+J''>S'+S''$, then case (c) does not violate analyticity as badly as cases (a) and (b). (There is no violation in the case of spinless external particles.) However, if an equal-mass channel is ever to couple to an unequal-mass channel, then the damping factor of $t^{J'+J''}$ is required.

Since the t-channel residues factorize, the damping factor must be split between incoming and outgoing channels. We have thus verified that the most general k dependence in the residue is given by

$$\beta_{kS\lambda}(t)=\sum_{n''M''J''} d_{n-k\,S\lambda}{}^{n''M''}(\phi_{ta})\times\begin{Bmatrix}J_+'' & J_-'' & n-k\\ J_- & J_+ & J''\end{Bmatrix}t^{J''}\bar a_{n''M''J''S}(t), \qquad (59)$$

where $\bar a$ is analytic at $t=0$.

We have previously shown that the summation over J'' is redundant. In the next section we shall use this redundancy to show that the 6-j symbols and $d_{\alpha S\lambda}{}^{nM}$ have expansions which make these functions much easier to handle.

V. SIMPLIFIED EXPANSION NEAR $t=0$

One of the main new results of our work so far is that the Regge vertex can be written in the following overcomplete manner (the sum over J is redundant):

$$\beta_{kS\lambda}(t)==\sum_{M',n',J} d_{\alpha_k S\lambda}{}^{n'M'}(\phi)a_{Jn'M'}(t)\times\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix} t^J, \quad (60)$$

where $J_\pm'=\frac{1}{2}(n'\pm M')$, $J_\pm=\frac{1}{2}(n\pm M)$. We achieve general helicity dependence by the sum over M'. There are two defects in this result which we would like to correct in this section. First, (60) is in a form that discourages practical applications, since the $O(4)$ matrices and 6-j symbols seem unnecessarily complicated to the person who would merely like to know the first few terms of the expansion for the first few daughters. Second, when we argued that (60) contained the most general k dependence, we had to conjecture that certain sum rules were still valid even after the angular momenta became complex.

We shall remove these defects in this section by focusing on the behavior near $t=0$ (that is, we shall ignore the behavior at pseudothresholds). We shall show that the vertex $\beta_{kS\lambda}$ for the case $M_a\neq M_c$ can be written in the form

$$\beta_{kS\lambda}(t)=\left[\frac{2\alpha_k+1}{t^{n-|\lambda-M|}k!\Gamma(2\alpha_k+k+2)}\right]^{1/2}\times\left[\frac{\Gamma(\alpha_k+M+1)\Gamma(\alpha_k-\lambda+1)}{\Gamma(\alpha_k+\lambda+1)\Gamma(\alpha_k-M+1)}\right]^{(M-\lambda)/2|M-\lambda|}\times f_{k,S\lambda}(t). \quad (61)$$

It turns out that (61) is valid even for $S<M$.

The function $f_{k,i}(t)$ is basic both to the discussion in this section and to the nonparallel case, which will be discussed in the next section. It is given by

$$f_{k,i}(t)=\sum_{J=0}^{k}\frac{k!\Gamma(2\alpha_k+k+2)}{(k-J)!\Gamma(2\alpha_k+k-J+2)}t^J a_{J,i}(t), \quad (62)$$

where $a_{J,i}(t)$ is analytic at $t=0$.

As our first step in the derivation of (61), we must consider the expansion of

$$g_k(t)=\sum_{J=0}^{\infty}\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix} t^J\bar{a}_J(t). \quad (63)$$

By using Eq. (62.12) of Edmonds,

$$\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix}\begin{Bmatrix} J_+ & J_- & \alpha_k \\ J_- & J_+ & 1\end{Bmatrix} =(-)^{J'+c_1}\sum_{J'}(2J'+1)\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J'\end{Bmatrix}\times\begin{Bmatrix} J_+' & J_+ & J \\ 1 & J' & J_+\end{Bmatrix}\begin{Bmatrix} J_-' & J_- & J \\ 1 & J' & J_-\end{Bmatrix}, \quad (64)$$

we can get the following recursion relation:

$$\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J+1\end{Bmatrix}c_2 =\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix}\left[\begin{Bmatrix} J_+ & J_- & \alpha_k \\ J & J_+ & 1\end{Bmatrix}(-)^{k+c_1}-c_3\right] +\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J-1\end{Bmatrix}c_4, \quad (65)$$

where c_i are coefficients that are independent of k.

We can make use of Table 5 of Edmonds, which gives

$$(-)^{k+c_1}\begin{Bmatrix} J_+ & J_- & \alpha_k \\ J_- & J_+ & 1\end{Bmatrix}=c_4+\alpha_k(\alpha_k+1)c_5, \quad (66)$$

to write (65) as

$$\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix}=\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J-1\end{Bmatrix}\times[c_6\alpha_k(\alpha_k+1)+c_7]+\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J-2\end{Bmatrix}c_8. \quad (67)$$

By induction we can immediately get

$$\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J\end{Bmatrix}=\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J_{\min}\end{Bmatrix}\sum_{i=0}^{J-J_{\min}}c_i[\alpha_k(\alpha_k+1)]^i, \quad (68)$$

where

$$J_{\min}=\max[|J_+'-J_+|,|J_-'-J_-|]=\tfrac{1}{2}(k'+|M'-M|).$$

We can now write (64) as

$$g_k(t)=\begin{Bmatrix} J_+' & J_-' & \alpha_k \\ J_- & J_+ & J_{\min}\end{Bmatrix}f_{k,i}(t), \quad (69)$$

where

$$f_{k,i}(t)=\sum_{J=0}^{\infty}a_J(t)[\alpha_k(\alpha_k+1)t]^J. \quad (70)$$

The easiest way to obtain the form (62) is to carry out an argument by induction. We first assume

$$f_{k,i}{}^{N}(t)\equiv\sum_{J=0}^{N}a_{J,1}(t)[\alpha_k(\alpha_k+1)t]^J$$

$$=\sum_{J=0}^{N}\frac{k!\Gamma(2\alpha_k+k+2)}{(k-J)!\Gamma(2\alpha_k+k-J+2)}a_{J,2}(t)t^J \quad (71)$$

for all $N\leq N_0$.

Now examine the sum for N_0+1:

$$f_{k,1}{}^{N_0+1}=\sum_{J=0}^{N_0}a_{J,1}(t)[\alpha_k(\alpha_k+1)t]^J+a_{N_0+1,1}(t)[\alpha_k(\alpha_k+1)t]^{N_0+1}. \quad (72)$$

The identity

$$\prod_{J=0}^{N_0+1}\{\alpha_k(\alpha_k+1)+c_J\}=\sum_{J=0}^{N_0+1}[\alpha_k(\alpha_k+1)]^J c_J' \quad (73)$$

with the choice

$$c_J=-n(n+1)+J(2n-J+1)$$

allows (72) to be written as

$$f_{k,1}{}^{N_0+1}(t)=f_{k,2}{}^{N}(t)+a_{N_0+1}(t)\frac{k!\Gamma(2\alpha_k+k+2)}{(k-J)!\Gamma(2\alpha_k+k-J+2)}t^{N_0+1}, \quad (74)$$

where

$$a_{J,2}(t)=a_{J,1}(t)+c_J t^{N_0+1-J}.$$

We are continuing our convention that $a_{J,i}(t)$ and c_J are coefficients that are independent of k. The first term on the right-hand side of (74) can be replaced by using (71), and we finally get

$$f_{k,1}{}^{N_0+1}(t)=\sum_{J=0}^{N_0+1}\frac{k!\Gamma(2\alpha_k+k+2)}{(k-J)!\Gamma(2\alpha_k+k-J+2)}t^J a_{J,3}(t), \quad (75)$$

which completes our induction proof.

Since (75) is valid for arbitrarily large N, and since the factor $1/(k-J)!$ vanishes for $J>k$, all terms in the sum with $J>k$ can be dropped, and we see that $f_{k,i}$ can be written as in (63).

It is clear from (71) that $f_{k,i}(t)$ has the following important property:

$$f_{k,1}(t)f_{k,2}(t)=f_{k,3}(t).$$

In fact, as long as $\alpha_k(\alpha_k+1)t$ is small, we have

$$F(f_{k,i}(t))=f_{k,j}(t), \quad (76)$$

where $F(f)$ stands for an arbitrary function of various $f_{k,i}(t)$'s.

By combining (69), (63), and (60), we see that the Regge vertex can be written as

$$\beta_{kS\lambda}(t)=\sum_{n'M'}d_{\alpha_k S\lambda}{}^{n'M'}(\phi)\begin{Bmatrix}J_+' & J_-' & \alpha_k\\ J_- & J_+ & J_{\min}\end{Bmatrix}\times t^{J_{\min}}f_{k,SM'n'}(t). \quad (77)$$

The k dependence of $f_{k,SM'n'}$ is unnecessary (since our expansion is overcomplete) and serves to remind us that the residue can always be multiplied by a function whose k dependence has the form $f_{k,i}(t)$ without disturbing the analyticity.

We can simplify (77) even further by expanding the remaining 6-j symbol in terms of gamma functions. Since $J_{\min}=\max\{|J_+'-J_+|,|J_-'-J_-|\}$, we can use (6.3.1) of Edmonds to write

$$t^{J_{\min}}\begin{Bmatrix}J_+' & J_-' & \alpha_k\\ J_- & J_+ & J_{\min}\end{Bmatrix}=\left[\frac{k!\Gamma(2\alpha_k+k+2)t^{k'}}{(k-k')!\Gamma(2\alpha_k+k-k'+2)}\right]^{1/2}\times\left[\frac{\Gamma(\alpha_k+M+1)\Gamma(\alpha_k-M'+1)t^{M-M'}}{\Gamma(\alpha_k+M'+1)\Gamma(\alpha_k-M+1)}\right]^{\pm 1/2}\times C_{k'M',M}. \quad (78)$$

The $\pm$ sign occurs for $M\gtrless M'$. The coefficient $C_{k'M'M}$ is independent of k and can therefore be absorbed into the dynamic coefficient of (77).

In order to show that (77) can be put in the form of (61), we shall use the results of the Appendix, where a brute-force expansion of the scattering amplitude in powers of t and s is examined. The brute-force method may not be the most convenient way to get a general expression for the residue, but it is a useful way to *count* the number of constraints on the residue.[14] The result that we find for the *unequal-mass* case can be stated as follows: The first k derivatives of $\beta_{kS\lambda}(t)$ are determined in terms of the first k derivatives of $\beta_{kS\lambda}'$ with $k'<k$. The higher derivatives of $\beta_{kS\lambda}(t)$ are free.

This constraint has important consequences for (77). Simply by counting the number of free parameters, we can conclude that a *complete* expansion of the residue is given by either of the following equations:

$$\beta_{kS\lambda}(t)=\sum_{n'M'}d_{\alpha_k S\lambda}{}^{n'M'}(\phi)\begin{Bmatrix}J_+' & J_-' & \alpha_k\\ J_- & J_+ & J_{\min}\end{Bmatrix}\times t^{J_{\min}}a_{SM'n'}(t),$$
$$\beta_{kS\lambda}(t)=d_{\alpha_k S\lambda}{}^{nM}(\phi)f_{kS\lambda}(t). \quad (79)$$

In the unequal-mass case, the form in (79) leads to the most convenient expansion of the residue. We can further simplify $\beta_{kS\lambda}$ by separating $d_{\alpha S\lambda}{}^{nM}(\phi)$ into two factors:

$$d_{\alpha S\lambda}{}^{nM}(\phi)=d_{\alpha S\lambda}{}^{nM}(t)\,|_0 g_{kS\lambda}(t)\,, \qquad (80)$$

where $d_{\alpha S\lambda}{}^{nM}(t)\,|_0=d_k\cdot p^n$ is the leading piece of $d_{\alpha S\lambda}{}^{nM}$ near $t=0$, and $g_{kS\lambda}(t)=\sum_i c_{ik}t^i$.

It can be shown that the constraints of the Appendix lead to the conclusion that $g_{kS\lambda}(t)$ must be of the form $f_k(t)$.

We shall illustrate how one shows $g_{kS\lambda}(t)=f_k(t)$ by showing that the first derivative of $g_{kS\lambda}(t)$ must have the same k dependence as the first derivative of $f_{kS\lambda}(t)$.

The logarithmic derivative of $\beta_{kS\lambda}(t)$ is given by

$$\frac{\beta_{kS\lambda}'(t)}{\beta_{kS\lambda}(t)}=\frac{np'}{p}+\frac{g_{kS\lambda}'}{g_{kS\lambda}}+\frac{f_k'}{f_k}=\frac{np'}{p}+c_{1k}+\frac{a_0'+a_1k(2n+1-k)}{a_0}\,, \qquad (81)$$

where the prime indicates a derivative with respect to t. It should be remarked that our arguments should really be made with the combination of residues, which has a definite parity.

The result of the Appendix implies that β_k'/β_k is determined once its value for $k=0$ and $k=1$ is known. Since a_0' and a_1 are free parameters, we can conclude that c_{1k} must have the same k dependence as $a_0'+a_1k\times(2n+1-k)$, or else β_2'/β_2 would be free. This argument can be extended to imply $g_{kS\lambda}(t)$ must have the k dependence given by $f_k(t)$. Thus the residue for unequal masses can be put in the form

$$\beta_{kS\lambda}(t)=d_{\alpha S\lambda}{}^{nM}(t)\,|_{t=0}f_{kS\lambda}(t)\,. \qquad (82)$$

We can now use (18) to find the singular part of $d_{\alpha S\lambda}{}^{nM}$. For $\lambda=M$ we can take $n=n'=J_++J_-$ and $M=M'=J_+-J_-$. We get

$$d_{\alpha_k SM}{}^{nM}(\phi)=\sum_\mu C(J_+J_-\alpha_k;\,\mu\;M-\mu\;M)\times C(J_+J_-S;\,\mu\;M-\mu\;M)e^{(2\mu-M)\phi}\,, \qquad (83)$$

where $e^\phi\sim t^{-1/2}$. Thus

$$d_{\alpha_k S\lambda}{}^{nM}(\phi)\,|_{t=0}=C(J_+J_-\alpha_k;\,J_+-J_-M)\times C(J_+J_-S;\,J_+-J_-M)t^{-n/2}\,. \qquad (84)$$

The k dependence is given by

$$C(J_+J_-\alpha_k;\,J_+-J_-\,M)=\left[\frac{(2\alpha_k+1)\Gamma(2J_++1)\Gamma(\alpha_k+k-M+1)}{k!\Gamma(2\alpha_k+k+2)}\right]^{1/2}. \qquad (85)$$

We are finally able to write (60) as

$$\beta_{kSM}(t)=\left[\frac{(2\alpha_k+1)}{t^nk!\Gamma(2\alpha_k+k+2)}\right]^{1/2}f_{k,SM}(t)\,. \qquad (86)$$

This result is *identical* to that found in the Appendix, which was obtained by the brute-force method. The equivalence of these results serves to verify the validity of our procedures in Sec. IV, where we analytically continued various Clebsch-Gordan sum rules.

If $\lambda\neq M$, then more than one term of (83) would contribute to the leading behavior, and the calculation would not be simple. However, since the vertex for $\lambda\neq M$ has an extra damping factor of $t^{|\lambda-M|}$, we can make use of our overcomplete expansion by setting $M'=\lambda$. We get

$$\beta_{kS\lambda}=d_{\alpha S\lambda}{}^{n\lambda}(\phi)\,|_{t=0}\begin{Bmatrix}J_+' & J_-' & \alpha_k\\ J_- & J_+ & J\end{Bmatrix}\times t^{|\lambda-M|/2}f_{k,S\lambda 1}(t)\,, \qquad (87)$$

with $J_\pm'=\frac{1}{2}(n\pm\lambda)$ and $J=\frac{1}{2}|M-\lambda|$,

$$\beta_{kS\lambda}=C(J_+'J_-'\alpha_k;\,J_+'-J_-'\lambda)t^{-n/2}\begin{Bmatrix}J_+' & J_-' & \alpha_k\\ J_- & J_+ & J\end{Bmatrix} t^{|\lambda-M|/2}f_{k,S\lambda 2}(t)\,, \qquad (88)$$

where the Clebsch-Gordan coefficient is given in (85), and the 6-j coefficient is given in (78).

Thus we finally get

$$\beta_{kS\lambda}(t)=\left[\frac{2\alpha_k+1}{t^{n-|\lambda-M|}k!\Gamma(2\alpha_k+k+2)}\right]^{1/2}\times\left[\frac{\Gamma(\alpha_k+M+1)\Gamma(\alpha_k-\lambda+1)}{\Gamma(\alpha_k+\lambda+1)\Gamma(\alpha_k-M+1)}\right]^{(M-\lambda)/2|M-\lambda|}f_{k,S\lambda}\,. \qquad (89)$$

We can use (80) to prove an interesting result connected with the signature factor. Let us examine one term of (8):

$$\bar{T}_{S'\lambda',S\lambda}=\frac{1+\tau e^{-i\pi n}}{\sin\pi n}\sum\beta_{kS'-\lambda'}(\phi_b)\times\beta_{kS\lambda}(\phi_a)D_{\lambda\lambda'}{}^{\alpha_k}(\phi_t)(-)^k\,. \qquad (90)$$

In order to simplify our considerations, let us take $\beta_{kS\lambda}=d_{\alpha_k S\lambda}{}^{nM}(\phi_a)$. We can modify (90) by using

$$\begin{aligned}e^{-i\pi n}d_{\alpha_k S\lambda}{}^{nM}(\phi_a)M_a{}^n&=e^{-i\pi n}d_{\alpha_k S\lambda}{}^{nM}(\phi_a)\,|_{t=0}M_a{}^nf_{k,1}(t)\\&=d_{\alpha_k S\lambda}{}^{nM}(\pi-\phi_c)\,|_{t=0}M_c{}^nf_{k,1}(t)\\&=d_{\alpha_k S\lambda}{}^{nM}(\pi-\phi_c)M_c{}^nf_{k,2}(t)\,. \qquad (91)\end{aligned}$$

We will need the sum rule

$$\sum_k d_{\alpha_k S\lambda}{}^{nM}(-\phi_b)d_{\lambda'\lambda}{}^{\alpha_k}(\theta_t)d_{\alpha_k S\lambda}{}^{nM}(\pi-\phi_c)=\sum_\tau d_{\lambda'\tau}{}^{S'}(-\chi_b{}^{tu})d_{S'S\tau}{}^{nM}(\phi_{bc})d_{\tau\lambda}{}^S(\chi_c{}^{tu})\,, \qquad (92)$$

where χ^{tu} is the crossing angle from the t-channel c.m. to the u-channel c.m. system, and

$$\cosh\phi_{bc}=(M_b{}^2+M_c{}^2-u)/2M_bM_c\,.$$

The amplitude $\bar{T}_{S'\lambda',S\lambda}$ can be written

$$\bar{T}_{S'\lambda',S\lambda} = \frac{1}{\sin\pi n} \sum_{\tau} [d_{\lambda'\tau}{}^{S'}(-\chi_b{}^{st}) d_{S'S\lambda}{}^{nM}(\phi_{ab}) \times d_{\tau\lambda}{}^{S}(\chi_a{}^{st})(M_a M_b)^n + \tau(M_b M_c)^n f_{k,2}(t) d_{\lambda'\tau}{}^{S'}(-\chi_b{}^{ut}) \times d_{S'S\tau}{}^{nM}(\phi_{cb}) d_{\tau\lambda}{}^{S}(\chi_c{}^{ut})]. \quad (93)$$

For large s and fixed t, we are able to extract the standard signature factor $1+\tau e^{-i\pi n}$. In addition, Durand, Fishbane, and Simmons[33] have shown that the form of the amplitude in (93) has the correct cut structure at the s- and u-channel thresholds as long as the background integrals are properly treated. Since (93) has the correct reality properties in the region $s>0$, $t>0$, $u>0$, we have overcome a problem that plagues simple Regge theory. In order to guarantee that the amplitude is real below threshold, it has been proposed[34] that the number of trajectories should be doubled (four degenerate trajectories in the case $M\neq 0$). However, our analysis shows that this extra doubling is unnecessary.

VI. EXTENSION TO NONPARALLEL TRAJECTORIES

In this section we shall show that our solution can be easily adapted to the case in which the trajectories are no longer parallel. We shall rely heavily on the results of the last two sections, which can be summarized as

$$h(n,t) = \sum_k g_k(n-k,\ t) f_i(n-k,\ t), \quad (94)$$

where

$$g_k(n-k,\ t) = \beta_{kS'\lambda'}(t)\beta_{kS\lambda}(t) \times D_{\lambda'\lambda}{}^{\alpha_k}(\theta_t) \frac{1+\tau(-)^k e^{-i\pi\alpha_k}}{\sin\pi\alpha_k}, \quad (95)$$

and

$$f_i(\alpha_k,t) = f_{k,i}(t) = \sum_{j=0} [(\alpha_k+\tfrac{1}{2})^2 t]^j a_{j,i}(t). \quad (96)$$

The main result of the previous sections is that $h(n,t)$ has the proper analytic structure in t and n.

If the trajectories are allowed to become nonparallel, we can define the trajectory function as

$$\alpha_k(n,t) = n-k+\Delta_k(n,t), \quad (97)$$

where n is a constant equal to the intercept of the leading trajectory.

The question before us in this section is to find the most general form for $\Delta_k(n,t)$ and $\beta_{kS\lambda}(t)$ that still guarantees the correct analyticity for $h(n,t)$.

Let us, for instance, simply replace $n-k$ in (94) by $\alpha_k(t)$. We can use a contour integral to rewrite (94):

$$\sum g_k(\alpha_k,t) f(\alpha_k,t) = \frac{1}{2\pi i} \oint dL \sum_k \frac{g_k(L-k,\ t) f(L-k,\ t)}{L-n-\Delta_k(n,t)} \quad (98)$$

$$= \frac{1}{2\pi i} \oint dL \sum_{k,p} \frac{g_k(L-k,\ t) f(L-k,\ t)[\Delta_k(n,t)]^p}{(L-n)^{p+1}}. \quad (99)$$

We would be able to carry out the sum on k without violating analyticity if the k dependence of $\Delta_k(n,t)$ could be written in the form $f(L-k,\ t)$. This is true because, as shown in (76),

$$f_1(L-k,\ t)[f_2(L-k,\ t)]^p = f_3(L-k,\ t). \quad (100)$$

Thus, if $\Delta_k(n,t) = f(L-k,\ t)$, we would be able to use (94) to do the summation in (99). However, $\Delta_k(n,t)$ does not even depend on L, so it cannot possibly have the needed k dependence.

The resolution of the difficulty was pointed out by various persons for the case $S=0$.[19,35] The trick is to include an extra factor of $[1-d\bar{\Delta}(\alpha_k,t)/d\alpha_k]^{-1}$, where $\bar{\Delta}(\alpha_k,t) \equiv \Delta(n,t)$. We have here written $\bar{\Delta}_k(\alpha_k,t)$ as a function of α_k and t rather than as a function of n and t. This is permissible, since $\alpha_k(t)$ is a function of n, k, and t. The summation (98) can now be written as

$$\sum_k \frac{g_k(\alpha_k,t) f(\alpha_k,t)}{1-d\bar{\Delta}(\alpha_k,t)/d\alpha_k} = \frac{1}{2\pi i} \oint dL \sum_k \frac{g_k(L-k,\ t) f(L-k,\ t)}{[L-n-\Delta_k(n,t)](1-d\bar{\Delta}/d\alpha_k)} \quad (101)$$

$$= \frac{1}{2\pi i} \oint dL \sum \frac{g_k(L-k,\ t) f(L-k,\ t)}{L-n-\bar{\Delta}_k(L,t)}$$

$$= \sum_p \frac{d^p}{dn^p} h_p(n,t), \quad (102)$$

where

$$h_p(n,t) = \sum_k g_k(n-k,\ t) f(n-k,\ t)[\bar{\Delta}_k(L,t)]^p, \quad (103)$$

and $\bar{\Delta}_k(L,t)$ must be of the form $f(L-k,\ t)$ in order for the summation over k to have the correct analyticity at $t=0$.

Let us now examine the summation more carefully in order to take parity into account. We can write (7)

[33] L. Durand, III, P. M. Fishbane, and L. M. Simmons, Jr., Phys. Rev. Letters **21**, 1654 (1968).

[34] G. Domokos, Bull. Am. Phys. Soc. **14**, 49 (1969).

[35] L. Durand, III, P. M. Fishbane, and L. M. Simmons, Jr., Phys. Rev. Letters **22**, 261 (1969); J. C. Taylor, Oxford University Report (unpublished); P. K. Kuo and P. Suranyi, Phys. Rev. Letters **22**, 1025 (1969).

as

$$T_{S'\lambda',S\lambda}=\oint dL\frac{1+\tau e^{-i\pi(L-v)}}{\sin\pi(L-v)}\times\sum_k B_k(L,t)D_{\lambda'\lambda}{}^{L-k}(\theta_t)(-)^k\,,\quad(104)$$

where

$$B_k(L,t)\equiv\frac{\beta_{kS'\lambda'}{}^+(t)\beta_{kS\lambda}{}^+(t)}{L-k-\alpha_k{}^+(n,t)}-\frac{\beta_{kS'\lambda'}{}^-(t)\beta_{kS\lambda}{}^-(t)}{L-k-\alpha_k{}^-(n,t)}\,,\quad(105)$$

$$\beta_{kS\lambda}{}^{\pm}(t)\equiv[\bar\beta_{kS\lambda}\pm\eta_a\bar\eta_c(-)^{S-v}\bar\beta_{kS-\lambda}]\times\left[1-\frac{d\bar\Delta_k{}^{\pm}(L,t)}{dL}\right]^{-1/2},\quad(106)$$

$$\bar\beta_{kS\lambda}=\sum_{k'M'}d_{L-k,S\lambda}{}^{L-k',M'}(\phi)\begin{Bmatrix}J_+' & J_-' & L-k\\ J_- & J_+ & J\end{Bmatrix}\times a_{k'M'}(t)t^J\,.\quad(107)$$

If we substitute (106) into (105) and use $\alpha_k=n-k+\bar\Delta_k$, we get

$$B_k(L,t)=\eta_d\bar\eta_d(-)^{S'-v}[\bar\beta_{kS'-\lambda'}(t)\bar\beta_{kS\lambda}+\eta\bar\beta_{kS'\lambda'}\bar\beta_{kS-\lambda}]F_k{}^+(L,t)+[\bar\beta_{kS'\lambda'}\bar\beta_{kS\lambda}+\eta\bar\beta_{kS'-\lambda'}\bar\beta_{kS-\lambda}]F_k{}^-(L,t)\,,\quad(108)$$

where

$$F_k{}^{\pm}(L,t)=1/[L-n-\bar\Delta_k{}^+(L,t)]\pm1/[L-n-\bar\Delta_k{}^-(L,t)]\quad(109)$$

and $\eta=\eta_a\eta_b\bar\eta_c\bar\eta_d(-)^{S-S'}$.

If we substitute the first of the two terms of (108) back into (104), we arrive at a summation that is identical to (8) except for the extra factor $F_k{}^+(L,t)$. We showed in Sec. III that, except for the extra factor $F_k{}^+(L,t)$, the sum over k has the correct analyticity at $t=0$ and pseudothreshold. In Sec. IV we showed that the residues could always be multiplied by a function whose k dependence is given by $f(L-k,\,t)$ [see (77)] without destroying the analyticity. The proper k dependence of $F_k{}^+(L,t)$ can thus be guaranteed by requiring

$$\bar\Delta_k{}^{\pm}(L,t)=f_{\pm}(L-k,\,t)\,.\quad(110)$$

It is a bit more difficult to guarantee that the *second* term of (108) has the proper analyticity, since the helicity indices differ from what was encountered in Sec. IV. In order to do the sum over k, we must reverse the sign of the helicity in one of the residues. This can be accomplished by multiplying one of the vertices by the factor

$$f_M'\equiv t^M\Gamma(L-k+M+1)/\Gamma(L-k-M+1)\,.\quad(111)$$

Since f_M' is a function whose k dependence is of the form $f(L-k,\,t)$, we do not distrub the analyticity by multiplying the residue by such a factor.

In order to see how the factor f_M' is able to reverse the sign of the helicity, we shall need the following identity, which can be easily checked by examining (78):

$$f_{\min}'\begin{Bmatrix}J_+' & J_-' & L-k\\ J_- & J_+ & J_{\min}\end{Bmatrix}a_{k'M'}(t)=\begin{Bmatrix}J_+' & J_-' & L-k\\ J_+ & J_- & J_{\min}'\end{Bmatrix}\bar a_{k'-M'}(t)\,,\quad(112)$$

where

$$\min=\min(M,M')\,,$$
$$J_{\min}=\tfrac12(k'+|M-M'|)\,,$$
$$J_{\min}'=\tfrac12(k'+|M+M'|)\,.$$

Upon multiplying the residue by f_M', we get

$$f_M'\bar\beta_{kS\lambda}=f_M'/f_{\min}'\sum_{k'M'}d_{L-k,S\lambda}{}^{L-k,'M'}(\phi)\times\begin{Bmatrix}J_-' & J_+' & L-k\\ J_- & J_+ & J_{\min}'\end{Bmatrix}\bar a_{k',-M'}(t)$$
$$=f_M'/f_{\min}'\sum_{k'M'}d_{L-k,S-\lambda}{}^{L-k',M'}(\phi)\times\begin{Bmatrix}J_+' & J_-' & L-k\\ J_- & J_+ & J_{\min}\end{Bmatrix}\bar a_{k',M'}(t)$$
$$=f_{M,M'}(L-k,\,t)\bar\beta_{kS-\lambda}(t)\,.\quad(113)$$

In order to arrive at (113), we have used the symmetry $d_{\alpha S\lambda}{}^{nM}(\phi)=d_{\alpha S-\lambda}{}^{n-M}(\phi)$ and the observation that the k dependence of $f_M'/f_{\min}'$ is of the form $f(L-k,\,t)$.

If we apply (113) to (108), we see that the second term of (108) has the proper analyticity only if $F_k{}^-(L,t)$ is able to provide the necessary factor of f_M'. From (109) and the argument following (102), it can be seen that the proper k dependence in $f_k{}^-(L,t)$ can be guaranteed by

$$\bar\Delta_k{}^+(L,t)-\bar\Delta_k{}^-(L,t)=f_M'f_2(L,t)\,.\quad(114)$$

We can now bring (110) and (114) together to give the most general expansion for a Regge trajectory:

$$k+\alpha_k{}^{\pm}(L,t)=f_1(L-k,\,t)\pm f_M'f_2(L-k,\,t)\,,\quad(115)$$

where $f_1(L-k,\,0)=n$. The contour integral over L always picks out the pole at $L=k+\alpha_k{}^{\pm}(L,t)$. At this pole the trajectory $\alpha_k{}^{\pm}$ becomes a function of n and t, and we have

$$k+\alpha_k{}^{\pm}=f_1(\alpha_k{}^{\pm},t)\pm\frac{\Gamma(\alpha_k{}^{\pm}+M+1)t^M}{\Gamma(\alpha_k{}^{\pm}-M+1)}f_2(\alpha_k{}^{\pm},t)\,.\quad(116)$$

In order to help understand the significance of (116), let us consider the trajectory for the case $M=\frac12$. It is convenient to introduce the variable $W=t^{1/2}$. The two terms of (116) can be combined to give

$$k+\alpha_k^{\pm}=\sum_J[\pm(\alpha_k^{\pm}+\tfrac{1}{2})W]^J a_J(t)\,,$$

where $a_0(0)=n$. We can also combine the two terms of (116) for *general M* in the following manner.

If M is a positive half-integer, we have

$$k+\alpha_k^{\pm}=\sum_J\frac{\Gamma(\alpha_k^{\pm}+\frac{1}{2}J+1)}{\Gamma(\alpha_k^{\pm}-\frac{1}{2}J+1)}(\pm W)^J a_J(t) \qquad (117)$$

and $a_J(t)=0$ for $J=1, 3, \cdots, 2M-2$. If M is a positive *integer*, then we have

$$k+\alpha_k^{\pm}=\sum_J\frac{\Gamma(\alpha_k^{\pm}+J+1)}{\Gamma(\alpha_k^{\pm}-J+1)}t^J a_{J,\pm}(t)\,, \qquad (118)$$

with the extra condition $a_{J,+}(t)=a_{J,-}(t)$ for $J<M$.

It is misleading to think of (116) as a simple parametrization for the trajectories in term of the coefficients $a_J(t)$. First, (116)–(118) are highly nonlinear implicit equations for $\alpha_k^{\pm}$ which must be solved by iteration. The explicit form for $\alpha_k^{\pm}$, which is given by Bronzan, is much more complicated in appearance. Second, even if one has an explicit expansion for $\alpha_k^{\pm}$, the coefficients may blow up.

As an example of this second problem, let us consider the top two trajectories for the case $M=0$:

$$\alpha_0(t)=a_0(t)\,, \qquad (119)$$

$$\begin{aligned}1+\alpha_1(t)&=a_0(t)+k[2\alpha_1(t)+k+1]ta_1(t)\\&=\alpha_0(t)+2[\alpha_1(t)+1]ta_1(t)\,,\\1+\alpha_1(t)&=\alpha_0(t)/[1-2ta_1(t)]\,.\end{aligned} \qquad (120)$$

We should not consider (120) to be a parametrization of $\alpha_1(t)$ in terms of $\alpha_0(t)$ and $a_1(t)$, since there are the following constraints on $a_1(t)$, which cannot be ignored. When $\alpha_0(t)$ goes through zero, we see that $ta_1(t)=\frac{1}{2}$ if the trajectories are nonparallel. Similarly, when $\alpha_1(t)=-1$, we must have $a_1(t)=\infty$. For the pion trajectory, it is likely that the first daughter goes through $\alpha_1(t)=-1$ quite close to $t=0$. It would thus be a mistake to consider $a_1(t)$ as a smoothly varying function near $t=0$.

Just as the dynamic coefficient $a_1(t)$ in (120) is not completely free, we can also see that the coefficients in the expansion of the vertex must also be constrained. The vertex is constrained to give the correct sense-nonsense behavior when α_k is an integer. In addition, the coefficients in (107) must be constrained to cancel singularities due to the "parallel-breaking" factor in (106).

In conclusion, we would like to present our main formula for the most general Regge residue (106), and (107) in an altered form. If we use the results of the previous section to expand the 6-j symbol of (107), we finally get

$$\begin{aligned}\beta_{kS\lambda}^{\pm}(t)\equiv\sum_{k'=0}^{k}\sum_{M'=-S}^{S}&[d_{\alpha_k\pm S\lambda}{}^{\alpha_k\pm+k-k',M'}(\phi)\pm\eta_a\bar{\eta}_c(-)^{S-v}d_{\alpha_k\pm S-\lambda}{}^{\alpha_k\pm+k-k',M'}(\phi)]\\&\times\left[\frac{\Gamma(\alpha_k^{\pm}+M+1)\Gamma(\alpha_k^{\pm}-M'+1)}{\Gamma(\alpha_k^{\pm}+M'+1)\Gamma(\alpha_k^{\pm}-M+1)}\right]^{(M-M')/2|M-M'|}\left[\frac{k!\Gamma(2\alpha_k^{\pm}+k+2)}{(k-k')!\Gamma(2\alpha_k^{\pm}+k-k'+2)}\right]^{1/2}\\&\times\left[1-\frac{d\bar{\Delta}^{\pm}(\alpha_k^{\pm},t)}{d\alpha_k^{\pm}}\right]^{-1/2}a_{k'M'}(t)t^{(|M'-M|+k')/2}.\end{aligned} \qquad (121)$$

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APPENDIX

In this appendix we shall examine the brute-force expansion of the t-channel amplitude (8) for the case of unequal masses and parallel trajectories. The amplitude with $\lambda=-\lambda'=M$ is particularly simple to analyze, since the discussion near (13) and (14) showed that $T_{S'-M,SM}$ and $T_{S'M,S-M}$ are the only helicity amplitudes that have the full Regge behavior. For other values of λ and λ', the scattering amplitude has extra factors of t or s^{-1}.

The scattering amplitude (8) can be written as

$$T_{S'-M,SM}=\sum_k\bar{\beta}_{k\lambda'\lambda}t^{-\alpha_0}D_{M-M}{}^{\alpha_k}(z_t)(-)^k(1+\tau_k e^{i\pi(\alpha_k-v)})/\sin\pi(\alpha_0-v)\,, \qquad (A1)$$

where

$$\bar{\beta}_{k\lambda'\lambda}(t)=[\beta_{kS'\lambda'}{}^{+}(t)\beta_{kS\lambda}{}^{+}(t)-\beta_{kS'\lambda'}{}^{-}(t)\beta_{kS\lambda}{}^{-}(t)]t^{\alpha_0}\,, \qquad (A2)$$

$$D_{-MM}{}^{J}(z)=\frac{\Gamma(2J+1)e^{i\pi(J-M)}}{\Gamma(J+M+1)\Gamma(J-M+1)}\left(\frac{1-z}{2}\right)^J F\left(-J+M,\ -J-M,\ -2J,\frac{2}{1-z}\right), \qquad (A3)$$

$$\frac{1-z}{2}=\frac{t(-s+u)+\Delta(t,a,c)\Delta(t,b,d)-\Delta(0,a,c)\Delta(0,b,d)}{\Delta(t,a,c)\Delta(t,b,d)}\equiv t\bar{S}. \qquad (A4)$$

We see that $\bar{S}$ is regular near $t=0$ in the unequal-mass case, since

$$\Delta(t,a,c)\equiv\{[t-(M_a+M_c)^2][t-(M_a-M_c)^2]\}^{1/2}. \quad (A5)$$

Combining (A1) and (A3) gives

$$T_{S'-M,SM}=t^{-\alpha_0}\Phi\sum_k(-)^k\Gamma(-\alpha_k-M)\Gamma(-\alpha_k+M)(t\bar{S})^{\alpha_k}\times F\left(-\alpha_k+M,\ -\alpha_k-M,\ -2\alpha_k,\ \frac{2}{1-z_t}\right)\Big/\times\Gamma(-2\alpha_k)2\pi\cos\pi(\alpha_k-M), \quad (A6)$$

where $\Phi\equiv\tau+e^{-i\pi(\alpha_0-v)}$. We now expand the hypergeometric function

$$T_{-MM}=\frac{\Phi t^{-\alpha_0}}{2\pi\cos\pi(\alpha_0-M)}\sum_{k=0}^{\infty}\beta_{kMM}(t\bar{S})^{\alpha_k}(-)^k\times\sum\frac{\Gamma(M-\alpha_k+J)\Gamma(-M-\alpha_k+\bar{J})}{\bar{J}!\Gamma(-2\alpha_k+\bar{J})(t\bar{S})^{\bar{J}}} \quad (A7)$$

$$=\Phi\bar{S}^{\alpha_0}\Gamma(M-\alpha_0)\sum_J c_J(t\bar{S})^{-J}\sum_{k=0}^{J}\bar{\beta}_{kMM}(-)^k\times\Gamma(2\alpha_0-J-k+1)/(J-k)!, \quad (A8)$$

where $J=\bar{J}+k$ and

$$c_J=\Gamma(\alpha_0-M+1)\times[\pi\Gamma(\alpha_0+M-J+1)\Gamma(\alpha_0-M-J+1)]^{-1}.$$

The summation over k must give a result that is able to cancel the factor t^{-J}. That is, we must have

$$c_J\sum_{k=0}^{J}\beta(-)^k\Gamma(2\alpha_0-J-k+1)/(J-k)!=a_J(t)t^J, \quad (A9)$$

where $a_J(t)$ is analytic at $t=0$.

If we examine the nth derivative of both sides of (A9), we arrive at the condition used in Sec. V: The first $n-1$ derivatives of the nth residue are determined, while the higher derivatives are free.

The combinatorial identity

$$\sum_J\frac{\Gamma(2\alpha_0-J-k+1)(-)^{k'-J}(2\alpha_0-2k'+1)}{(J-k)!(k'-J)!\Gamma(2\alpha_0-k'-J+2)}=\delta_{kk'} \quad (A10)$$

allows us to solve (A9) for $\bar{\beta}_k(t)$:

$$\bar{\beta}_k(t)=(2\alpha_k+1)\sum_J\frac{a_{J,r}(t)t^J}{(k-J)!\Gamma(2\alpha_0-k-J+2)}=(2\alpha_k+1)[k!\Gamma(2\alpha_0-k+2)]^{-1}f_{k,r}(t), \quad (A11)$$

where

$$a_{J,r}(t)=a_J(t)/c_J \quad (A12)$$

and $f_{J,i}(t)$ is the function discussed in Sec. VII. We have chosen $a_J(t)$ in such a way that (A8) can be written as

$$T_{-MM}=\Phi\Gamma(M-\alpha_0(t))\sum_J a_J(t)\bar{S}^{\alpha_0-J}. \quad (A13)$$

The form of (A13) shows that T_{-MM} has the desired analytic properties and also shows the role of the dynamical coefficient $a_J(t)$.

The expansion in (A13) will have a pole whenever $\alpha_0=M+2n+\xi$ for $n=0, 1, 2, \cdots$, and where $\xi=\frac{1}{2}[1-(-)^{M-v}\tau]$. Also, (A13) gives zeros at $\alpha_0=M-2n-1+\xi$, where $n=1, 2, \cdots$. These are the characteristics of the sense-choosing mechanism. Other ghost-eliminating mechanisms can be obtained by modifying the factors in c_J.

We will now obtain the *factored* residue $\beta_{kS\lambda}{}^{\pm}$ from (A11). However, we see in (A2) that $\bar{\beta}_k$ cannot be simply factorized. In order to separate $\beta_{kS\lambda}{}^{+}$ from $\beta_{kS\lambda}{}^{-}$, we must examine $T_{S'M,SM}$ in addition to $T_{S'-M,SM}$. This allows us to find $\bar{\beta}_{kM-M}$.[14,19]

$$\bar{\beta}_{kM-M}=\eta_a\bar{\eta}_c(-)^{S-v}\times(\beta_{kS'M}{}^{+}\beta_{kSM}{}^{+}+\beta_{kS'M}{}^{-}\beta_{kSM}{}^{-})t^{\alpha_0}. \quad (A14)$$

These results can be applied to elastic scattering ($M_a=M_c$, $M_b=M_d$), where, by combining $\bar{\beta}_{kMM}$ and $\bar{\beta}_{kM-M}$, we are able to determine $\beta_{kSM}{}^{+}$ and $\beta_{kSM}{}^{-}$. The results are in agreement with (86), which serves to verify the correctness of the analytic continuations used in Sec. IV.

Once the residues for the parallel-trajectory case have been determined, the extension to nonparallel trajectories is straightforward. The discussion of Sec. VI concerning nonparallel trajectories applies to the brute-force method as well as to the method involving the Lorentz-group sum rule.

The main limitation of the brute-force method is that it does not have the proper behavior at pseudothreshold even for the case $S=0$. This means that the unequal-mass case does not go smoothly to the equal-mass case. The residues given by (86), on the other hand, are valid for all mass configurations.