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Description of Particles in Scalar Hyperplane-Dependent Field Theory. I. The General Formalism*

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The concept of scalar hyperplane-dependent fields is defined on the basis of the author's earlier treatments of the position operators for relativistic particles. Such fields are a special case of fields over homogeneous spaces of the Poninearé group and are related to infinite families of integer spin particles with spin-dependent mass spectra. The intrinsic parity of the particles and degeneracy in the mass spectrum is examined. The Lagrangian formalism, including Noether's theorem, is developed and explored. It is shown that the existence of a Lagrangian formalism yields a normalization condition that is incompatible with the existence of spacelike solutions. All is in preparation for the sequel to this paper which considers a specific model with Bose statistics and nondegenerate mass spectrum.

1. INTRODUCTION

THIS is the first[1] of a series[2] of papers in which a particular version of nonlocal quantized field theory will be studied. The version in question is motivated by the author's earlier studies of the properties of position as a dynamical variable in relativistic quantum theory.[3] It amounts to defining the fields over a seven-dimensional continuous manifold, the points of which are labeled by the ordered pair (x,η), where x is a position four-vector of Minkowski space and η is a *dimensionless timelike unit vector* with positive time component.[4]

The vector η is to be geometrically understood as designating the normal direction to a three-dimensional *spacelike hyperplane* in Minkowski space which includes x. The introduction of such *hyperplane parameters* is prompted by the realization that the results of a precise position measurement of a particle, e.g., the resulting state vector (which is almost a position eigenvector), depends dramatically on the spacelike hyperplane over which the search for the particle was made as well as on that particular point in the hyperplane at which the particle is found.[3,4] This means that a unique specification of the results of a position measurement requires at least the parameters of the ordered pair (x,η). Inasmuch as field theory is just the study of quantities defined over the manifold of all possible results of position measurements, the study of quantized fields over the seven-dimensional Minkowski-hyperplane space is strongly indicated.

The position taken here is similar in spirit to those expressed in the studies of field theories for extended particles.[5] Since extended particles have, besides a conventional position, a configuration or orientation as well, the parameters for describing that configuration become new arguments of the associated fields.[6] Such approaches require the investigator to go out on a limb in hypothesizing the character of the structural features of the extended particles. The nonlocal field theory to be presented here, however, enjoys the advantage that no commitment to the structural features of particles is required. The additional arguments of the field emerge, instead, from a very general study of the nature of position itself in relativistic quantum theory.[7]

The results that emerge from the study of such *hyperplane-dependent fields*[8] are very similar to the prop-

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[1] The present paper may be regarded as a revised version of an earlier report [G. N. Fleming, Pennsylvania State University report, 1968 (unpublished)].

[2] The second paper in the series will treat a boson model with nondegenerate linear mass spectrum [G. N. Fleming and F. Ardalan, Pennsylvania State University report, 1969 (unpublished)].

[3] G. N. Fleming, Phys. Rev. **137**, B188 (1965); **139**, B963 (1965).

[4] G. N. Fleming, J. Math. Phys. **7**, 1959 (1966). Thus the space of the (x,η) is a homogeneous space of the Poincaré group; mathematically, this field theory is just a special case of field theories over homogeneous spaces which have received renewed attention from C. Chi, D. van Dyck, and N. van Hieu, Ann. Phys. (N. Y.) **49**, 173 (1968); H. Bacry and A. Kihlberg, Institute for Theoetical Physics, Goteborg, Sweden, Report, 1968 (unpublished); A. Kihlberg, Institute for Theoretical Physics, Goteborg, Sweden, Report, 1969 (unpublished). The latter authors favor the eight-dimensional homogeneous space for its ability to accomodate half-integral spin without the introduction of spinor fields. The seven-dimensional homogeneous space is favored here for the independent physical interpretation that is provided by the study of position operators.

[5] H. Yukawa, in *Proceedings of the International Conference on Elementary Particles, 1965, Kyoto*, pp. 139–158; Y. Katayama, in *Proceedings of the 1967 International Conference on Particles and Fields*, edited by C. R. Hagen, G. Guralnik, and V. A. Mathur (Wiley-Interscience, Inc., New York, 1968), pp. 157–165; T Takabayashi, Progr. Theoret. Phys. (Kyoto) **36**, 187 (1966); **36**, 660 (1966); **36**, 662 (1966); **37**, 765 (1967); **37**, 767 (1967); in *Proceedings of the 1967 International Conference on Particles and Fields*, edited by C. R. Hagen, G. Guralink, and V. A. Mathur (Wiley-Interscience, Inc., New York, 1968), pp. 413–425.

[6] H. C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden Day Publishing Co., New York, 1968).

[7] A closely related development is indicated in F. Lurçat, Physics **1**, 95 (1964). Lurçat's ideas greatly influenced the thinking of the present author, but a recent presentation in F. Lurçat, Phys. Rev. **173**, 1461 (1968), indicates a wider divergence from the position taken here.

[8] From a formal point of view, hyperplane-dependent fields are very similar to the frame-dependent fields of R. L. Ingraham [Nuovo Cimento **24**, 1117 (1962); **26**, 328 (1962); **27**, 303 (1963); **32**, 323 (1964); **34**, 182 (1964); **39**, 131 (1965); Phys. Rev. **152**, 1290 (1966)], whose articles convinced the present author of the internal consistency of the concepts of frame- or hyperplane-dependent fields. The motivation for Ingraham, however, is primarily that of getting a convergent perturbation theory. For a systematic exposition, see R. L. Ingraham, *Renormalization Theory of Quantum Field Theory with a Cutoff* (Gordon and Breach, Science Publishers, Inc., New York, 1968).

erties of both extended-particle fields and the more algebraically motivated infinite-component field theories.[9] The quanta of the fields comprise infinite families of particles lying on Regge-like trajectories. Infinite degeneracy[10] in the masses is easily come by but can be avoided in examples with realistic spectra. Consecutive particles lying on the same Regge-like trajectory have opposite parities reminiscent of the signature property[11] of Regge theory. Examples can be found of theories with well-defined and well-behaved vertex functions which, however, are not analytic functions of their arguments.[12] Finally, hyperplane-dependent spinor fields (to be discussed in a later paper in this series) display a version of parity doubling[13] in their solutions.

The most striking and novel result is the demonstration that within the context of a Lagrangian formalism[14] there can be no acceptable spacelike solutions[15] in scalar hyperplane-dependent field theory. Thus the major plague of infinite mutliplet theories with nondegenerate mass spectra is not present here.

In Secs. 2 and 3 the particle aspect of scalar hyperplane-dependent fields is explored in a general way. This is followed by the development of the Lagrangian formalism and the determination of the form of currents with vanishing four-divergence that emerge from Noether's theorem[16] in a seven-dimensional field theory. In Sec. 5 a brief discussion of equal-hyperplane commutation relations as the analog of equal-time commutation relations in local field theory is presented. It is to be noted that the conventional notions of locality and microcausality have no weight here for the fields by virtue of the physical interpretation provided above. Microcausality of the conserved currents is another question that will be considered in the sequel. Finally, the paper closes with a demonstration of the impossibility of particles with spacelike momenta occurring as acceptable solutions. It must be emphasized that this result depends crucially on the presumed existence of a Lagrangian formalism.

The problem of the statistics[17] of the particles will also be taken up in the sequel.

[9] E. Majorana, Nuovo Cimento **9**, 335 (1932); I. M. Gelfand and A. M. Yaglom, Zh. Eksperim. i Teor. Fiz. **18**, 703 (1948). These papers provide the historical origins of the subject.
[10] H. D. I. Abarbanel and Y. Frishman, Phys. Rev. **171**, 1442 (1968); R. F. Streater, Commun. Math. Phys. **5**, 88 (1967).
[11] E. J. Squires, *Complex Angular Momentum in Particle Physics* (W. A. Benjamin, Inc., New York, 1963).
[12] C. Fronsdal and R. White, Phys. Rev. **163**, 1835 (1967).
[13] V. Barger, University of Wisconsin Report, 1968 (unpublished).
[14] A Lagrangian approach to infinite component fields has been presented by C. Fronsdal, Phys. Rev. **181**, 1881 (1968).
[15] Y. Nambu, Phys. Rev. **160**, 1171 (1967); S. J. Chang and L. O'Raifeartaigh, *ibid.* **170**, 1316 (1968).
[16] E. Noether, Nachr. Kgl. Ges. Wiss. Göttingen, p. 235 (1918).
[17] E. Abers, I. F. Grodksy, and R. E. Norton, Phys. Rev. **159**, 1222 (1967). For a more recent evaluation of the spin-statistics problem, see Ref. 14.

2. BASIC EQUATIONS FOR SCALAR FIELD

The scalar hyperplane-dependent field operator to be studied in this and a future paper will be denoted by $\phi(x,\eta)$. Under the Poincaré transformation,[18]

$$x_\mu' = \Lambda_\mu{}^\nu x_\nu + a_\mu\,, \tag{2.1a}$$

$$\eta_\mu' = \Lambda_\mu{}^\nu \eta_\nu\,, \tag{2.1b}$$

the field is assumed to satisfy

$$\phi'(x',\eta') = \phi(x,\eta)\,. \tag{2.2}$$

These relations immediately lead to the commutation relations between the field and the generators of the Poincaré group, P_μ and $M_{\mu\nu}$,

$$[\phi, P_\mu] = i\hbar \partial_\mu \phi\,, \tag{2.3a}$$

$$[\phi, M_{\mu\nu}] = i\hbar(x_\mu\partial_\nu - x_\nu\partial_\mu + \eta_\mu\delta_\nu - \eta_\nu\delta_\mu)\phi\,, \tag{2.3b}$$

where

$$\delta_\mu \equiv \frac{\partial}{\partial\eta^\mu} - \frac{\eta_\mu\eta^\lambda}{\eta^2}\frac{\partial}{\partial\eta^\lambda} \tag{2.4}$$

is the differential operator in η space that allows η to always remain in the unit hyperboloid,

$$\eta^2 = 1\,. \tag{2.5}$$

When manipulating this differential operator, it is necessary to employ the relations

$$\eta^\mu \delta_\mu \equiv 0\,, \tag{2.6a}$$

$$\delta^\mu \eta_\nu = g_{\mu\nu} - \eta_\mu\eta_\nu\,, \tag{2.6b}$$

$$\delta_\mu\delta_\nu - \delta_\nu\delta_\mu = \eta_\mu\delta_\nu - \eta_\nu\delta_\mu\,. \tag{2.6c}$$

From the appearance of the η variable and derivative on the right-hand side of (2.3b), one naturally expects the hyperplane dependence of the field to relate to the spin of the associated particles.[19] This is borne out by the multiple commutation relation

$$-\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}[[\phi, M^{\alpha\beta}], P^\gamma] = \hbar^2\Omega_\mu\phi\,, \tag{2.7}$$

where

$$\Omega_\mu \equiv \epsilon_{\mu\alpha\beta\gamma}\eta^\alpha\delta^\beta\partial^\gamma\,, \tag{2.8}$$

for if (2.7) is applied to the vacuum state $|0\rangle$, one obtains

$$W_\mu\phi|0\rangle = \hbar^2\Omega_\mu\phi|0\rangle \tag{2.9a}$$

and then by iteration

$$W^2\phi|0\rangle = \hbar^4\Omega^2\phi|0\rangle\,, \tag{2.9b}$$

where W_μ is the Pauli-Lubanski operator[20] and the eigenvalues of W^2 are $-m^2c^2\hbar^2 s(s+1)$.

Thus, when applied to the vacuum, the scalar hyperplane-dependent field yields a vector describing nonvanishing spin, provided only that the hyperplane

[18] G. N. Fleming, Bull. Am. Phys. Soc. **13**, 39 (1968).
[19] G. N. Fleming, Bull. Am. Phys. Soc. **13**, 1696 (1968).
[20] J. K. Lubanski, Physica **9**, 310 (1942).

dependence of the field is sufficiently involved. In particular, the familiar correspondences

$$P_\mu \sim (\hbar/i)\partial_\mu, \quad P^2 \sim -\hbar^2\Box, \tag{2.10a}$$

are here supplemented by

$$W_\mu \sim \hbar^2\Omega_\mu, \quad W^2 \sim \hbar^4\Omega^2. \tag{2.10b}$$

3. PARTICLE ASPECT

Upon introducing the Fourier transform

$$\tilde{\phi}(k,\eta) = (2\pi)^{-2}\int d^4x\, e^{ikx}\phi(x,\eta), \tag{3.1}$$

one obtains the results

$$[\tilde{\phi},P_\mu] = \hbar k_\mu \tilde{\phi} \tag{3.2}$$

and

$$-\tfrac{1}{2}\epsilon_{\mu\alpha\beta\gamma}[[\tilde{\phi},M^{\alpha\beta}],P^\gamma] = -\hbar^2\omega_\mu(k)\tilde{\phi}, \tag{3.3}$$

where

$$\omega_\mu(k) \equiv \epsilon_{\mu\alpha\beta\gamma}\eta^\alpha\delta^\beta k^\gamma. \tag{3.4}$$

Consequently, $\tilde{\phi}(k,\eta)$ acts as a creation operator for momentum $-\hbar k_\mu$ and as an annihilation operator for momentum $\hbar k_\mu$ in the familiar way, and the spin content of the Fourier transform is determined by its relation to the differential operator $\omega_\mu(k)$.

To explore this spin content, it is convenient to expand $\tilde{\phi}$ as a power series in the components of η. Specifically, the expansion

$$\tilde{\phi}(k,\eta) = \sum_{s=0}^{\infty} h_{\alpha_1}\cdots h_{\alpha_s}\tilde{\chi}^{\alpha_1\cdots\alpha_s}(k), \tag{3.5}$$

where the first term is $\tilde{\chi}(k)$ and

$$h_\alpha \equiv \eta_\alpha - k_\alpha(\eta\cdot k/k^2), \tag{3.6}$$

will be employed. The use of h rather than η itself as the expansion variable is dictated by the convenience of a linear constraint among the expansion variables,

$$k\cdot h = 0, \tag{3.7}$$

as opposed to the quadratic constraint (2.5). If one so chooses, (3.5) could be written as a power series in independent components of h, but a power series in η could not be similarly written. Also, the components of h can vanish while $\eta_0 > 1$, so that (3.5) is truly an expansion about the origin in h space.

One serious limitation on the expansion, however, is that it cannot be employed for null k. The definition (3.6) breaks down for $k^2=0$, and so a study based on (3.5) can only apply to the case of massive particles. The massless case will be treated separately.[2]

Ignoring questions of convergence, one can rearrange the terms in (3.5) to obtain the expansion

$$\tilde{\phi}(k,\eta) = \sum_{s=0}^{\infty} h_{\alpha_1}\cdots h_{\alpha_s}\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k), \tag{3.8}$$

where the operator coefficients, $\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k)$, may now depend on η through the scalar $\eta\cdot k$ and are, themselves, fully *symmetric*, *traceless*, and *transverse* s-rank tensor fields with $(2s+1)$ independent components. By transverse is meant that

$$k_{\alpha_r}\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k) = 0 \quad (1\le r\le s). \tag{3.9}$$

At this point, one expects the s term in the expansion (3.8) to be associated with particles of definite spin as well as definite momentum. To confirm this, note that

$$\delta_\mu(h_{\alpha_1}\cdots h_{\alpha_s}) = -s\eta_\mu(h_{\alpha_1}\cdots h_{\alpha_s}) + \sum_{r=1}^{s} h_{\alpha_1}\cdots h_{\alpha_{r-1}}\left(g_{\alpha_r\mu} - \frac{k_{\alpha_r}k_\mu}{k^2}\right)h_{\alpha_{r+1}}\cdots h_{\alpha_s}, \tag{3.10}$$

from which it follows that

$$(k\cdot\delta)(h_{\alpha_1}\cdots h_{\alpha_s}) = -s(\eta\cdot k)(h_{\alpha_1}\cdots h_{\alpha_s}), \tag{3.11}$$

$$(k\delta)^2(h_{\alpha_1}\cdots h_{\alpha_s}) = \{s^2(\eta\cdot k)^2 - s[k^2-(\eta\cdot k)^2]\} \times (h_{\alpha_1}\cdots h_{\alpha_s}), \tag{3.12}$$

and

$$(\delta^2)(h_{\alpha_1}\cdots h_{\alpha_s}) = -s(s+2)(h_{\alpha_1}\cdots h_{\alpha_s}) + 2\sum_{r<t} h_{\alpha_1}\cdots h_{\alpha_{r-1}}h_{\alpha_{r+1}}\cdots h_{\alpha_{t-1}}h_{\alpha_{t+1}}\cdots h_{\alpha_s} \times\left(g_{\alpha_r\alpha_t} - \frac{k_{\alpha_r}k_{\alpha_t}}{k^2}\right). \tag{3.13}$$

Hence, employing

$$\omega^2(k) = [(\eta\cdot k)^2 - k^2]\delta^2 + (k\cdot\delta)^2 - (\eta\cdot k)(k\cdot\delta), \tag{3.14}$$

which follows from the definition (3.4), one obtains

$$[\omega^2(k)h_{\alpha_1}\cdots h_{\alpha_s}]\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k) = s(s+1)k^2(h_{\alpha_1}\cdots h_{\alpha_s})\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k). \tag{3.15}$$

Finally, since

$$\omega_\mu(k)(h_{\alpha_1}\cdots h_{\alpha_s}\tilde{\phi}^{\alpha_1\cdots\alpha_s}) = (\omega_\mu(k)h_{\alpha_1}\cdots h_{\alpha_s})\tilde{\phi}^{\alpha_1\cdots\alpha_s} \tag{3.16a}$$

and

$$\omega^2(k)(h_{\alpha_1}\cdots h_{\alpha_s}\tilde{\phi}^{\alpha_1\cdots\alpha_s}) = [\omega^2(k)h_{\alpha_1}\cdots h_{\alpha_s}]\tilde{\phi}^{\alpha_1\cdots\alpha_s}, \tag{3.16b}$$

it follows that

$$\omega^2(k)\tilde{\phi} = \sum_s s(s+1)k^2 h_{\alpha_1}\cdots h_{\alpha_s}\tilde{\phi}^{\alpha_1\cdots\alpha_s}, \tag{3.17}$$

thereby confirming the suspected spin content of the terms in the expansion. The scalar field describes particles of integral spin.

It is still not possible to identify the $\tilde{\phi}^{\alpha_1\cdots\alpha_s}$ with the conventional creation and annihilation operators of spin and momentum eigenstates for single particles because of the residual hyperplane dependence. In general, an expansion of the form

$$\tilde{\phi}^{\alpha_1\cdots\alpha_s}(\eta\cdot k,k)=\sum_n \rho_{sn}(\eta\cdot k)\tilde{\psi}_n{}^{\alpha_1\cdots\alpha_s}(k) \quad (3.18)$$

is possible, with the functions ρ_{sn} chosen as appropriately normalized functions forming a complete set which may or may not have to satisfy a differential equaion derived from the original field equation for ϕ. The $\tilde{\psi}^{\alpha_1\cdots\alpha_s}$ will then have the form [see (3.21) as an example]

$$\tilde{\psi}_n{}^{\alpha_1\cdots\alpha_s}(k)=\delta(k^2-\kappa_{sn}{}^2)[\theta(k_0)a_n{}^{\alpha_1\cdots\alpha_s}(\mathbf{k}) +\theta(-k_0)b_n{}^{\alpha_1\cdots\alpha_s\dagger}(-\mathbf{k})], \quad (3.19)$$

and the a's and b's are the particle-antiparticle annihilation operators.

If the free-field masses κ_{sn} are degenerate with respect to n, i.e., $\kappa_{sn}=\kappa_s$, then there is no physical basis, in the absence of interactions, for distinguishing between $\tilde{\psi}_n$ for given s, and the expansion functions ρ_{sn} are entirely conventional. The equation of motion does not determine them. If the equation of motion does determine them, then κ_{sn} is nondegenerate for given s. Both kinds of field theories can be found.

Thus any field equation of the form

$$f(\Box,\Omega^2)\phi=0 \quad (3.20a)$$

yields

$$f(-k^2,\ -\omega^2(k))\tilde{\phi}=0, \quad (3.20b)$$

and

$$\sum_{s=0}^{\infty} h_{\alpha_1}\cdots h_{\alpha_s}f(-k^2,\ -k^2s(s+1))\tilde{\phi}^{\alpha_1\cdots\alpha_s}=0. \quad (3.20c)$$

From this,

$$\tilde{\phi}^{\alpha_1\cdots\alpha_s}\propto\delta(f(-k^2,\ -k^2s(s+1))) \propto\delta(k^2-\kappa_s{}^2), \quad (3.21)$$

where

$$f(-\kappa_s{}^2,\ -\kappa_s{}^2s(s+1))\equiv0, \quad (3.22)$$

and so the mass spectrum is infinitely degenerate.

The general fourth-order differential equation of the type[19]

$$(\Omega^2+\lambda\Box+l_0{}^2\Box^2+l_1{}^{-2})\phi=0 \quad (3.23)$$

yields the mass spectrum

$$-\kappa_s{}^2s(s+1)-\lambda\kappa_s{}^2+l_0{}^2\kappa_s{}^4+l_1{}^{-2}=0$$

or

$$\kappa_s=[\lambda+s(s+1)]/2l_0{}^2\pm(1/2l_0{}^2) \times\{[\lambda+s(s+1)]^2-4(l_0/l_1)^2\}^{1/2} \quad (3.24)$$

for $l_0\neq0$ and $l_1\neq\infty$. If $l_0=0$, then

$$\kappa_s{}^2=1/l_1{}^2[\lambda+s(s+1)], \quad (3.25)$$

and if $l_1=\infty$, then

$$\kappa_s{}^2=[\lambda+s(s+1)]/l_0{}^2. \quad (3.26)$$

Consider the parity transformation

$$\begin{aligned}x&\equiv(x_0,\mathbf{x})\to(x_0,\ -\mathbf{x})\equiv x',\\ \eta&\equiv(\eta_0,\boldsymbol{\eta})\to(\eta_0,\ -\boldsymbol{\eta})\equiv\eta'.\end{aligned} \quad (3.27)$$

If one assumes

$$\mathcal{P}\phi(x',\eta')\mathcal{P}^{-1}=\lambda\phi(x,\eta)\quad(\lambda=\pm1), \quad (3.28a)$$

then

$$\mathcal{P}\tilde{\phi}(k',\eta')\mathcal{P}^{-1}=\lambda\tilde{\phi}(k,\eta) \quad (3.28b)$$

and

$$h_{\alpha_1}'\cdots h_{\alpha_s}'\rho_{sn}(\eta'\cdot k')\mathcal{P}\tilde{\psi}_n{}^{\alpha_1\cdots\alpha_s}(k')\mathcal{P}^{-1} =\lambda h_{\alpha_1}\cdots h_{\alpha_s}\rho_{sn}(\eta\cdot k)\tilde{\psi}_n{}^{\alpha_1\cdots\alpha_s}(k). \quad (3.28c)$$

But since

$$h_{m_1}'\cdots h_{m_s}'=(-1)^s h_{m_1}\cdots h_{m_s},\quad(m_r=1,2,3) \quad (3.29)$$

and

$$\rho_{sn}(\eta'\cdot k')=\rho_{sn}(\eta\cdot k), \quad (3.30)$$

one finds

$$\mathcal{P}\tilde{\psi}_n{}^{m_1\cdots m_s}(k')\mathcal{P}^{-1}=\lambda(-1)^s\tilde{\psi}_n{}^{m_1\cdots m_s}(k), \quad (3.31)$$

and the intrinsic parity of the sn particle or antiparticle is

$$\lambda(-1)^s. \quad (3.32)$$

The intrinsic parity alternates with integer changes in spin.

4. LAGRANGIAN FORMALISM

To account for the most general field equation of fourth order, it is necessary to consider Lagrangian densities of the form

$$L(x,\eta)=L(\phi,\partial_\mu\phi,\delta_\mu\phi,\partial_\mu\partial_\nu\phi,\partial_\mu\delta_\nu\phi,\delta_\mu\delta_\nu\phi; \phi^\dagger,\partial_\mu\phi^\dagger,\delta_\mu\phi^\dagger,\partial_\mu\partial_\nu\phi^\dagger,\partial_\mu\delta_\nu\phi^\dagger,\delta_\mu\delta_\nu\phi^\dagger,\eta_\mu), \quad (4.1)$$

where the last entry indicates a possible explicit dependence of L on the hyperplane vector η. Such a dependence is possible since η is not subject to translations under the Poincaré group.

The variational principle is

$$\delta\int d^4\eta\ \delta(\eta^2-1)\theta(\eta_0)d^4x\ L(x,\eta)=0, \quad (4.2)$$

and in deriving the Euler-Lagrange equation that follows from such a principle it is necessary to note that, for *any* function of η, $f(\eta)$ satisfying

$$\int d^4\eta\frac{\partial}{\partial\eta^\mu}[\delta_+(\eta^2-1)f(\eta)]=0, \quad (4.3)$$

it follows that

$$\int d^4\eta\ \delta_+(\eta^2-1)\delta_\mu f(\eta)=3\int d^4\eta\ \delta_+(\eta^2-1)\eta_\mu f(\eta). \quad (4.4)$$

Consequently, when doing integration by parts in (4.2) on the derivatfve δ_μ, one cannot convert the η integral completely into vanishing surface integrals.

The resulting Euler-Lagrange equation is then

$$\frac{\partial L}{\partial\phi}-\partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)}+\partial_\mu\partial_\nu\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}+\partial_\mu(\delta_\nu-3\eta_\nu)\frac{\partial L}{\partial(\partial_\mu\delta_\nu\phi)}$$
$$-(\delta_\mu-3\eta_\mu)\frac{\partial L}{\partial(\delta_\mu\phi)}+(\delta_\nu-3\eta_\nu)(\delta_\mu-3\eta_\mu)\frac{\partial L}{\partial(\delta_\mu\delta_\nu\phi)}=0. \qquad (4.5)$$

In the next paper in this series,[2] the Lagrangian density

$$L=\partial^\mu\delta^\nu\phi^\dagger\partial_\mu\delta_\nu\phi-\partial^\mu\delta^\nu\phi^\dagger\partial_\nu\delta_\mu\phi+\lambda\partial_\mu\phi^\dagger\partial_\mu\phi+l_0{}^2\partial^\mu\partial^\nu\phi^\dagger\partial_\mu\partial_\nu\phi \qquad (4.6)$$

will be studied. The corresponding Euler-Lagrange equation is

$$[\Box\delta^2-(\partial\cdot\delta)^2+3(\eta\cdot\delta)(\partial\cdot\partial)-\lambda\Box+l_0{}^2\Box^2]\phi=0. \qquad (4.7)$$

Noether's[16] theorem applies in a form appropriate to the seven-dimensional space of the (x,η) and must be manipulated slightly to yield conserved "currents" in Minkowski space. First note that the variation in the *form* of L due to a variation in the *form* of ϕ is

$$\bar\delta L\equiv\frac{\partial L}{\partial\phi}\bar\delta\phi+\frac{\partial L}{\partial(\partial_\mu\phi)}\bar\delta\partial_\mu\phi+\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}\bar\delta\partial_\mu\partial_\nu\phi$$
$$+\frac{\partial L}{\partial(\delta_\mu\phi)}\bar\delta\delta_\mu\phi+\frac{\partial L}{\partial(\partial_\mu\delta_\nu\phi)}\bar\delta\partial_\mu\delta_\nu\phi+\frac{\partial L}{\partial(\delta_\mu\delta_\nu\phi)}$$
$$\times\bar\delta\delta_\mu\delta_\nu\phi+\text{H.c.}$$
$$=\partial_\mu\bigg(\frac{\partial L}{\partial(\partial_\mu\phi)}\bar\delta\phi+\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}\overleftrightarrow{\partial}_\nu\bar\delta\phi+\frac{\partial L}{\partial(\partial_\mu\delta_\nu\phi)}$$
$$\times\delta_\nu\bar\delta\phi+\text{H.c.}\bigg)+(\delta_\mu-3\eta_\mu)\bigg(\frac{\partial L}{\partial(\delta_\mu\phi)}\bar\delta\phi-\partial_\nu\frac{\partial}{\partial(\partial_\nu\delta_\mu\phi)}$$
$$\times\bar\delta\phi-(\delta_\nu-3\eta_\nu)\frac{\partial L}{\partial(\delta_\nu\delta_\mu\phi)}\bar\delta\phi+\frac{\partial L}{\partial(\delta_\mu\delta_\nu\phi)}\delta_\nu\bar\delta\phi+\text{H.c.}\bigg), \qquad (4.8)$$

where (4.5) and the interchangability of differential coefficients and $\bar\delta$ have been used. Integration of the hyperplane-dependent variation over all hyperplanes passing through the point x yields, from (4.4),

$$\int d^4\eta\ \delta_+(\eta^2-1)\bar\delta L(x,\eta)$$
$$=\partial_\mu\int d^4\eta\ \delta_+(\eta^2-1)\bigg(\frac{\partial L}{\partial(\partial_\mu\phi)}\bar\delta\phi+\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}\overleftrightarrow{\partial}_\nu\bar\delta\phi$$
$$+\frac{\partial L}{\partial(\partial_\mu\delta_\nu\phi)}\delta_\nu\bar\delta\phi+\text{H.c.}.\bigg) \qquad (4.9)$$

For the special case of gauge transformations

$$\phi'=\phi e^{i\Lambda}\approx\phi(1+i\Lambda), \qquad (4.10)$$

the guage invariance of L yields

$$\bar\delta L=0 \qquad (4.11)$$

and thus

$$\partial_\mu j^\mu(x)=0, \qquad (4.12)$$

where

$$j^\mu=i\int d^4\eta\ \delta_+(\eta^2-1)\bigg(\frac{\partial L}{\partial(\partial_\mu\phi)}\phi+\frac{\partial L}{\partial(\partial_\mu\partial_\nu\phi)}\overleftrightarrow{\partial}_\nu\phi$$
$$+\frac{\partial L}{\partial(\partial_\mu\delta_\nu\phi)}\delta_\nu\phi-\text{H.c.}\bigg). \qquad (4.13)$$

In the succeeding paper, Eq. (4.13) will be used as the electromagnetic current and the circumstances under which

$$Q\equiv\sum_{s,n}\int d^3k[n_{sn}(\mathbf{k})-\bar n_{sn}(\mathbf{k})]=\int d^3x\ j_0(\mathbf{x},x_0) \qquad (4.14)$$

holds will be determined. Bose statistics will be required.

In the case of infinitesimal translations, we have

$$L'(x',\eta)=L(x,\eta) \qquad (4.15a)$$

and

$$\phi'(x',\eta)=\phi(x,\eta), \qquad (4.15b)$$

so that

$$\bar\delta L=-\partial_\mu L\bar\delta a^\mu \qquad (4.16a)$$

and

$$\bar\delta\phi=-\partial_\mu\phi\bar\delta a^\mu. \qquad (4.16b)$$

Upon substitution of (4.16) into (4.9), one obtains

$$\partial^\mu\mathcal{T}_{\mu\nu}(x)=0, \qquad (4.17)$$

where

$$\mathcal{T}_{\mu\nu}(x)=\int d^4\eta\ \delta_+(\eta^2-1)\bigg(\frac{\partial L}{\partial(\partial^\mu\phi)}\partial_\nu\phi+\frac{\partial L}{\partial(\partial^\mu\partial^\nu\phi)}\overleftrightarrow{\partial}^\lambda\partial_\nu\phi$$
$$+\frac{\partial L}{\partial(\partial^\mu\delta^\lambda\phi)}\delta^\lambda\partial_\nu\phi+\text{H.c.}-g_{\mu\nu}L\bigg). \qquad (4.18)$$

Again, the relation

$$\mathcal{P}_\mu\equiv\sum_{s,n}\int d^3k\ hk_\mu[n_{sn}(\mathbf{k})+\bar n_{sn}(\mathbf{k})]$$
$$=\int d^3x\ \mathcal{T}_{0\mu}(\mathbf{x},x_0) \qquad (4.19)$$

for the model Lagrangian (4.6) will be examined in the sequel to this paper.

Finally, for the infinitesimal homogeneous transformation, we have

$$\bar\delta L=(x_\mu\partial_\nu-x_\nu\partial_\mu+\eta_\mu\delta_\nu-\eta_\nu\delta_\mu)L\bar\delta\omega^{\mu\nu} \qquad (4.20a)$$

and

$$\bar{\delta}\phi=(x_\mu\partial_\nu-x_\nu\partial_\mu+\eta_\mu\delta_\nu-\eta_\nu\delta_\mu)\phi\bar{\delta}\omega^{\mu\nu}\,, \tag{4.20b}$$

so that substitution into (4.9) yields

$$\partial^\mu M_{\mu\nu\lambda}=0\,, \tag{4.21}$$

where

$$M_{\mu\nu\lambda}=\int d^4\eta\;\delta_+(\eta^2-1)\bigg(\frac{\partial L}{\partial(\partial^\mu\phi)}(x_\nu\partial_\lambda-x_\lambda\partial_\nu-\eta_\nu\delta_\lambda-\eta_\lambda\delta_\nu)\phi + \frac{\partial L}{\partial(\partial^\mu\delta^\rho\phi)}\overleftrightarrow{\partial}^\rho(x_\nu\partial_\lambda-x_\lambda\partial_\nu+\eta_\nu\delta_\lambda-\eta_\lambda\delta_\nu)\phi + \frac{\partial L}{\partial(\partial^\mu\delta^\rho\phi)}\delta^\rho(x_\nu\partial_\lambda-x_\lambda\partial_\nu+\eta_\nu\delta_\lambda-\eta_\lambda\delta_\nu)\phi+\text{H.c.} -(x_\nu g_{\mu\lambda}-x_\lambda g_{\mu\nu})L\bigg). \tag{4.22}$$

In a consistent model the generator of homogeneous Lorentz transformations $M_{\mu\nu}$ must, of course, be given by

$$M_{\mu\nu}\equiv\int d^3x\,M_{0\mu\nu}(\mathbf{x},x_0)\,. \tag{4.23}$$

5. EQUAL-HYPERPLANE COMMUTATION RELATIONS

The nonlocal character of the hyperplane-dependent field removes the preferred status of equal-time and generally spacelike commutators enjoyed in local field theory. On the other hand, a similar but not equivalent role is placed by equal-hyperplane commutation relations. Specificially, it will be shown here that under standard assumptions for the commutation relations between the momentum-space creation and annihilation operators, one obtains

$$\delta(\eta\cdot x-\eta\cdot x')[\phi(x',\eta),\phi(x,\eta)]=0\,, \tag{5.1a}$$

$$\delta(\eta\cdot x-\eta\cdot x')[\phi(x',\eta),\phi^\dagger(x,\eta)]=0\,. \tag{5.1b}$$

The *standard assumptions* are[21]

$$[a_{n'}{}^{\alpha_1\cdots\alpha_{s'}}(\mathbf{k}'),a_n{}^{\beta_1\cdots\beta_s\dagger}(\mathbf{k})]=\delta_{n'n}\delta_{s's}2k_0{}^{sn}\delta^3(\mathbf{k}-\mathbf{k}')\times G^{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}(k)\,, \tag{5.2a}$$

$$[b_{n'}{}^{\alpha_1\cdots\alpha_{s'}}(\mathbf{k}'),b_n{}^{\beta_1\cdots\beta_s\dagger}(\mathbf{k})]=\delta_{n'n}\delta_{s's}2k_0{}^{sn}\delta^3(\mathbf{k}-\mathbf{k}')\times G^{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}(k)\,, \tag{5.2b}$$

where all other commutators vanish and

$$G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(k)\equiv g_{\beta_1\gamma_1}\cdots g_{\beta_s\gamma_s}G^{\alpha_1\cdots\alpha_s;\gamma_1\cdots\gamma_s}(k) \tag{5.3}$$

is the unique $2s$-rank tensor function of k which is symmetric, traceless, and transverse in both its upper and lower indices and is also a projection operator, i.e.,

$$G_{\gamma_1\cdots\gamma_s}{}^{\alpha_1\cdots\alpha_s}(k)G_{\beta_1\cdots\beta_s}{}^{\gamma_1\cdots\gamma_s}(k)=G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(k)\,. \tag{5.4}$$

Examples of lower rank are

$$G_\beta{}^\alpha=g_\beta{}^\alpha-k^\alpha k_\beta/k^2 \tag{5.5a}$$

and

$$G_{\beta_1\beta_2}{}^{\alpha_1\alpha_2}=\tfrac{1}{2}\{G_{\beta_1}{}^{\alpha_1}G_{\beta_2}{}^{\alpha_2}+G_{\beta_2}{}^{\alpha_1}G_{\beta_1}{}^{\alpha_2}-\tfrac{2}{3}G^{\alpha_1;\alpha_2}G_{\beta_1;\beta_2}\}\,. \tag{5.5b}$$

Note that

$$G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(-k)=G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(k)\,. \tag{5.6}$$

From (5.2) the commutation relation (5.1a) is trivial since it is a combination of vanishing commutators. For (5.1b) one must employ (3.1), (3.8), and (3.18) to obtain

$$\delta(\eta\cdot x-\eta\cdot x')[\phi(x',\eta),\phi^\dagger(x,\eta)] =(2\pi)^{-4}\int d^4k'd^4k\;e^{-ik'x'}e^{ikx}\sum_{s'n'}\sum_{s,n}h_{\alpha_1}{}'\cdots h_{\alpha_{s'}}{}' \times\rho_{s'n'}(\eta\cdot k')[\tilde{\psi}_{n'}{}^{\alpha_1\cdots\alpha_{s'}}(k'),\tilde{\psi}_n{}^{\beta_1\cdots\beta_s\dagger}(k)] \times\rho_{sn}{}^*(\eta\cdot k)h_{\beta_1}\cdots h_{\beta_s}\delta(\eta x-\eta x')\,. \tag{5.7}$$

But (3.19) yields

$$\begin{aligned}[\tilde{\psi}_{n'}{}^{\alpha_1\cdots\alpha_{s'}}(k'),\tilde{\psi}_n{}^{\beta_1\cdots\beta_s\dagger}(k)] &=\delta(k'^2-\kappa_{s'n'}{}^2)\delta(k^2-\kappa_{sn}{}^2)[\theta(k_0')a_{n'}{}^{\alpha_1\cdots\alpha_{s'}}(\mathbf{k}') +\theta(-k_0')b_{n'}{}^{\alpha_1\cdots\alpha_{s'}\dagger}(-\mathbf{k}'),\,\theta(k_0)a_n{}^{\beta_1\cdots\beta_s\dagger}(\mathbf{k}) +\theta(-k_0)b_n{}^{\beta_1\cdots\beta_s}(-\mathbf{k})]\\ &=\delta(k'^2-\kappa_{sn}{}^2)\delta(k^2-\kappa_{sn}{}^2)\delta_{n'n}\delta_{s's}[\theta(k_0')\theta(k_0)2|k_0| \times\delta^3(\mathbf{k}-\mathbf{k}')-\theta(-k_0')\theta(-k_0)2|k_0|\delta^3(\mathbf{k}-\mathbf{k}')] G^{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}(k)\\ &=\delta_{n'n}\delta_{s's}\delta^4(k-k')\delta(k^2-\kappa_{sn}{}^2) \times G^{\alpha_1\cdots\alpha_s;\beta_1\cdots\beta_s}(k)\epsilon(k_0)\,,\end{aligned} \tag{5.8}$$

where

$$\epsilon(k_0)=\theta(k_0)-\theta(-k_0)\,. \tag{5.9}$$

Substituting (5.8) into (5.7) and performing the k' integration and the $s'n'$ sum yields

$$\delta(\eta x-\eta x')[\phi(x',\eta),\phi^\dagger(x,\eta)]=(2\pi)^{-4}\int d^4k\;e^{ik(x-x')} \times\sum_{s,n}h_{\alpha_1}\cdots h_{\alpha_s}G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(k)h^{\beta_1}\cdots h^{\beta_s} \times\delta(k^2-\kappa_{sn}{}^2)|\rho_{sn}(\eta\cdot k)|^2\epsilon(k_0)\delta(\eta\cdot x-\eta\cdot x')\,. \tag{5.10}$$

Now

$$h_{\alpha_1}\cdots h_{\alpha_s}G_{\beta_1\cdots\beta_s}{}^{\alpha_1\cdots\alpha_s}(k)h^{\beta_1}\cdots h^{\beta_s}\equiv f_s \tag{5.11}$$

is a scalar function of k and η and thus can depend only on $\eta\cdot k$ and k^2. In fact, from (3.11) we have

$$-2s(\eta\cdot k)f_s=(k\delta)f_s=[k^2-(\eta\cdot k)^2](\partial/\partial\eta\cdot k)f_s \tag{5.12}$$

[21] It is customary to state these commutation relations for, say, helicity eigenvector creation and annihilation operators. Clearly, a linear transformation of the form $a_{ns\lambda}(\mathbf{k})=C_\lambda{}^{\alpha_1\cdots\alpha_s}(k)a_{n;\alpha_1\cdots\alpha_s}(\mathbf{k})$ exists, and if properly normalized will yield the standard commutation relations for the helicity operators.

or

$$f_s \propto [(\eta\cdot k)^2 - k^2]^s. \tag{5.13}$$

In fact,

$$f_s = \left(1 - \frac{(\eta\cdot k)^2}{k^2}\right)^s = (h^2)^s. \tag{5.14}$$

Coupling this with

$$\epsilon(k_0) = \epsilon(\eta\cdot k) \tag{5.15}$$

yields

$$\int d\eta\cdot k \sum_{s,n} (h^2)^s |\rho_{sn}(\eta\cdot k)|^2 \delta(k^2 - \kappa_{sn}{}^2)\epsilon(\eta\cdot k) = 0, \tag{5.16}$$

and Eq. (5.1b) follows.

Clearly, the result is dependent upon the equal weighting of particles and antiparticles in the decomposition of the hyperplane field.[22]

6. NORMALIZATION AND SPACELIKE SOLUTIONS

The Lagrangian density (4.1) must be bilinear in ϕ and $\phi^\dagger$ if the action principle is to yield a linear field equation. The fields, at the same time, are linear superpositions of the quantities,

$$h_{\alpha_1} \cdots h_{\alpha_s} \rho_{sn}(\eta\cdot k), \tag{6.1}$$

as far as their η dependence is concerned. Consequently, the action integral in (4.2) and the currents (4.13), (4.18), and (4.22) are linear superpositions of the bilinear integrals

$$\int d^4\eta\, \delta(\eta^2 - 1) h_{\alpha_1}{}' \cdots h_{\alpha_{s'}}{}' \rho_{s'n'}{}^*(\eta\cdot k') \times \rho_{sn}(\eta\cdot k) h_{\beta_1} \cdots h_{\beta_s}, \tag{6.2}$$

where a prime on an h indicates dependence on k' rather than k. If these integrals do not converge, then the existence of the action integral and the currents, even at the free-field level, is very doubtful and is bound up with integrals and infinite series. The convergence of (6.2) does not guarantee the existence of the action integral and the currents but it does remove an obvious obstacle to their existence.

Now, the convergence of (6.2) for all k, k', s, s', n, and n' implies the convergence of the quadratic integral

$$\int d^4\eta\, \delta(\eta^2 - 1)(h^2)^s |\rho_{sn}(\eta\cdot k)|^2. \tag{6.3}$$

[22] S. Weinberg, Phys. Rev. 133, B1318 (1964).

The converse implication is obscured by the nondefinite character of the Minkowski metric but can in fact be derived. Thus the convergence of (6.3) will be taken as the normalization condition on the ρ functions.

For timelike k the integral (6.3) is reduced to

$$(-1)^s \pi \int_1^\infty \frac{dy}{[y(y-1)]^{1/2}} (y-1)^{s+1} |\rho_{sn}((yk^2)^{1/2})|^2, \tag{6.4}$$

where

$$y \equiv (\eta\cdot k)^2 / k^2. \tag{6.5}$$

Convergence at the upper limit is assured if ρ_{sn} vanishes faster than

$$\rho_{sn} \underset{y\to\infty}{\lesssim} \mathcal{O}(y^{-\frac{1}{2}s - \frac{1}{2} - \epsilon}), \quad \epsilon > 0. \tag{6.6a}$$

Convergence at the lower limit requires

$$\rho_{sn} \underset{y\to 1+}{\lesssim} \mathcal{O}((y-1)^{-\frac{1}{2}s - \frac{3}{4} + \epsilon}), \quad \epsilon > 0. \tag{6.6b}$$

These are the boundary conditions to be imposed on the ρ functions that will be studied in specific models.

In obtaining the form (6.4) for the invariant integral (6.3), a transformation to the rest frame of the timelike vector k was employed. For spacelike k, a transformation to the frame in which

$$k_0 = 0, \quad \mathbf{k} = \hat{e}_3 k_3 \tag{6.7}$$

yields the form

$$\int_{-\infty}^{\infty} d\eta_1 \eta_2 \int_{-\infty}^{0} dy \frac{(1-y)^s}{2[y(y - \eta_1{}^2 - \eta_2{}^2 + 1)]^{1/2}} \times |\rho_{sn}((yk^2)^{1/2})|^2 \tag{6.8}$$

for (6.3). This integral is always at least *linearly divergent* in the integration over (η_1, η_2). Hence *scalar hyperplane-dependent field theory can have no spacelike solutions with normalizable* ρ *functions.*

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