PHYSICAL REVIEW D VOLUME 1, NUMBER 2 15 JANUARY 1970

# Formal Description of Measurements in Local Quantum Field Theory*

K.-E. HELLWIG AND K. KRAUS
*Institut für Theoretische Physik der Universität Marburg, Marburg, Germany*
(Received 25 February 1969; revised manuscript received 22 July 1969)

Ideal measurements in quantum mechanics can be described formally as instantaneous changes of the state vector or, more generally, of the density matrix $W$. By its lack of covariance, this formalism is not very adequate for relativistic quantum theories such as local field theory. For the latter, we propose a modified formalism, according to which a field measurement in a space-time region $C$ changes the field state $W$ in the future and side cone of $C$. This proposal is justified by physical considerations.

## 1. INTRODUCTION

SOME questions regarding the measurement of particle positions in relativistic quantum theory have been raised recently by Bloch.[1] Our paper intends to solve similar problems for field measurements in local quantum field theory.

Consider a quantum mechanical system with state space $\mathcal{H}$. Its state is described by a positive Hermitian operator $W$ on $\mathcal{H}$ with $\mathrm{Tr}W=1$, the density operator. In the Heisenberg picture used here, $W$ does not depend on time $t$ as long as no measurements are performed. The expectation value in state $W$ of an observable, corresponding to a Hermitian operator A, is $\mathrm{Tr}(AW)$.

Let a property of the system, corresponding to a projection operator $P$, be measured at time $t_0$.[2] According to the usual formalism,[3] this measurement changes the state $W$ into a new one for times later than $t_0$, and leaves unaffected the state $W$ before $t_0$. The new state is

$$W_P'=PWP+(1-P)W(1-P), \tag{1}$$

if the systems are not selected with regard to the outcome "yes" or "no" of the $P$ measurement (nonselective measurement). It is

$$W_P''=PWP/\mathrm{Tr}(PW), \tag{2}$$

if the systems are selected which have given the result "yes" (selective measurement).[4]

This formalism works well in nonrelativistic quantum mechanics. However, in relativistic theories it leads to difficulties since an instantaneous change of state is not a Lorentz covariant description of the process of measurement.

Such difficulties have been discussed in detail in Ref. 1. A suitably modified formalism, which is described in the following section, will eliminate these difficulties at least for the particular case of local quantum field theory.

As already indicated in the title of our paper, our subject is the *formal* description of field measurements. We will not discuss the highly controversial question of the physics (or philosophy) behind equations like (1) and (2).[5] The equations themselves are notoriously invariant with respect to their interpretation. We believe the same to hold true for the relativistic formalism proposed here.

Section 3 describes in some detail the practical application of our formalism, and compares our proposal with a similar one due to Schlieder.[6] Remarks on causality and vacuum state are the content of Sec. 4. State changes more general than those given by (1) and (2) will be discussed briefly in the Appendix.

## 2. DESCRIPTION OF FIELD MEASUREMENTS

The system under consideration will be a relativistic quantized field. Its observables are, for instance, quantized electromagnetic field strengths averaged over finite space-time regions.[7] We shall require only some very general assumptions.[8]

(i) There exist sets $\mathfrak{N}_C$ of field observables (Hermitian operators $A$ on $\mathcal{H}$), and in particular of field properties (projection operators $P$), for sufficiently many finite space-time regions $C$.[9]

(ii) There is a unitary representation $U(a,\Lambda)$ of the inhomogeneous Lorentz group in $\mathcal{H}$, so that

* Work supported in part by the Deutsche Forschungsgemeinschaft.

[1] I. Bloch, Phys. Rev. **156**, 1377 (1967).

[2] Actually the measurement will last a certain time $\Delta t$ around $t_0$, but $\Delta t$ is usually assumed to be very small.

[3] G. Lüders, Ann. Phys. (Leipzig) (6) **8**, 322 (1951); G. Ludwig, in *Werner Heisenberg und die Physik unserer Zeit* (F. Vieweg, Braunschweig, 1961), p. 150.

[4] It should be mentioned that von Neumann, *Mathematische Grundlagen der Quantenmechanik* [Julius Springer-Verlag, Berlin, 1932 (English transl.: Princeton University Press, Princeton, N. J., 1955)], uses a description of ideal selective $P$ measurements which differs from the one adopted in Ref. 3 if $P\mathcal{H}$ has more than one dimension. He defines, with some complete orthonormal system $\psi_\nu$ in $P\mathcal{H}$ and the projection operators $P_\nu=|\psi_\nu\rangle\langle\psi_\nu|$ onto $\psi_\nu$, $W_P''=\sum_\nu P_\nu W P_\nu/\mathrm{Tr}(PW)$. In the language used here, this corresponds to a simultaneous measurement of all $P_\nu$ which is nonselective with respect to $\nu$. This proposal is not adequate for field theory, since one-dimensional projections $P_\nu$ are not locally observable. More arguments in this direction may be found in Ref. 6.

[5] Some recent papers about this problem are: L. Rosenfeld, Progr. Theoret. Phys. (Kyoto) (Extra Number) p. 222, 1965; J. M. Jauch, E. P. Wigner, and M. M. Yanase, Nuovo Cimento **48B**, 144 (1966); J. Bub, *ibid.* **57B**, 503 (1968); L. Rosenfeld, Nucl. Phys. **A108**, 241 (1968); A. Loinger, *ibid.* **A108**, 245 (1968).

[6] S. Schlieder, Commun. Math. Phys. **7**, 305 (1968).

[7] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **12**, No. 8 (1933); Phys. Rev. **78**, 794 (1950).

[8] The usual assumptions about quantized fields are much more specific. (Compare, e.g., Ref. 22.)

[9] The discussion of observables $A$ may be reduced, by the spectral theorem for Hermitian operators, to a discussion of the properties $P$.

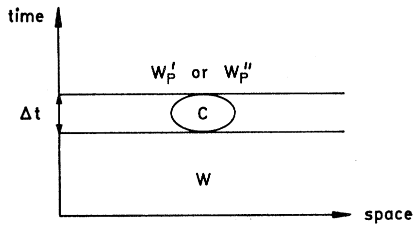


FIG. 1. Nonrelativistic description of measurement.

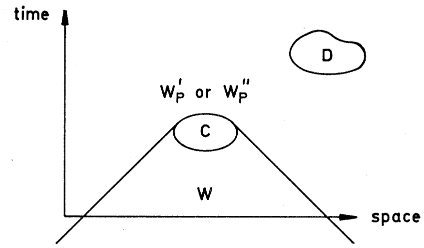


FIG. 2. Proposed relativistic description of measurement.

$U(a,\Lambda)\mathfrak{N}_C U^*(a,\Lambda)$ is the set of observables belonging to the Lorentz transform of the region $C$ (Lorentz invariance).

(iii) Field observables $A\in\mathfrak{N}_C$, $B\in\mathfrak{N}_D$ commute, $AB=BA$, if the regions $C$ and $D$ are spacelike with respect to each other (locality).

Assume the field to be originally in state $W$, and perform a measurement of a field property $P$ in some space-time region $C$, i.e., $P\in\mathfrak{N}_C$. This region $C$ accounts for a certain duration $\Delta t$ of the measuring process around a time $t_0$, as well as for the spatial extension of the measuring device.

With $W_P'$ and $W_P''$ as defined above, the "nonrelativistic" formalism described in Sec. 1 will ascribe the state $W$ to the field before the time $t_0-\frac{1}{2}\Delta t$, and the states $W_P'$ or $W_P''$ for nonselective or selective measurement, respectively, after the time $t_0+\frac{1}{2}\Delta t$. The state in the time interval $\Delta t$ in which the measurement proceeds will be left undetermined (Fig. 1). The expectation value for a subsequent measurement of a field property $Q$ in a space-time region $D$ later than $t_0+\frac{1}{2}\Delta t$ is $\mathrm{Tr}(QW_P')$ or $\mathrm{Tr}(QW_P'')$, respectively.

This description, however, seems to violate Lorentz invariance. With our assumption (ii) above, a successive measurement of the field properties

$$P'=U(a,\Lambda)PU^*(a,\Lambda)$$

and

$$Q'=U(a,\Lambda)QU^*(a,\Lambda)$$

in a field state

$$W'=U(a,\Lambda)WU^*(a,\Lambda)$$

is, for any inhomogeneous Lorentz transformation, essentially the same experiment as the one considered above. $P'$ and $Q'$ are now measured in the regions $C'$ and $D'$ onto which the regions $C$ and $D$, respectively, are mapped by the Lorentz transformation $(a,\Lambda)$. If now $C$ and $D$ are spacelike with respect to each other, a suitable Lorentz transformation will change their temporal order. In this case, however, the formal description of the second experiment changes drastically, and the question arises whether the expectation values are invariant, as they have to be.

Although the invariance of the expectation values may be proved,[10] it is preferable to have a manifestly covariant description of the relativistic measurement process.

We propose, therefore, the following covariant formalism: The field state remains $W$ in the past cone of $C$, and it changes into $W_P'$ or $W_P''$ in the future and side cone[11] of $C$ (Fig. 2). We shall illustrate this proposal by two examples, which are very similar to the ones discussed in Ref. 1.

Consider the measurement, in a given field state $W$, of two properties $P\in\mathfrak{N}_C$ and $Q\in\mathfrak{N}_D$. Assume $D$ to be contained in the union of the future and side cone of $C$ (Fig. 2), and let the $P$ measurement be selective. The expectation value for $Q$ is then, according to our formalism,

$$\mathrm{Tr}(QW_P'')=\frac{\mathrm{Tr}(QPWP)}{\mathrm{Tr}(PW)}.$$

The same result follows immediately from the "nonrelativistic" formalism if $D$ is in the future cone of $C$, but its derivation in the general case requires some care.[1]

Now consider the more complicated example[1] of three measurements of field properties $P$, $Q$, and $R$, where the corresponding space-time regions are indicated schematically in Fig. 3. Assume selective measurements of $P$ and $Q$, and ask for the expectation value (conditional probability) of $R$. Our formalism applies as follows. The $P$ measurement produces $W_P''$ in the regions 3 and 4, and leaves $W$ in the regions 1 and 2 unaffected. The $Q$ measurement does not affect $W$ in region 1 and $W_P''$ in region 3. It changes $W$ into $W_Q''$ in region 2, and $W_P''$ into

$$(W_P'')_Q''=\frac{QW_P''Q}{\mathrm{Tr}(QW_P'')}=\frac{QPWPQ}{\mathrm{Tr}(QPW)} \tag{3}$$

in region 4. The space-time map of field states before

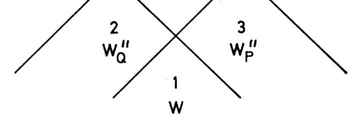


FIG. 3. Sequence of three field measurements.

[10] This follows from locality by arguments very similar to the ones used to derive Eq. (7) in Sec. 4 below. Compare also Ref. 1.

[11] The side cone of $C$ is the set of all points which are spacelike with respect to $C$.

the $R$ measurement is indicated in Fig. 3. Since $R$ is measured in region 4, its expectation value is

$$\mathrm{Tr}(R(W_P'')_Q'')=\frac{\mathrm{Tr}(RQPWPQ)}{\mathrm{Tr}(QPW)}. \tag{4}$$

Note that, in the case considered here, locality implies $QP=PQ$, and thus

$$(W_P'')_Q''=\frac{QPWPQ}{\mathrm{Tr}(QPW)}=\frac{PQWQP}{\mathrm{Tr}(PQW)}=(W_Q'')_P''. \tag{5}$$

Therefore, it does not matter whether the $Q$ or the $P$ measurement is considered to be the first one.[12] The ambiguity of temporal order of spacelike separated measurements thus does not produce any ambiguities[1] of the field state.

We assumed $W$ to remain unchanged in the past cone of $C$. This assumption has no physical implications at all, and is therefore a pure convention.[13] If we imagine the field state in the past cone to be tested by measurements, these measurements would already have changed the field state in $C$, and these state changes have to be taken into account before considering the $P$ measurement in $C$. In other words, sequences of field measurements have always to be considered in the proper causal order.

This taken into account, joint probabilities[14] for highly arbitrary sets of field properties may be calculated, provided only that the regions of measurements do not intersect with the light cones across which the field state changes. We hope our last example has already shown the formal simplicity of such calculations, as compared to similar ones[1] using the "nonrelativistic" formalism. Although the particle positions discussed in Ref. 1 are not directly related to the field observables discussed here, an appropriate modification of our arguments should also apply to particle observables.

### 3. PRACTICAL USE OF FORMALISM

In this section, we want to discuss how the formalism described above may be applied to actual experiments.[15] In view of this practical application, our formalism shall then be compared with a similar one proposed by Schlieder.[6]

Consider again, as an instructive example, the experiment corresponding to Fig. 3. Assume some experimentalists plan to perform such an experiment in a given field state $W$. (A few remarks about the original field state $W$ will follow in Sec. 4.)

According to our formalism, a unique space-time map of field states (Fig. 3) may be drawn. This field map may be assumed to be known to all observers before the experiment begins. The field map predicts the following quantities:

(i) The expectation value $\mathrm{Tr}(PW)$ of $P$ in the original state $W$. This coincides with the transition probability $p(W\rightarrow W_P'')$ from the state $W$ to the state $W_P''$.

(ii) The expectation value of $Q$ in state $W_P''$ [which is equal to the transition probability from $W_P''$ to $(W_P'')_Q''$],

$$\mathrm{Tr}(QW_P'')=p(W_P''\rightarrow(W_P'')_Q'')=\frac{\mathrm{Tr}(QPW)}{\mathrm{Tr}(PW)}.$$

(iii) The expectation value of $R$ in state $(W_P'')_Q''=(W_Q'')_P''$,

$$\mathrm{Tr}(R(W_P'')_Q'')=\frac{\mathrm{Tr}(RQPWPQ)}{\mathrm{Tr}(QPW)};$$

see Eq. (4).

Since the succession of $P$ and $Q$ is ambiguous, the same field map also predicts the quantities[16]:

(iv) The expectation value (transition probability)

$$\mathrm{Tr}(QW)=p(W\rightarrow W_Q'').$$

(v) The expectation value (transition probability)

$$\mathrm{Tr}(PW_Q'')=p(W_Q''\rightarrow(W_Q'')_P'')=\frac{\mathrm{Tr}(QPW)}{\mathrm{Tr}(QW)}.$$

The actual experiment may be performed as follows. The observers go to the preassigned space-time regions, measure $P$, $Q$, and $R$, and write down the experimental results, i.e., numbers $p$, $q$, and $r$ which are either 0 or 1.

Since the predictions of quantum theory are statistical ones, they can be tested only by very many repetitions of the same experiment. Imagine, in our case, the same experiment to be repeated $N$ times, always starting from the same field state $W$. (The meaning of the phrase "the same" is explained in more detail in Sec. 4.) This yields $N$ recordings, each consisting of three digits $p_i$, $q_i$, and $r_i$.

The predictions (i)–(v) of the theory then have to be compared with the following numbers [summations

[12] Since $(W_P'')_Q''=W_{PQ}''$, one may also speak of a selective measurement of the field property "$P$ and $Q$"$=PQ$ in the union of the regions belonging to $P$ and $Q$.

[13] This remark applies also to the unchanged state $W$ before the time $t_0$ of measurement in the nonrelativistic formalism, and to the states in the regions 1 to 3 of Fig. 3. For similar reasons, the state between $t_0-\frac{1}{2}\Delta t$ and $t_0+\frac{1}{2}\Delta t$ in the nonrelativistic formalism (Fig. 1) cannot be tested experimentally. One may also assume, without observable consequences, the existence of similar "transition regions" bounded by suitable light cones in the relativistic formalism. To simplify discussions, we have adopted as an idealization that the state jumps on the upper boundary of the transition region (Fig. 2).

[14] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[15] "Actual experiments" if invented by theoreticians are in most cases "*Gedanken* experiments" which are never performed by experimentalists. This is especially true for all experiments discussed here. Nevertheless, they may help to clarify the operational meaning of theoretical concepts.

[16] For a similar experiment, in which a $Q$ measurement is performed in the future cone of a $P$ measurement, the "predictions" (iv) and (v) are meaningless. In this case the field can *not* be assumed to be in state $W$ during the $Q$ measurement, or in state $W_Q''$ during the $P$ measurement.

over (i) always from 1 to $N$]:

(i) $$\sum_i p_i/N\,,$$

(ii) $$\sum_i p_iq_i/\sum_i p_i\,,$$

(iii) $$\sum_i p_iq_ir_i/\sum_i p_iq_i\,,$$

(iv) $$\sum_i q_i/N\,,$$

(v) $$\sum_i p_iq_i/\sum_i q_i\,.$$

Expressions (i) and (iv) are obvious. In (ii) we have selected, according to the definition of a selective $P$ measurement,[17] the subset of recordings with $p_i=1$, and have calculated the mean value of $q_i$ in this subset. An analogous calculation with $Q$ and $P$ interchanged yields (v). Finally, (iii) is the mean value of $r_i$ for those recordings, for which both $p_i=1$ and $q_i=1$. This corresponds to a measurement of $R$ after selective measurements of both $P$ and $Q$. The predictions of the theory are expected to agree with the experimental results for sufficiently large $N$.

We hope that the practical application of our formalism to an arbitrary set of field measurements has become obvious.

The formalism proposed by Schlieder[6] is somewhat more complicated. First of all, Schlieder considers also experiments for which the measurement program (i.e., the space-time regions for measurement $C$ and field properties $P\in\mathfrak{N}_C$ to be measured) is not fixed from the beginning. We think this to be an unnecessary complication. Any theoretical prediction that can be verified is a prediction about the result of actual experiments. In quantum theory, such a verification even requires a large number of repetitions of the same experiment. Therefore, we may always assume a fixed measurement program which then has to be known to all observers participating in the experiment.

The second complication in Ref. 6 arises from a problem which may again be illustrated by the experiment of Fig. 3. The $P$ measurement was assumed to be selective, thereby producing a field state $W_P''$. During the subsequent $Q$ measurement, however, the corresponding observer does not yet know the result $p$ (1 or 0) of the $P$ measurement, since this information can travel at most with the velocity of light. Therefore, while measuring $Q$ the observer does not know whether or not the field is actually in the state $W_P''$ (which requires $p=1$). The question of whether or not the result for $Q$ has been obtained in the required state $W_P''$ can, however, be decided afterwards by looking at the result $p$ of the $P$ measurement. This kind of temporary ignorance of some observers thus has no influence on the numerical evaluation of the experiment.

Therefore, in our opinion, the above-described information problem may be considered as unessential for the construction of a state map.[18] In fact, the desire for a formalism which yields unique, observer-independent field states was our principal motivation in writing this paper. The formalism described here is especially simple, since Schlieder's "$N$-map"[6] which describes the knowledge of observers becomes superfluous. Then, the remaining ingredient of Schlieder's formalism, the so called "$M$-covering,"[6] coincides with the state map proposed here. We leave it to the interested reader to check the details concerning the physical equivalence of Schlieder's formalism with ours.

## 4. REMARKS ON CAUSALITY AND VACUUM STATE

An ideal measurement of a property $P$ at some quantum system is achieved physically by a suitable interaction with a measuring apparatus. For properties $P\in\mathfrak{N}_C$ of a field, the interaction is assumed to take place in the corresponding space-time region $C$. This interaction produces correlations between the properties of the field and a macroscopic property of an apparatus (e.g., a pointer position). The replacement of the field state $W$ by $W_P'$ is a formal description of these correlations. The further reduction of $W_P'$ to $W_P''$ for selective measurements accounts for the decision of the observer to count only experiments in which the property $P$ has been found.

These replacements are certainly justified in the future cone of $C$. At first sight, however, our proposal of replacing $W$ also in the side cone of $C$ seems to contradict causality. Perhaps one would prefer a different formalism, with $W$ unchanged also in the side cone of $C$.

Actually, such formalism is equivalent to the one proposed here. Consider a $P$ measurement in $C$, and a subsequent $Q$ measurement in $D$, spacelike with respect to $C$. For a nonselective $P$ measurement, the expectation values of $Q$ in both formalisms are the same, since locality implies $PQ=QP$, and therefore,[19]

$$\begin{aligned}\mathrm{Tr}(QW_P')&=\mathrm{Tr}(QPWP)+\mathrm{Tr}[Q(1-P)W(1-P)]\\&=\mathrm{Tr}(PQWP)+\mathrm{Tr}[(1-P)QW(1-P)]\\&=\mathrm{Tr}(PQW)+\mathrm{Tr}[(1-P)QW]\\&=\mathrm{Tr}(QW)\,.\end{aligned}\tag{6}$$

Equation (6) is indeed a necessary consequence of causality, since a nonselective measurement of $P$ can

[17] An apparatus for selective measurements of a *particle* property $P$ may be visualized as a filter which absorbs particles with $P=0$ and leaves unaffected particles with $P=1$. Similar filters corresponding to local *field* properties $P$ obviously do not exist. Accordingly, the selection procedure adopted here does not absorb the fields with $P=0$, but merely neglects all experiments with $p_i=0$.

[18] Note that the whole experiment may even be performed by a single observer with the help of automatic measuring and recording devices.

[19] Generally speaking, expectation values for field observables belonging to a proper subset of space-time do not fix a unique density operator, but an equivalence class of density operators. In this sense, $W$ and $W_P'$ are equivalent in the side cone of $C$. We have chosen $W_P'$ for reasons of simplicity.

never change the result of a $Q$ measurement at a spacelike distance. This reasoning, however, does not apply to selective $P$ measurements, since the field properties $P$ and $Q$ may be correlated in some field states $W$. Thus, in general,

$$\mathrm{Tr}(QW_P'')\neq\mathrm{Tr}(QW).$$

Correlations of this type are responsible for the famous Einstein-Podolski-Rosen paradox (compare also Ref. 6).

In a formalism with $W$ unchanged in the side cone of $C$, the selective $P$ measurement in $C$ and the subsequent measurement of $Q$ in $D$ may be described as a simultaneous measurement of $P$ and $Q$ in state $W$. We ask for the expectation value of $Q$ in such experiments for which the $P$ measurement gives "yes." Since the field property "$P$ and $Q$," for commuting $P$ and $Q$, corresponds to the projection operator $QP$, the required expectation value of $Q$ is

$$\frac{\mathrm{Tr}(QPW)}{\mathrm{Tr}(PW)}=\frac{\mathrm{Tr}(QPWP)}{\mathrm{Tr}(PW)}=\mathrm{Tr}(QW_P''). \qquad (7)$$

Equations (6) and (7) indicate the consistency of our formal description of the state changes due to measurements. Moreover, our description seems to be the simplest one, since the states produced in the future and side cone of the region of measurement $C$ have the same form.

We conclude with some remarks about the field state $W$ prior to any experiment.[20] A comparison of quantum theory and experiment is only possible if the same experiment is repeated many times. This may be achieved here by exploiting Lorentz (or translation) invariance (Sec. 3). Using the same experimental equipment in space-time regions which are mapped onto each other by suitable inhomogeneous Lorentz transformations $(a,\Lambda)$, these experiments may be considered as identical, since $U(a,\Lambda)$ maps the corresponding local field properties onto each other.

Two conditions have to be fulfilled. First, the different experiments should not disturb each other. Second, the original field state $W$ should be invariant with respect to the mappings $U(a,\Lambda)$. Since the $(a,\Lambda)$ are highly arbitrary (although somewhat restricted by the first condition, e.g., to large $a$), $W$ should be invariant with respect to $U(a,\Lambda)$ for *all* $a$ and $\Lambda$.[21] The only invariant state $W$ which is normalizable to $\mathrm{Tr}W=1$ is, in the usual framework of field theory,[22] the projection $W=P_\omega=|\omega\rangle\langle\omega|$ onto the unique vacuum vector $\omega$.

This choice also fulfills the first condition. The easiest way to achieve this is to choose $(a,\Lambda)$ with $a$ spacelike and sufficiently large. However, actual experiments are usually repeated in time. Regeneration of the original vacuum state $P_\omega$ then occurs since $\omega$ is the ground state of the field and excitations of $\omega$ caused by local measurements will die away after a sufficiently long time.

With this assumption about the original field state, our state map becomes completely fixed by the measurement program.

## APPENDIX

Instead of the highly idealized measurements considered above, we will now briefly describe a more general class of state changes, called operations. (A more detailed discussion is presented elsewhere.[23])

Assume, in addition to the postulates (i) to (iii) of Sec. 2, that the set $\mathfrak{N}_C$ of field observables in $C$ is the set of Hermitian operators of a von Neumann algebra $\mathfrak{R}_C$. With this assumption, the field becomes a so-called Haag field.[24] A local operation is then described by a set of operators $A_{ki}\in\mathfrak{R}_C$, $k$, $i=1\cdots n$ (including the possibility $n=\infty$), with

$$\sum_{k=1}^{n} A_{ki}^*A_{kj}=\delta_{ij}1\,,\quad \sum_{i=1}^{n} A_{ki}A_{li}^*=\delta_{kl}1\,, \qquad (\mathrm{A}1)$$

a sequence of numbers

$$c_i\geqslant 0 \quad\text{with}\quad \sum_{i=1}^{n} c_i=1\,,$$

and a subset $K$ of the index set $\{1\cdots n\}$.

The field state $W$ then changes into

$$W'=\sum_{k=1}^{n}\sum_{i=1}^{n} c_iA_{ki}WA_{ki}^* \qquad (\mathrm{A}2)$$

if the operation is nonselective, and into

$$W''=\hat{W}/\mathrm{Tr}\hat{W},\quad \hat{W}=\sum_{k\in K}\sum_{i=1}^{n} c_iA_{ki}WA_{ki}^* \qquad (\mathrm{A}3)$$

in case of a selective operation. This state change is again assumed to occur in the future and side cone of the region of operation $C$.

Ideal measurements of a field property $P\in\mathfrak{R}_C$ are particular cases. [Eqs. (A2) and (A3) reduce to (1) and (2), e.g., for $n=2$, $A_{11}=A_{22}=P$, $A_{12}=A_{21}=1-P$, $c_1=1$, $c_2=0$, $K=\{1\}$.] Local operations are more general than ideal measurements in two respects. First, the field property $P\in\mathfrak{R}_C$ is replaced by a "local effect," i.e., a Hermitian operator

$$F=\sum_{k\in K}\sum_{i=1}^{n} c_iA_{ki}^*A_{ki} \qquad (\mathrm{A}4)$$

with $0\leqslant F\leqslant 1$, $F\in\mathfrak{R}_C$. Secondly, even if $F$ is a projec-

[20] Compare A. L. Licht, J. Math. Phys. **9**, 1468 (1968).

[21] One may confirm this by using the group property of $U(a,\Lambda)$.

[22] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That* (W. A. Benjamin, Inc., New York, 1964).

[23] K.-E. Hellwig and K. Kraus, Commun. Math. Phys. **11**, 214 (1968); *ibid.* (to be published).

[24] B. Misra, Helv. Phys. Acta **38**, 189 (1965).

tion operator $P$, the state change need not be described by Eqs. (1) and (2). (In other words, such $P$ measurements are not ideal, as defined in Ref. 3. A trivial example is the case $n=1$, with a unitary $A_{11}=U\in\mathcal{R}_C$, and $K=\{1\}$. The property measured is $P=1$, whereas the new state is $UWU^*\neq W$. This type of state change may be due to external fields in $C$.)

A local effect describes the occurrence of "clicks" in a suitable "counter" which interacts with the field in the space-time region $C$.[23] The rate of occurrence of the effect $F$ in the field state $W$ is $\mathrm{Tr}(FW)$.[25]

With this interpretation, our generalized formalism allows the calculation of joint probabilities for such clicks $F$ in the same way as does the formalism for ideal measurements. As is evident from Eq. (A4), a local effect $F$ corresponds to many different local operations. Therefore, joint probabilities for local effects depend not only on the effects themselves but also, via (A2) and (A3), on the particular local operations performed.

[25] The state $W''$ [Eq. (A3)] describes the field if it has produced the effect $F$, whereas the state $W'$ [Eq. (A2)] is generated in experiments in which no selection is made with regard to the occurrence or nonoccurrence of $F$. For the particular case $F=1$, $W'$ and $W''$ are identical since every field produces the effect corresponding to $F=1$, and a nontrivial selection becomes impossible. Therefore, selective operations with $F=1$ are equivalent to nonselective ones.

PHYSICAL REVIEW D VOLUME 1, NUMBER 2 15 JANUARY 1970

# Quantum Theory of the Infinite-Component Majorana Field and the Relation of Spin and Statistics*

E. C. G. SUDARSHAN
*Physics Department, Syracuse University, Syracuse, New York 13210*

AND

N. MUKUNDA
*Tata Institute for Fundamental Research, Colaba, Bombay, India*

(Received 23 December 1968)

The quantum theory of the infinite-component Majorana field is formulated. The present paper discusses the three classes (slower than light, lightlike, and faster than light) of solutions to this equation and their Wigner classification. Particular attention is paid to the question of the normalization of the faster-than-light solutions. The current operator is shown to be timelike even for the spacelike solutions, and it is shown to lead to a finite process of emission of light by charged Majorana particles. The quantum theory of the Majorana field is formulated in accordance with the substitution law, and the usual connection between spin and statistics is recovered.

## I. INTRODUCTION

THE usefulness of the local covariant field description of particle phenomena has been amply demonstrated in the successes of quantum electrodynamics and of the chiral $V-A$ weak interactions. It has been conventional in such treatments to use local fields, each of which describes only one kind of particle, with a definite mass and a definite spin. Many years ago the late H. J. Bhabha systematically investigated the possibility of describing a family of particles with varying masses and spins by a single irreducible equation.[1] As a special class of such equations, Bhabha studied relativistic wave equations of the form

$$(i\Gamma^\mu\partial/\partial x^\mu-\kappa)\psi=0\,, \tag{1.1}$$

where the matrices $\Gamma^\mu$ together with the spin matrices $S^{\mu\nu}$ satisfied the commutation relations of the de Sitter group. Of course, the $S^{\mu\nu}$ themselves satisfy the commutation relations of the homogeneous Lorentz group, and the matrices $\Gamma^\mu$ constitute a four-vector operator with respect to this group. Bhabha's additional assumption was that the matrices $\Gamma^\mu$ among themselves satisfied the commutation relations of the form

$$[\Gamma^\mu,\Gamma^\nu]=i\lambda S^{\mu\nu}.$$

From this de Sitter structure, Bhabha was able to obtain a mass-spin spectrum in which the mass decreased as the spin increased. These equations include the spin-$\frac{1}{2}$ Dirac equation and the spin-0 and -1 Duffin-Kemmer-Petiau equations. But except for these special cases, the Bhabha equations lead to the necessity of introducing an indefinite metric of an unsatisfactory kind. This difficulty can be traced to the unfortunate restriction to finite-component fields, which necessarily correspond to nonunitary representations of the homogeneous Lorentz group. We should therefore relax this restriction

* Supported in part by the U. S. Atomic Energy Commission.
[1] H. J. Bhabha, Rev. Mod. Phys. 17, 200 (1945).