ρ Bootstrap Using the Veneziano Representation*

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A ρ bootstrap calculation, using the Balázs method, with the input crossed-channel amplitude given by the Veneziano representation, was performed. Results for the self-consistent mass and width are in good agreement with experiment. The results are consistent with the amplitude being given by only a single Veneziano term, although they do not exclude the presence of additional terms with lower-order asymptotic behavior.

THE Veneziano model[1–3] has the great advantage of allowing one to write an explicit expression for a scattering amplitude which has many of the properties, such as crossing symmetry, Regge asymptotic behavior, and resonances where the trajectories rise through integers, which the exact amplitude is believed to have. Since the resonances are treated in the narrow-width approximation so that the model is not unitary, it cannot provide a correct point-by-point representation of the amplitude; however, it is reasonable to believe that it may represent the physical amplitude correctly on the average, so that corresponding integrals over the Veneziano and the exact amplitudes would be equal. This suggests that the Veneziano amplitude would be a suitable input to a bootstrap calculation using the N/D equations. One could then solve these by the Balázs method[4–7] which avoids the severe cutoff problems associated with the exchange of higher Regge recurrences. We report here on the results of carrying out such a calculation of the parameters of the ρ meson. This procedure might be especially advantageous, in that it gives one a way of including contributions from the intermediate energy region in the crossed channel, assuming the Veneziano amplitude is, indeed, a reasonable approximation to the average behavior of the physical amplitude. Previous results[8,9] have indicated that the inclusion of f^0 exchange may be important in calculations of the ρ, and this suggests that the inclusion of the force due to the exchange of additional intermediate energy resonances may also be important.

The foregoing is the program which we carry through in this paper. Before describing our procedure and results, however, we first discuss one difficulty. If Regge trajectories in nature rise without limit, as they do in the Veneziano model (where they are strictly linear), then an N/D-type bootstrap cannot be carried out.[10] This is due to the fact that if the Regge trajectories never return to the left half of the j plane, the value of the real part of the phase shift in the jth partial wave at infinite energy, instead of being 0, will be $n\pi$, where n is the number of trajectories; n is infinity in the Veneziano model, but the same difficulties would present themselves if there were only one infinitely rising trajectory. This in turn means that more than one subtraction is required in the dispersion relation for the Omnès function,[11] or, equivalently, that Castillejo-Dalitz-Dyson (CDD) poles are present in the denominator function. If such a situation occurs, it indicates a failure of the bootstrap approach, at least in its conventional formulation; it is not clear how one could then carry out the bootstrap program of calculating scattering amplitudes without the introduction of arbitrary parameters. The same conclusions arise from the point of view of Levinson's theorem.[12] Since we wish to pursue here the consequences of the bootstrap approach, we adopt the point of view that the trajectories are not completely linear and eventually turn over and recede to the left half j plane, or at least that the infinitely rising trajectories are due to multichannel effects which may be neglected in a low-energy calculation. We suppose that the Veneziano representation is a reasonable approximation to the amplitude in the region where the trajectories are linear, and that this region extends to relatively high energies, so that the

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10 One might ask whether the N/D approach even has meaning in the case of infinitely rising trajectories, since the partial-wave amplitude $A_l(s)=\int_{-(s-4)}^{0}A(s,t)P_l(1+2t/(s-4))dt$ is unbounded as $s\to-\infty$. Therefore, the Cauchy integral formula for $A_l(s)$ does not take the form of a dispersion relation and there are nonvanishing contributions from the circle at infinity. However, the Uretsky form of the N/D equations [J. L. Uretsky, Phys. Rev. **123**, 1459 (1961)], can be obtained by writing once-subtracted dispersion relations over the right-hand, or physical, cut only for the denominator function $D_l(s)$, and for the function $H_l(s)=R_l(s)D_l(s)$, where $R_l(s)$ is the contribution to $A_l(s)$ from the integral over the right-hand cut. $D_l(s)$ is essentially the Omnès function and hence well behaved, while $R_l(s)$ is asymptotically no worse than constant because of unitarity, i.e., because $\alpha(0)\leq 1$, so that both of these dispersion relations are valid. The kernel of the integral equation for the numerator function involves the function $L_l(s)=A_l(s)-R_l(s)$, but only for $s>0$, where it is governed by the unitarity bounds on $A_l(s)$ and $R_l(s)$. It makes no difference for the validity of the N/D equations whether $L_l(s)$ satisfies a dispersion relation or whether there are contributions to the Cauchy integral formula for L_l from parts of the infinite circle.

11 For example, see G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).

12 R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

important contributions to the input "force" are well represented by using the Veneziano model for the crossed-channel amplitude. The Veneziano model must, in any event, partially break down at very high energies, where the trajectory functions develop appreciable imaginary parts, so that they can no longer be exactly linear, and where a narrow-width approximation for the amplitude is no longer reasonable. We assume that the departure of the physical amplitude from the Veneziano model is such as to render a bootstrap calculation meaningful.

In performing the ρ bootstrap, we follow the procedure of II with two exceptions: $A_l^{I(L)}(\nu)$ is now calculated from the Veneziano model and ν_D is here the point of separation of the elastic and inelastic regions, while the point above which the Regge representation holds is now designated by ν_{DR}. In II we had $\nu_{DR}=\nu_D$. We consider two models. In both of them, $A_l^{I(L)}(\nu)$, the contribution from the low-energy region of the crossed channel, is represented by the contribution of the first N Veneziano resonances. $A_l^{I(H)}(\nu)$, the high crossed-channel energy contribution, is equal to zero in the first model, which we call the Veneziano model. In the second model we use a Regge representation for $A_l^{I(H)}(\nu)$ in the same way as in II. ν_{DR} is set to a value corresponding to the mass of the highest resonance retained in calculating $A_l^{I(L)}(\nu)$. This second model we refer to as the Veneziano-Regge model. For both models, we demand self-consistency in the sense of requiring the output ρ parameters to equal the parameters of the ρ in the input force. We have not attempted to obtain self-consistency for the other resonances, although we have verified that there is an output f^0 whose parameters are in at least fair agreement with the input. As discussed in Ref. 9, we feel that the output f^0 may be fairly sensitive to the details of the treatment of inelasticity.

The input force into the N/D equations through Eqs. (I1), (I2), and (I9), is given by Eqs. (II14)–(II20) for $A_l^{I(H)}(\nu)$ in the Veneziano-Regge model. For $A_l^{I(L)}(\nu)$, we have

$$A_l^{I(L)}(\nu)=\frac{1}{2\pi\gamma}\Bigg\{[C_{st}^I+(-1)^I C_{su}^I]\int_4^{t_{DR}}dt'\,\mathrm{Im}A(s,t')\times Q_l\left(1+\frac{t'}{2\nu}\right)+[1+(-1)^I]\times C_{ut}^I\int_4^{t_{DR}}dt'\,\mathrm{Im}A(4-s-t',t')\times Q_l\left(1+\frac{t'}{2\nu}\right)\Bigg\}, \quad (1)$$

where $A(s,t)$ is the amplitude for the s-channel reaction $\pi^++\pi^-\rightarrow\pi^++\pi^-$ and C_{xy}^I is given by[2]

$I \diagdown$ st	su	ut	
0 $\frac{3}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	
1 1	-1	0	(2)
2 0	0	1.	

We take $A(s,t)$ to be given by the Veneziano representation with the possible presence of one "satellite" term. That is, we write

$$A(s,t)=\sum_{k=1}^{2}\gamma_k\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(k-\alpha(s)-\alpha(t))}. \quad (3)$$

The γ_k are constants to be determined. The asymptotic behavior of the amplitude is controlled by the first term in (3). We have not included possible additional terms of the form $\Gamma(m-\alpha(s))\Gamma(m-\alpha(t))/\Gamma(2m-k-\alpha(s)-\alpha(t))$, with $m>1$ and $m>k\geqq 0$, which likewise do not affect the asymptotic behavior. Substituting the imaginary part of (3) into (1) and integrating over the δ functions, one finds

$$\begin{aligned}A_l^{I(L)}(\nu)=\sum_{n=0}^{N}Q_l\left(1+\frac{t_n}{2\nu}\right)\frac{1}{2\alpha'\nu n!}&[C_{st}^I+(-1)^I C_{su}^I]\\ &\times[\gamma_1\alpha(s)\phi_n(\alpha(s))-\gamma_2\phi_n(\alpha(s)-1)]\\ &+C_{ut}^I[1+(-1)^I](-1)^{n+1}\\ &\times[\gamma_1 a(s)+\gamma_2]\phi_n(a(s)), \end{aligned}\quad (4)$$

where

$$\begin{aligned}\phi_n(z)&=1 &&\text{for}\quad n=0\\ &=\prod_{j=1}^{n}(j+z) &&\text{for}\quad n\geqq 1;\end{aligned}\quad (5)$$

$\alpha(s)$ is the Veneziano trajectory and is given by

$$\alpha(s)=\alpha(0)+\alpha's; \quad (6)$$

and

$$\begin{aligned}a(s)=1-\alpha(u)-\alpha(t),\quad \alpha(t_n)=n+1,\\ N=\alpha(4\nu_{DR}+4)-1.\end{aligned}\quad (7)$$

We determine γ_1 by comparing the ρ term in Eqs. (I10) and (4) and find

$$\gamma_1=3\Gamma_\rho. \quad (8)$$

γ_2 is determined by a choice of the relative widths of the ρ and the ϵ, the $j=0$ resonance roughly degenerate with the ρ. Taking $\gamma_2=0$, i.e., keeping only one Veneziano term, corresponds to an ϵ width about $4\frac{1}{2}$ times that of the ρ; we refer to this as 100% ϵ. The ϵ may be canceled entirely by choosing

$$\gamma_2=\gamma_1[\alpha(0)-2\alpha'\nu_\rho]. \quad (9)$$

The experimental status of the ϵ is highly uncertain;

TABLE I. Self-consistent values (in MeV) of the ρ mass m_ρ and the ρ full width Δm_ρ for the Veneziano and Veneziano-Regge models. γ_2 is chosen in accordance with Eq. (9) to eliminate the ϵ. ν_F is the optimum matching point, in units of m_π^2, chosen as in Ref. 9. N is defined by Eq. (4) so that $N+1$ is the angular momentum of the highest-spin resonances included in the input force.

	Veneziano model			Veneziano-Regge model		
N	ν_F	m_ρ	Δm_ρ	ν_F	m_ρ	Δm_ρ
1				−5.40	711	23
3				−5.40	711	26
5				−5.40	711	31
9	−5.40	711	32			
12	−5.40	711	41	−5.35	711	64
13	−5.40	711	44	−5.35	711	105
15	−5.40	711	56	−5.30	711	106
18	−5.37	711	65	−5.30	711	106
20	−5.37	711	72	−5.30	711	100
25	−5.37	711	74			

its existence has been reported with widths which are poorly determined but tend to be quite broad.[13]

The behavior of the solutions of the ρ bootstrap as a function of n, the number of poles approximating the left-hand cut, shows the same features as that of the double ρ and f^0 bootstrap reported in II. Thus we restrict our attention to the seven-pole case, at which point we believe the solutions to have leveled off appreciably as a function of n. The exact form of the output width as given by Eqs. (II8a) is used throughout. The choice of parameters is the same as in II, except for $\alpha(0)=0.483$. This choice is suggested by the Adler self-consistency condition for the case $\gamma_2=0^2$. If one wishes to preserve the Adler condition, $\alpha(0)$ should, strictly speaking, be varied with γ_2. We have not done this, since the results turn out to be quite insensitive to variations of $\alpha(0)$ within the range allowed by experiment. $\nu_{DR}=[N+1-\alpha(4)]/4\alpha'$, where N is defined by Eq. (4).

Table I shows the effect of increasing the number of terms retained in the Veneziano series, with 0% ϵ included, on the ρ bootstrap solution. There are some blank spaces in the table, since we did not always expend the computer time to obtain solutions where we felt the additional information would reveal nothing of significance. The solution becomes comparatively insensitive to N for large N, as expected, since the convergence of the sum in Eq. (5), i.e., of the integrals in Eq. (1), is guaranteed by the fact that, for large t, the behavior of the Veneziano amplitude is given by $A(s,t)\sim t^{\alpha(s)}$, with the integral being carried out at a value of s where $\alpha(s)<1$. It should be pointed out, however, that large N does not necessarily represent the best approximation to the physical amplitude, since, as we have already noted, we expect the linear-trajectory and narrow-width approximations to break down at some point. In the Veneziano-Regge case, especially, one might expect it would be a better approximation to keep only a relatively few Veneziano resonances, i.e., to use a fairly small value of ν_{DR}, since the Regge representation for the t-channel absorptive part may well be superior to the Veneziano representation at moderate t-channel energies. The duality concept, of course, suggests that the Veneziano-Regge results should be quite insensitive to the choice of ν_{DR}, that is, to the choice of the dividing line in energy between the regions where the amplitude is given, respectively, by a resonance and by a Regge representation. As seen in Table I, however, the bootstrap results for the ρ width are fairly sensitive to the choice of N, or of ν_{DR}, up to about $N=12$ for the 0% ϵ case.

Table II shows the effect of the ϵ particle on the ρ bootstrap solution in both models. In the Veneziano model we take $N=20$, which, according to Table I, is the value at which the results seem to become quite insensitive to N. For the Veneziano-Regge model, we take $N=4$. This corresponds to taking the amplitude to be given by its asymptotic form above about 2 BeV; we judge from Table I that these results should not be too sensitive to the precise choice of N. For comparison, we also give some Veneziano-Regge results for $N=20$.

The effect of the ϵ in both models is very similar. Above 70% ϵ the ρ mass starts to decrease at an accelerated rate, up to about 85% ϵ, where the solution disappears altogether and instead of it there appear two solutions which at 100% ϵ are well defined. One, with a ρ mass of 750 MeV, has the properties of series 1 as discussed in I, and the other solution with a ρ mass of about 700 MeV has the properties of what would be, in the notation of I, series 0. Arguments of the type used in I, based on a criterion proposed by Williamson and Everett,[14] give preference to series 1 over series 0. We did not carry out calculations for γ_2 corresponding to ϵ widths greater than 100%.

As a number N of terms retained in the Veneziano series is increased, the sensitivity of the values of the output mass and width to the input width decreases. At the same time, it becomes more and more difficult to require agreement between input and output widths to within a few percent, the output width always being wider, and one has to settle for about 10% agreement (about 20% for 100% ϵ) as giving a bootstrap. These two effects combined result in a bootstrap width which is not precisely determined.

From the results in Tables I and II one concludes that, using the Veneziano representation for the input to the N/D equations, solved by the Balázs procedure, one can obtain a solution in which the input and output parameters of the ρ are approximately self-consistent. The Veneziano-Regge calculation with $N=4$ and with

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TABLE II. Self-consistent ρ parameters for Veneziano and Veneziano-Regge models, with the indicated values of N, as a function of the width of the ϵ included in the input force. 100% ϵ means an ϵ width $4\frac{1}{2}$ times that of the ρ, corresponding to the retention of only a single Veneziano term in Eq. (3). Units and notation are the same as in Table I. Errors, where shown, indicate an appreciable range of values yielding equally satisfactory self-consistency. For 100% ϵ, there are two somewhat different solutions, as explained in the text; the first is preferred to some extent by the criteria of Refs. 7 and 9.

	Veneziano-Regge $N=4$			Veneziano-Regge $N=20$			Veneziano $N=20$		
% ϵ	ν_F	m_ρ	Δm_ρ	ν_F	m_ρ	Δm_ρ	ν_F	m_ρ	Δm_ρ
00	-5.40	711	29	-5.30	711	100 ± 6	-5.37	711	72
10	-5.40	711	32	-5.35	711	89 ± 6	-5.40	711	89
20	-5.40	711	36				-5.40	711	98
30	-5.40	711	42				-5.45	711	93
50	-5.50	711	42	-5.45	711	83 ± 6	-5.50	711	89 ± 6
70	-5.60	711	52				-5.60	711	83 ± 8
76	-5.65	711	64						
80	-5.65	710	77	-5.65	710	61 ± 6	-5.70	710	77 ± 8
83	-5.67	706	91				-5.70	706	77 ± 8
100	-6.50	753	100 ± 25	-6.51	753	90 ± 40	-6.51	753	80 ± 30
100	-5.80	712	86 ± 9	-5.80	712	72 ± 36	-5.80	712	72 ± 36

100% ϵ, i.e., with $\gamma_2=0$ and only a single Veneziano term retained in Eq. (3), yields results in excellent agreement with the experimental parameters of the ρ, although, as mentioned above, there is some uncertainty in the theoretical value for the ρ width. With $N=4$, if γ_2 is chosen so as to significantly reduce the ϵ width from the value for a single Veneziano term, then the results in Table II indicate that the bootstrap value for the mass is slightly reduced, and the value for the width becomes appreciably too narrow. For $N=20$, the results are in reasonable agreement with experiment throughout the range of ϵ widths investigated for both the Veneziano and Veneziano-Regge cases. They are still perfectly consistent with keeping only one Veneziano term, and, in fact, doing so gives a theoretical mass in somewhat better agreement with experiment.

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General Solution for Regge Residues and Trajectories

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A parametrization for Regge vertices is presented. These vertices have the most general t dependence consistent with constraints at $t=0$ and pseudothresholds, and are valid for general spins and general masses and for nonparallel trajectories. The assumptions upon which this work is based are analyticity, crossing symmetry, factorization (unitarity), and Regge asymptotic behavior. In the unequal-mass case, we find that the general Regge vertex has a particularly simple expansion around $t=0$.

I. INTRODUCTION

THE problem of constructing a Regge expansion that has the proper kinematic singularities (the conspiracy problem) has received much attention during the last two years.[1] One reason why so much work has been expended by so many people is that different cases have been treated separately. The equal-mass case[2–4] was thought to be entirely separate from the unequal-mass case,[5–7] daughters separate from conspirators. Some authors consider only low value of spin and Lorentz number M, others only consider residues for the parent and first daughter, or only the most singular parts of the residue. The approaches range from elegant group theory,[2,5,8] which makes use of special symmetries at $t=0$, through techniques using Feynman diagrams[9] or Bethe-Salpeter models,[5,10] and finally brute-force

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