PHYSICAL REVIEW D VOLUME 1, NUMBER 2 15 JANUARY 1970

High-Energy Limit of Production Amplitudes*

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(Received 14 July 1969)

A simplified version of the Bali-Chew-Pignotti amplitude for multi-Regge-pole particle production is given using the recently developed projection-operator technique to construct many-particle states. This amplitude is generalized to a multi-Toller-pole expansion to avoid singularities which otherwise occur when any momentum-transfer variable goes to zero. It is shown how to determine the M value of a Toller pole by measuring the distribution as a function of a Treiman-Yang angle.

1. INTRODUCTION

THE formal theory of the multi-Regge model of particle production has been given by Bali, Chew, and Pignotti[1] as a development of earlier work by Toller.[2] In Sec. 2[3] we give a somewhat simplified rederivation of their result, using the projection-operator technique for constructing many-particle states that has recently been developed by the authors.[4]

A formal defect of the multi-Regge expansion is that it is intended for use as an approximation to the particle-production amplitude in the limit of high subenergies and small momentum transfer. However, in the usual physical situation, the limit of the amplitude when the momentum transfer in any part of the multi-Regge diagram tends to zero is nonuniform, just as in the two-particle scattering case (see Freedman and Wang[5]). This difficulty is removed in Sec. 4, by the techniques previously applied to two-body scattering (see Ref. 4 for further references), in which we go over to the multi-Toller expansion. The expression is given in terms of $O(3,1)$ functions with arguments which are angles between four vectors. The required limit is then well defined.

We discover that if a subprocess is dominated by a Toller pole with quantum numbers M and σ, the value of M can be determined by measuring the distribution in an azimuth angle ϕ in a suitably chosen kinematic region of the multiparticle process. This angle is the Treiman-Yang angle, or is related to it by crossing.

For simplicity the argument is given in the first place for zero-spin particles, but the generalization to arbitrary spin is straightforward and is outlined in the final section. We also make there the trivial extension from the production of particles to the production of particle clusters.

* Research sponsored in part by the Air Force Office of Scientific Research (OAR) through the European Office of Aerospace Research, U. S. Air Force.

† On leave of absence from Johns Hopkins University, Baltimore, Md. Supported in part by the National Science Foundation.

[1] N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. **163**, 1572 (1967).

[2] M. Toller, Nuovo Cimento **37**, 631 (1965).

[3] Most of this section is a brief resume of Sec. 2 and 3 of Ref. 4, to which we refer the reader for further details.

[4] G. Feldman and P. T. Matthews, Ann. Phys. (N. Y.) **55**, 506 (1969). Referred to hereafter as I.

[5] D. Z. Freedman and Jiunn-Ming Wang, Phys. Rev. **153**, 1596 (1967).

2. MULTIPARTICLE STATES

A relativistic single-particle state is conventionally given as

$$|p,j,\lambda\rangle, \tag{2.1}$$

in which the labels specify the four-momentum (with $p^2=m^2$), spin, and helicity, respectively. The spin label j is defined so that

$$W^2(p)|p,j,\lambda\rangle=-m^2j(j+1)|p,j,\lambda\rangle, \tag{2.2}$$

where

$$W_\mu(p)=-\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}J^{\nu\lambda}p^\rho, \tag{2.3}$$

and λ is the eigenvalue of some component of $W_\mu(p)$. Under a general Lorentz transformation parametrized by η, which transforms p into p', the spin is transformed by a (Wigner) rotation. Thus

$$\begin{aligned}\exp[-\tfrac{1}{2}i\eta^{\mu\nu}J_{\mu\nu}]|p,j,\lambda\rangle&\equiv U(\eta)|p,j,\lambda\rangle\\&=\sum_{\lambda'}|p',j,\lambda'\rangle d_{\lambda'\lambda}{}^j[\theta(p',\eta,p)].\end{aligned} \tag{2.4}$$

An alternative construction has recently been proposed by the authors[4] in terms of the projection operators $O(p)$, defined by the relation

$$[J_{\mu\nu},O(p)]=i(p_\mu\partial_\nu-p_\nu\partial_\mu)O(p), \tag{2.5}$$

where

$$\partial_\mu=\partial/\partial p^\mu \tag{2.6}$$

and

$$O(p)O(q)=O(q)O(p)=O(p)\delta^4(p-q). \tag{2.7}$$

From (2.5) it follows that

$$U(\eta)O(p)U^{-1}(\eta)=O(p'), \tag{2.8}$$

and from (2.7) that

$$\int O(p)d^4p=1. \tag{2.9}$$

If we now define

$$P_\mu\equiv\int p_\mu O(p)d^4p, \tag{2.10}$$

it is easy to show that P_μ has the required commutation relations with $J_{\mu\nu}$ to be interpreted as the energy-momentum operator.[6]

[6] Formally these equations are satisfied by $O(p)\equiv\delta^4(P-p)$. It sometimes avoids ambiguity to write $O(p)\equiv O(P,p)$.

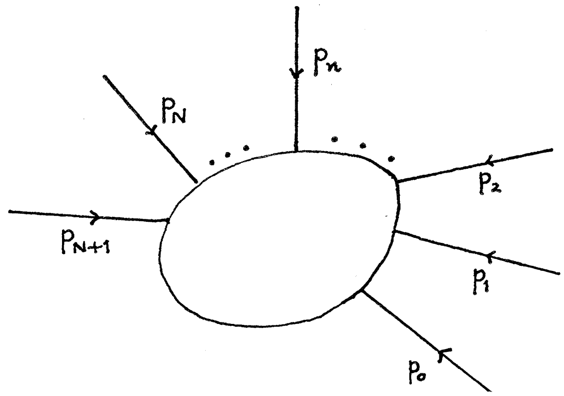

FIG. 1. The $(N+2)$-particle amplitude.

We now define representations of the homogeneous Lorentz group

$$|M,\sigma;\, j,\lambda\rangle, \tag{2.11}$$

where the labels and states are defined in (A21)–(A24). Then, single-particle states[7] may alternatively be defined as

$$|p,(M,\sigma),j,\lambda\rangle \equiv O(p)U(p)|M,\sigma;\, j,\lambda\rangle, \tag{2.12}$$

where $U(p)$ is a boost which takes the four-vector to the value p_μ from its rest value $m_\mu=(m,\mathbf{0})$ in some particular frame; if

$$U(p)=e^{-iJ_2\theta}e^{-iK_3\epsilon}, \tag{2.13}$$

$$p_\mu=p(\cosh\epsilon,\, \sinh\epsilon\,\sin\theta, 0,\, \sinh\epsilon\,\cos\theta). \tag{2.14}$$

It is easy to show that (2.12) is an eigenstate of the same operators as (2.1), and transforms in the same way [see (2.4)] under Lorentz transformations.

The N-particle state is simply an outer product of single-particle states. Thus (with σ_n standing for the pair M_n, σ_n)

$$\begin{aligned}|p_1,(\sigma_1),j_1,\lambda_1;\, p_2,(\sigma_2),j_2,\lambda_2;\, \ldots;\, p_N,(\sigma_N),j_N,\lambda_N\rangle \\ =O(p_1)O(p_2)\cdots O(p_N) \\ \times U^{[1]}(p_1)U^{[2]}(p_2)\cdots U^{[N]}(p_N) \\ \times|\sigma_1,j_1,\lambda_1;\, \sigma_2,j_2,\lambda_2;\, \ldots;\, \sigma_N,j_N,\lambda_N\rangle,\end{aligned} \tag{2.15}$$

where

$$O(p_n)\equiv O(P_n,p_n), \tag{2.16}$$

and $U^{[n]}(p_n)$ are constructed from the operators[8] $P_n{}^\mu$ and $J_{\mu\nu}{}^{[n]}$ which operate on the nth particle only (and commute with the corresponding operators for the other particles).

[7] In this paper, we shall not discuss the labelling of intrinsic parity (which requires reducible representations M, σ) but refer the reader to I.

[8] A more consistent notation might be to denote the single-particle momentum operators by $P_{[n]}{}^\mu$, but we find in practice that this notation is unnecessarily clumsy. However, for the $O(3,1)$ generators, we must distinguish between operators on particular particles, $J_{\mu\nu}{}^{[n]}$, and operators specified in particular frames $J_{\mu\nu}{}^{(n)}$.

If all the particles have zero spin, the factors $U^{[n]}$ reduce to unity, and

$$M_n=j_n=\lambda_n=0,\qquad \sigma_n=1 \quad (\text{for all } n), \tag{2.17}$$

which is the trivial representation of the homogeneous Lorentz group for all particles. We denote such a state by $|0\rangle$. Thus the state of N spinless particles is

$$|p_1,p_2,\ldots,p_N\rangle=O(p_1)O(p_2)\cdots O(p_N)|0\rangle. \tag{2.18}$$

3. MULTI-REGGE MODEL

Consider the $N+2$ spinless particle amplitude, illustrated in Fig. 1. If we allow the time components of the four-vectors $p_n{}^\mu$ to take on positive or negative values, this figure has many physical interpretations (provided the masses are such as to allow for energy-momentum conservation). Three processes which interest us particularly are:

(a) the decay of p_{N+1} into $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_N$;
(b) the multiple production process

$$p_0+p_{N+1}\to\bar{p}_1+\bar{p}_2+\cdots+\bar{p}_N; \tag{3.1}$$

and

(c) the multiple production process

$$p_1+p_2\to\bar{p}_0+\bar{p}_3+\cdots+\bar{p}_{N+1},$$

where

$$\bar{p}_n=-p_n. \tag{3.2}$$

The processes (b) and (c) differ only in their ordering, which is defined below and in Fig. 2. In addition to its dependence on the $N+2$ particle masses, the amplitude may be described in terms of

$$V\equiv3(N+2)-10=3N-4 \tag{3.3}$$

independent scalar variables. For the processes (a)–(c), it is convenient to define

$$k_n=p_n+k_{n-1},\quad n=0,\ldots,N \tag{3.4}$$

where

$$k_0=p_0,$$

and to take the independent scalar variables to be

$$t_n=k_n{}^2\quad (n=1,2,\ldots,N-1), \tag{3.5}$$

$$s_n=(p_n+p_{n+1})^2\quad (n=1,2,\ldots,N-1), \tag{3.6}$$

and

$$S_n=(p_n+p_{n+1}+p_{n+2})^2\quad (n=1,2,\ldots,N-2). \tag{3.7}$$

There are, respectively, $(N-1)$, $(N-1)$, and $(N-2)$ such variables, which form a possible complete set. The significance of these variables is apparent from Fig. 2. In the decay process (a), the variables t_n are the masses of the intermediate particles if the $(N+1)$-fold decay takes place as a cascade of two-particle decays. In the multiproduction process (b), the t's and s's are generalized momentum-transfer and energy variables,

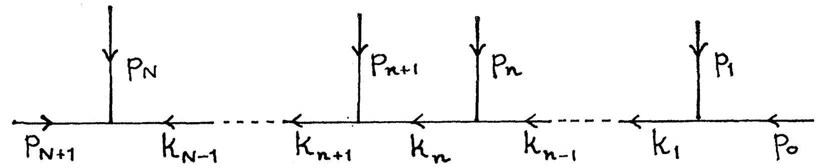

FIG. 2. The configuration of the amplitude of Fig. 1 which leads to the chosen sets of scalar variables t_n, s_n, and S_n or t_n, θ_n, and ϕ_n.

respectively. We will be interested in (b) and (c), particularly in the limit of $s_n \to \infty$, $t_n \to 0$ for some or all of the (s_n, t_n), and will obtain them by analytic continuation from the amplitude for process (a). The required amplitude[9] is

$$\langle \bar{p}_{N+1} | T | p_N, \ldots, p_n, \ldots, p_2, p_1, p_0 \rangle \equiv \langle T \rangle = \langle 0 | O(\bar{p}_{N+1}) T O(p_N) \cdots O(p_n) \cdots O(p_2) O(p_1) O(p_0) | 0 \rangle , \quad (3.8)$$

where the $O(p_n)$ are defined in (2.15) and (2.16). Define operators[10] K_n analogously to (3.4),

$$K_n = P_n + K_{n-1} , \quad (3.9)$$

and the operator

$$Q_n = a_n K_n + b_n P_n , \quad (3.10)$$

where a_n and b_n ($\neq 0$) are constants chosen so that the variable

$$q_n = a_n k_n + b_n p_n \quad (n = 1 \cdots N) , \quad (3.11)$$

is positive timelike. It is important to note that for arbitrary a_n and b_n, q_n^2 is a function only of the "masses" at the nth vertex, t_n, t_{n-1} and p_n^2. Now by (2.16) and (3.9)–(3.11)

$$O(p_n) O(k_{n-1}) = O(q_n) O(k_n) \quad (n = 1, 2, \ldots, N) . \quad (3.12)$$

By repeated use of this relation in (3.8),

$$\langle T \rangle = \langle 0 | O(P, \bar{p}_{N+1}) T O(P, k_N) O(q_N) \cdots O(q_n) \cdots O(q_1) | 0 \rangle , \quad (3.13)$$

where we have used the fact that for the initial state

$$P | 0 \rangle = | P_0 + P_1 + \cdots + P_N | 0 \rangle \equiv K_N | 0 \rangle , \quad (3.14)$$

and for the final state

$$\langle 0 | P = \langle 0 | P_{N+1} . \quad (3.15)$$

Now since

$$[T, P] = 0 , \quad (3.16)$$

$$O(P, \bar{p}_{N+1}) T O(P, k_N) = O(P, \bar{p}_{N+1}) T \delta(k_N - \bar{p}_{N+1}) . \quad (3.17)$$

Thus,

$$\langle T \rangle = \langle 0 | O(P, \bar{p}_{N+1}) T O(q_N) \cdots O(q_n) \cdots O(q_1) | 0 \rangle \times \delta\Big(\sum_{n=0}^{N+1} p_n\Big) . \quad (3.18)$$

In the subsequent equations we omit the δ function contained in $\langle T \rangle$.

To display the dependence of the amplitude $\langle T \rangle$ on the variables (3.5)–(3.7), consider Fig. 2 as a succession of pseudo "scattering" processes of which the nth typical example is

$$k_{n-1} + p_n \to k_{n+1} + \bar{p}_{n+1} . \quad (3.19)$$

Define the corresponding nth c.m. scattering frame to have its time axis ν_n in the direction of k_n (assumed positive timelike), z axis γ_n along the "incoming beam" k_{n-1}, and y axis β_n perpendicular to the "scattering plane." The $(n+1)$th frame is related to the nth frame by the boost $B_n(\chi_n)$. [See (A12)–(A15) for details.]

We also introduce the boost of q_n in the nth frame,[11] namely,

$$U_{(n)}(q_n) \equiv e^{-iK_3^{(n)} \epsilon(k_n, q_n)} , \quad (3.20)$$

where by (2.13)

$$\cosh \epsilon(k_n, q_n) = \hat{k}_n \cdot \hat{q}_n . \quad (3.21)$$

This boost has the property that

$$O(q_n) = U_{(n)}(q_n) O(q_n^{(r)}) U_{(n)}^{-1}(q_n) , \quad (3.22)$$

where[11]

$$q_n^{(r)} \equiv [(q_n^2)^{1/2}, \mathbf{0}] , \quad (3.23)$$

in the nth frame.

Now consider two typical operators in (3.18). Using (3.22),

$$\begin{aligned} O(q_{n+1}) O(q_n) &= U_{(n+1)}(q_{n+1}) O(q_{n+1}^{(r)}) U_{(n+1)}^{-1}(q_{n+1}) \times U_{(n)}(q_n) O(q_n^{(r)}) U_{(n)}^{-1}(q_n) \\ &= U_{(n+1)}(q_{n+1}) O(q_{n+1}^{(r)}) B_n(\chi_n) U_{(n)}^{-1}(q_{n+1}) \times B_n^{-1}(\chi_n) U_n(q_n) O(q_n^{(r)}) B_{n-1}(\chi_{n-1}) \\ &\quad \times U_{(n-1)}^{-1}(q_n) B_{n-1}^{-1}(\chi_{n-1}) , \end{aligned} \quad (3.24)$$

where in writing (3.24) we have used (A20).

Between the factors $U_{(n)}(q_n)$ and $O(q_n^{(r)})$ one can insert a complete set of states in the nth frame based on (2.11) and the eigenvalues ω of the other operators necessary to complete the set:

$$U_{(n)}(q_n) O(q_n^{(r)}) = \sum U_{(n)}(q_n) | \omega, M, \sigma, j, \lambda \rangle_{(n)} \times {}_{(n)}\langle \omega, M, \sigma, j, \lambda | O(q_n^{(r)}) , \quad (3.25)$$

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[9] In order to obtain the formal expansions given below, we could equally have started with the amplitude for the process $p_0 + p_1 + \cdots + p_n \to \bar{p}_{n+1} + \bar{p}_{n+2} + \cdots \bar{p}_{N+1}$.

[10] The reader should not confuse these momentum operators K_n with Lorentz transformation generators $K^{(n)}$ defined in the Appendix.

[11] The operator $K_3^{(n)}$ is defined in (A10). From the definitions of q_n, (3.11), and the nth frame, q_n has only a "0" and "3" component. That is, $q_n^\mu = \nu_n^\mu (q_n \cdot \nu_n) + \gamma_n^\mu (q_n \cdot \gamma_n)$. If $(q_n \cdot \gamma_n)$ is negative (i.e., if the three-vector q_n points in the negative "3" direction), (3.20) must be multiplied by an additional factor $\exp(-i\pi J_2^{(n)})$. We shall, for simplicity, omit writing this extra factor. Note also that the rest vector [Eq. (3.22)] is $q_n^{(r)\mu} = \nu_n^\mu \times (q_n^2)^{1/2}$.

where the summation is over the labels appearing in the states and the states are defined[12] in (A21)–(A24).

Similarly, between the factors $B_n(\chi_n)$ and $U_{(n)}^{-1}(q_{n+1})$ one can insert a complete set of states in the nth frame:

$$\begin{aligned} B_n(\chi_n)U_{(n)}^{-1}(q_{n+1}) &= \sum B_n(\chi_n)\,|\omega,M,\sigma,j,\lambda\rangle_{(n)} \\ &\qquad\times{}_{(n)}\langle\omega,M,\sigma,j,\lambda|\,U_{(n)}^{-1}(q_{n+1}) \\ &= \sum |\omega,M,\sigma,j,\lambda\rangle_{(n+1)} \\ &\qquad\times{}_{(n)}\langle\omega,M,\sigma,j,\lambda|\,U_{(n)}^{-1}(q_{n+1})\,, \end{aligned} \tag{3.26}$$

where we have used (A25). By making such insertions in (3.18), it can be seen that $\langle T\rangle$ is made up of a product of two types of factor, one of which is the form factor of the nth vertex,

$$_{(n)}\langle\omega',M',\sigma',j,\lambda|\,O(q_n{}^{(r)})\,|\omega,M,\sigma,j,\lambda\rangle_{(n)} \equiv G_{(n)}{}^{\omega',M',\sigma',\omega,M,\sigma,j,\lambda}. \tag{3.27}$$

From the remark following (3.11) and the definition (3.23), it is clear that this is a function only of the vertex variables:

$$G_{(n)}=G_{(n)}(t_n,t_{n-1},p_n{}^2)\,. \tag{3.28}$$

The other basic factor[13] is

$$\begin{aligned} &_{(n)}\langle\omega',M',\sigma',j',\lambda'|\,U_{(n)}^{-1}(q_{n+1})B_n^{-1}(\chi_n) \\ &\qquad\times U_{(n)}(q_n)\,|\omega,M,\sigma,j,\lambda\rangle_{(n)} \\ &= {}_{(n)}\langle\omega',M',\sigma',j',\lambda'|\,e^{iJ_3{}^{(n)}\phi_n}e^{iK_3{}^{(n)}\epsilon(k_n,q_{n+1})} \\ &\qquad\times e^{iJ_2{}^{(n)}\theta(q_{n+1},q_n;k_n)}e^{-iK_3{}^{(n)}\epsilon(q_n,k_n)}\,|\omega,M,\sigma,j,\lambda\rangle_{(n)} \\ &\equiv D_{M,\sigma,j',\lambda',j,\lambda}(\psi_n)\delta_{MM'}\delta_{\sigma\sigma'}\delta_{\omega\omega'}\,, \end{aligned} \tag{3.29}$$

where the only dependence[14] on n is through the angles

$$\psi_n=(\phi_n,\epsilon(q_{n+1},k_n),\theta_n,\epsilon(q_n,k_n))\,, \tag{3.30}$$

which are defined in (3.29). [For the notation see (A18) and (A28).]

Some care must be taken of end effects, which, however, are quite simple. Thus

$$\begin{aligned} O(q_1)|0\rangle &= U_{(1)}(q_1)O(q_1{}^{(r)})U_{(1)}^{-1}(q_1)|0\rangle \\ &= \sum_{\omega,\sigma} U_{(1)}(q_1)\,|\omega,0,\sigma,0,0\rangle_{(1)} \\ &\qquad\times{}_{(1)}\langle\omega,0,\sigma,0,0|\,O(q_1{}^{(r)})\,|0\rangle\,, \end{aligned} \tag{3.31}$$

where we have used the relations

$$U_{(1)}^{-1}(q_1)|0\rangle=|0\rangle\,, \tag{3.32}$$

where $|0\rangle$ is the state with

$$M=j=\lambda=0\,,\quad \sigma=1\,, \tag{3.33}$$

and that

$$[(\mathbf{J}^{(1)})^2,O(q_1{}^{(r)})]=[J_3{}^{(1)},O(q_1{}^{(r)})]=0\,.$$

The final term in (3.31) is $G_{(1)}$, and $U_{(1)}(q_1)$ is a required factor in $D(\psi_1)$.

At the other end,

$$\begin{aligned} &\langle 0|O(P,\bar{p}_{N+1})TO(q_N) \\ &= \sum\langle 0|O(P,\bar{p}_{N+1})TU_{(N)}(q_N)\,|\omega,0,\sigma,j,0\rangle_{(N)} \\ &\qquad\times{}_{(N)}\langle\omega,0,\sigma,j,0|\,O(q_N{}^{(r)})\,|\omega',M',\sigma',j,0\rangle_{(N)} \\ &\qquad\times{}_{(N)}\langle\omega',M',\sigma',j,0|\,U_{(N)}^{-1}(q_N)\,, \end{aligned} \tag{3.34}$$

where the sum is over ω, σ, ω', M', σ', and j. The values of the labels entered in the states follow since, first, $\bar{p}_{N+1}=k_N$, second, T is an $O(3)$ scalar, and, finally, $U_{(N)}(q_N)$, $O(q_N{}^{(r)})$, $O(P,\bar{p}_{N+1})$ commute with $J_3{}^{(N)}$. The matrix element of $O(q_N{}^{(r)})$ is just $G_{(N)}$ and $U_{(N)}^{-1}(q_N)$ is a required factor in $D(\psi_{N-1})$. The first matrix element in (3.34) refers only to the Nth vertex. We define

$$\langle 0|O(P,\bar{p}_{N+1})TU_{(N)}(q_N)\,|\omega,0,\sigma,j,0\rangle_{(N)} \equiv F^{\omega,\sigma,j}(p_{N+1}{}^2,p_N{}^2,t_N)\,. \tag{3.35}$$

Thus finally,

$$\langle T\rangle=FG_{(N)}D(\psi_{N-1})G_{(N-1)}\cdots G_{(2)}D(\psi_1)G_{(1)}\,, \tag{3.36}$$

which is to be understood as a matrix product in the space labelled by ω, M, σ, j, λ.

The factors F and G in (3.36) depend only on vertex variables. Note that since in ψ_n the variables $\epsilon(k_n,q_{n+1})$ and $\epsilon(q_n,k_n)$ also refer only to the $(n+1)$th and nth vertices, respectively, the entire s_n, S_n dependence of the amplitude is contained through θ_n and ϕ_n. We may, in fact, regard t_n, θ_n, and ϕ_n as a complete set of scalar variables alternative to (3.5)–(3.7). To obtain an alternative form of the amplitude, we can collect together all the t_n dependences by combining the pure boost factors in $D(\psi_n)$ with the G factors. Thus we define a new vertex factor

$$\begin{aligned} &g_{(n)}{}^{\omega',M',\sigma',\omega,M,\sigma,\bar{j}',\bar{j},\lambda} \\ &= \sum_{\bar{j}}\langle M',\sigma',j',\lambda|\,e^{-iK_3\epsilon(q_n,k_n)}\,|M',\sigma',\bar{j},\lambda\rangle \\ &\qquad\times G_{(n)}{}^{\omega',M',\sigma',\omega,M,\sigma,\bar{j},\lambda} \\ &\qquad\times\langle M,\sigma,\bar{j},\lambda|\,e^{iK_3\epsilon(k_{n-1},q_n)}\,|M,\sigma,j,\lambda\rangle\,, \end{aligned} \tag{3.37}$$

which also depends only on the scalar variables of the nth vertex, t_n, t_{n-1}, and $p_n{}^2$. Similarly, we define

$$d_{j,\lambda',\lambda}(\theta_n,\phi_n)\equiv\langle j,\lambda'|\,e^{iJ_3\phi_n}e^{iJ_2\theta_n}\,|j,\lambda\rangle = e^{i\lambda'\phi_n}d_{\lambda'\lambda}{}^j(\theta_n)\,, \tag{3.38}$$

which contains all the dependence of the amplitude on s_n and S_n. The $d_{\lambda'\lambda}{}^j$ are the usual $O(3)$ functions. The

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[12] The boosts $U_{(n)}(q_n)$ do not affect the degeneracy label ω and the states defined in the Appendix are labelled only by the homogeneous Lorentz-group parameters.

[13] In deriving the right-hand side of Eq. (3.29), we have used (A29) which gives $\exp[iK_3{}^{(n)}\epsilon(k_{n+1},q_{n+1})]\exp[iK_3{}^{(n)}\epsilon(k_n,k_{n+1})] = \exp[iK_3{}^{(n)}\epsilon(k_n,q_{n+1})]$.

[14] This is so since matrix elements of the operators $J^{(n)}$, $K^{(n)}$ in the nth frame are independent of n and depend only on the labels in the states. From now on, when no confusion arises, we shall drop the labels n on states and operators.

factor $U_{(N)}(q_N)$ in F [see (3.35)] is absorbed in $g_{(N)}$ and the remaining factors are defined to be f. Then

$$\langle T\rangle = f g_{(N)} d(\theta_{N-1},\phi_{N-1}) g_{(N-1)} \cdots g_{(2)} d(\theta_1,\phi_1) g_{(1)}\,, \quad (3.39)$$

which is again to be understood as a matrix product.[15]

If some or all of the summations over the j's that are implied in (3.39) are replaced by integrals of the Sommerfeld-Watson transform, this expansion over $O(3)$ functions [in the decay region-process (a) of (3.1)] can be continued into regions of multiparticle production processes such as (b) and (c) of (3.1). The expansion then would be over $O(2,1)$ functions and we would pick up the Regge poles in the vertex factors $g_{(n)}$ in the standard way. To obtain the multi-Regge model for process (b), we would perform the Watson-Sommerfeld transform in the j's canonical to all of the θ_n.

From (A13) we have, for large s_n, s_{n+1}, S_n, and finite t_n,

$$\cos\phi_n \sim a + bS_n/s_n s_{n+1}\,, \quad (3.40)$$

and (for small but nonzero t_n),

$$d_{\lambda'\lambda}{}^j(\theta_n) \sim s_n{}^j\,. \quad (3.41)$$

If the high-energy limit for process (b) is taken so that $\cos\phi_n$ is kept constant, and it is assumed that the integrals over j are each dominated by a single Regge trajectory $\alpha(t)$ with factorized residues, the high-energy behavior of the amplitude is of the form

$$\langle T\rangle \sim \prod_{n=1}^{N} \Psi(\phi_n,t_n,t_{n-1},p_n{}^2) s_n{}^{\alpha_n(t_n)}\,. \quad (3.42)$$

This is in agreement with the result of Bali, Chew, and Pignotti.[1]

The analytic continuation into the region of process (c) will be useful when s_1 is large but the other s_n not necessarily so. Since here the variable t_2 is timelike, the process we are describing is the high-energy production of a "k_2" resonance

$$p_1 + p_2 \to \bar{p}_0 + k_2\,, \quad (3.43)$$

followed by the cascading decay of "k_2."

4. MULTI-TOLLER MODEL

The above expansion has been applied to particle production in the limit

$$s_n \to \infty\,, \quad t_n \to 0\,. \quad (4.1)$$

However, this limit of $\cos\theta_n$ is not uniform, since it depends on (st) and the boosts in $D(\psi_n)$ [see (3.29)] that have been combined with $G_{(n)}$ to form $g_{(n)}$ are singular. This is the well-known difficulty pointed out by Freedman and Wang[5] for two-body scattering. It may be solved in a similar manner by returning to the expansion (3.36).

According to (A27), we can write (3.29) as

$$\begin{aligned} D_{M,\sigma,j',\lambda',j,\lambda}(\psi_n) &= \langle M,\sigma,j',\lambda'|\exp(iJ_3\phi_n)\exp(-iJ_2\theta_n{}^L) \\ &\quad\times \exp(iK_3\epsilon_n{}^C)\exp(iJ_2\theta_n{}^R)|M,\sigma,j,\lambda\rangle \quad (4.2)\\ &\equiv \sum_m e^{i\lambda'\phi_n} d_{\lambda' m}{}^{j'}(\theta_n{}^L) d_{j'mj}{}^{M\sigma}(\epsilon_n{}^C) d_{m\lambda}{}^j(\theta_n{}^R)\,, \quad (4.3)\end{aligned}$$

where [see (A18)]

$$\theta_n{}^L = \theta(k_n,q_n;q_{n+1})\,, \quad (4.4)$$

$$\theta_n{}^R = \theta(k_n,q_{n+1};q_n)\,, \quad (4.5)$$

and

$$\epsilon_n{}^C = \epsilon(q_n,q_{n+1})\,. \quad (4.6)$$

Note that

$$|M| \leqslant \min(j',j)\,, \quad (4.7)$$

and

$$|m| \leqslant \min(j',j)\,. \quad (4.8)$$

In the limit $s_n \to \infty$, $t_n \to 0$, with t_{n-1}, t_{n+1} held finite and fixed,

$$\theta_n{}^L,\quad \theta_n{}^R \sim t_n{}^{1/2}\,, \quad (4.9)$$

$$\cosh\epsilon_n{}^C \sim s_n\,, \quad (4.10)$$

so that[16]

$$D(\psi_n)_{M,\sigma,j',\lambda',j,\lambda} \sim \sum_m \exp(i\lambda'\phi_n) t_n{}^{\frac{1}{2}|\lambda'\mp m|} \times s_n{}^{\sigma-1-|m-M|} t_n{}^{\frac{1}{2}|\lambda\mp m|}\,, \quad (4.11)$$

which is nonsingular at $t=0$. The leading behavior for large s_n is given by the term $m=M$, which gives as the leading term in this limit

$$D_{M,\sigma,j',\lambda',j,\lambda}(\psi_n) \sim \exp(i\lambda'\phi_n) t_n{}^{\frac{1}{2}|\lambda'\mp M|} t_n{}^{\frac{1}{2}|\lambda\mp M|} s_n{}^{\sigma-1}\,. \quad (4.12)$$

Performing all summations implicit in (3.36) except over those labels on which $D(\psi_n)$ depends, we can write (3.36) as

$$\langle T\rangle = \sum_{M,\sigma,j',\lambda',j,\lambda} T^{M,\sigma,j',\lambda',j,\lambda} D_{M,\sigma,j',\lambda',j,\lambda}(\psi_n)\,, \quad (4.13)$$

where the entire s_n and ϕ_n dependence[17] is contained in the explicit functions $D(\psi_n)$, and the coefficients T depend on all the other scalar variables. We shall be interested in using (4.13) in regions in which s_n is an energy and t_n is a momentum transfer. There are many processes for which this is true [see, e.g., (4.16) and

[15] Multi-$O(3)$ expansions of the type given by (3.39) have been obtained previously by many authors. See, e.g., A. J. Macfarlane, Rev. Mod. Phys. 34, 41 (1962); and W. H. Klink and G. J. Smith, II, Phys. Rev. 175, 2010 (1968).

[16] For the behavior of the variables $\theta_n{}^L$, $\theta_n{}^R$, and $\epsilon_n{}^C$ for large s_n and small t_n, see (A33), (A34), and (A36). For the behavior of the $O(3,1)$ and $O(3)$ d functions in this region, see, e.g., Appendix C of I or K. M. Bitar and G. L. Tindle, Phys. Rev. 175, 1835 (1968). The $\mp$ sign in (4.11) depends on whether $\cos\theta^L$ ($\cos\theta^R$) $\to \pm 1$ in this limit. This in turn depends on the size of the masses $p_n{}^2$, $p_{n+1}{}^2$ relative to t_{n+1}. The subsequent discussion does not depend on this sign.

[17] We are here choosing the independent variables to be the set t_i, s_i (or θ_i), and ϕ_i.

(4.17) below]. With this in mind, we make the assumption that, after performing a suitable Watson-Sommerfeld transformation in the σ plane, we obtain a representation for the amplitude describing processes such as (b) and (c) of (3.1). We further assume that $T^{M,\sigma,j',\lambda',j,\lambda}$ is dominated by a Toller pole with quantum numbers M and $\sigma(t_n)$ and that it is not singular for t_n small. It follows using (4.12) that the leading terms for $s_n\to\infty$ and t_n small coming from the sums over λ and λ' are those for which

$$\lambda=\pm M,\quad \lambda'=\pm M.$$

Substituting into (4.13), this determines the dependence[18] of $\langle T\rangle$ on ϕ_n to be of the form

$$(T^M+T^{-M})\cos M\phi_n+i(T^M-T^{-M})\sin M\phi_n, \quad (4.14)$$

for $s_n\to\infty$, $t_n\to 0$.

It is instructive to consider the special case of $N=3$, illustrated in Fig. 2, with the momenta ordered as shown. The five independent variables can be taken to be

$$t_1=k_1^2,\quad t_2=k_2^2,\quad s_1=(p_1+p_2)^2,\quad s_2=(p_2+p_3)^2,$$

and

$$S_1\equiv s=(p_1+p_2+p_3)^2=(p_4+p_0)^2. \quad (4.15)$$

In place of s, we use[17] the angle $\phi_1\equiv\phi$ defined generally by (A13). One can see that the $\beta_n{}^\mu$ defined in the Appendix are all spacelike unit vectors for all physical processes implied by Fig. 2 (i.e., all the $p_n{}^\mu$ are timelike, positive or negative). Thus $\cos\phi$ as a function of the variables defined in (4.15) will (for physical processes) always lie on the range $(-1, 1)$. In fact, for whatever process one discusses, ϕ is the angle between the planes whose directions are given by $\mathbf{p}_1\times\mathbf{p}_0$ and $\mathbf{p}_3\times\mathbf{p}_4$ in the frame $\mathbf{p}_2=0$. For the process (b),

$$p_0+p_4\to\bar{p}_1+\bar{p}_2+\bar{p}_3, \quad (4.16)$$

ϕ is what we shall call[19] the "Toller angle," whereas for the process (c),

$$p_1+p_2\to\bar{p}_0+\bar{p}_3+\bar{p}_4, \quad (4.17)$$

ϕ is the Treiman-Yang angle.[20]

For $N=3$ the general formula (3.36) becomes

$$\langle T\rangle=FG_3D(\psi_2)G_2D(\psi_1)G_1. \quad (4.18)$$

We use the results of Sec. 3, especially Eqs. (3.31), (3.34), and (3.35), to obtain the labels which will appear on the $D(\psi_1)$ matrix in the expansion (4.13). Thus, using (4.3), (4.4), and (4.6), we have

$$D_{0,\sigma,j,\lambda,0,0}(\psi_1)=\sum_m e^{i\lambda\phi}d_{\lambda m}{}^j(\theta_1{}^L)d_{j,m,0}{}^{0\sigma}(\epsilon_1{}^C), \quad (4.19)$$

and the expansion becomes

$$\langle T\rangle=\sum_{\sigma,j,\lambda} T^{0,\sigma,j,\lambda,0,0}D_{0,\sigma,j,\lambda,0,0}(\psi_1), \quad (4.20)$$

where the ϕ and s_1 variables appear only in the explicit $D(\psi_1)$ function and not in the T. We now assume that we can take the limits $s_1\to\infty$ and $t_1\to 0$ inside the summation sign[21] in (4.20). Using (4.11) and (4.12), it follows immediately that $\langle T\rangle$ is independent of ϕ in this region of the (s_1,t_1) plane.[22] That $\langle T\rangle$ is independent of ϕ is because we have been assuming the particles to have zero spin. When one includes spin (see Sec. 5), the sum in (4.20) will also run over M, with

$$|M|\leqslant j_0+j_1, \quad (4.21)$$

where j_0 and j_1 are the spins of the particles labelled "0" and "1," respectively. Thus, in general we will have

$$\langle T\rangle_{s_1\to\infty,t_1\to 0}=\sum_{M=0(\frac{1}{2})}^{j_0+j_1}\left[f_M(s_2,t_2)e^{iM\phi}+f_{-M}(s_2,t_2)e^{-iM\phi}\right]s_1{}^{\sigma-1}, \quad (4.22)$$

and the assumption of the dominance of the amplitude by a single Toller pole with quantum number M will lead to a unique ϕ distribution in the high-s_1, low-t_1 region of the process.

In particular, let us consider the specific reaction

$$N_1+\pi_2\to N_0+\pi_3+\pi_4, \quad (4.23)$$

corresponding to the labelling (4.17) where the N's are nucleons and the π's are pions. In this case s_1 is the total energy of process, t_1 the momentum transfer from initial to final nucleon, and t_2 the energy squared of the (π_3,π_4) system. If we assume that this amplitude is dominated by a pion Toller pole in the variable t_1, the M value of this pole can be obtained by studying the distribution in the Treiman-Yang angle[20] for the large-energy, low-momentum-transfer region of the process. The value $t_1=0$ is not physically accessible in this reaction. However, the minimum value of $|t_1|$ is

$$m_N{}^2(t_2-m_\pi{}^2)^2/s_1{}^2, \quad (4.24)$$

for large s_1. For t_2 in the neighborhood of $m_\rho{}^2$, say, small values of $|t_1|/m_N{}^2$ are reasonably accessible.

Processes in which the value $t_1=0$ lies within the physical boundary can easily be found. Choosing the

[18] To obtain this result, we have used the fact that if a Toller pole leads to a term in the amplitude of the form $T^M\exp(iM\phi_n)$, parity conservation requires the existence of the term $T^{-M}\times\exp(-iM\phi_n)$. Here, T^M is an abbreviation for the amplitude in (4.13) multiplied by $s_n{}^{\sigma-1}$.

[19] The decomposition of a process such as (4.16) has been written down earlier by T. W. B. Kibble [Phys. Rev. **131**, 2282 (1963)] and K. M. Ter-Martirosyan [Nucl. Phys. **68**, 591 (1965)], who introduce the angle ϕ.

[20] S. B. Treiman and C. N. Yang, Phys. Rev. Letters **8**, 140 (1962).

[21] If the sum over σ appearing in (4.20) is only over a finite number of terms (i.e., if there are a finite number of Toller poles), then this interchange is clearly possible.

[22] The angle ϕ is the angle between the normals to the (p_0,p_1,p_2) and (p_2,p_3,p_4) "scattering planes." The ϕ dependence of the matrix elements has been obtained for those events for which the angles $\theta_1{}^L$ and $\theta_1{}^R$ are small (i.e., t_1 small as $s_2\to\infty$). It is important that neither $\theta_1{}^L$ or $\theta_1{}^R$ is actually zero, since then the "scattering planes" are not defined.

labelling (4.16), the necessary conditions are

$$m_0^2 > m_1^2\,, \tag{4.25}$$

and

$$m_4^2 < s_2\,. \tag{4.26}$$

For example,

$$N_0 + \bar{N}_4 \to \pi_1 + \pi_2 + \pi_3 \tag{4.27}$$

would be such a reaction provided that we stick to events[23] for which the energy in the (π_2,π_3) system is larger than m_N. Note that in this case, ϕ is the Toller angle, i.e., the angle between the $(\bar{N}_4,\pi_3)$ plane and the (N_0,π_1) plane in the rest frame of the π_2. Also large s_1 means large (π_1,π_2) energies, and t_1 is the momentum transfer from N_0 to π_1.

5. CLUSTER APPROXIMATION AND SPIN

Only trivial changes are necessary if, instead of a single particle at each vertex, there is a cluster of, say, C_n+1 particles at the nth vertex with specified masses and four-moments $b_1, b_2, \ldots, b_{C_n}, b_{C_n+1}$,

$$p_n = \sum_{i=1}^{C_n+1} b_i\,. \tag{5.1}$$

Each cluster gives rise to $3C_n$ scalar variables—to be referred to as the cluster variables—additional to those considered in Sec. 2 which may be taken to be the directions of the vectors b_i^μ $(i=1, 2, \ldots, C_n)$ in the nth frame. The other variables can be taken exactly as before, with the new definition[24] (5.1) of p_n. (This, of course, reduces to the old definition when $C_n=0$.) The dependence of the amplitude on p_n^2 may still be regarded as dependence on the external masses, since the value of p_n^2 is determined by the specification of the mass of the (C_n+1)th particle in the cluster (which reduces correctly to $p_n^2=m_n^2$ in the no-cluster limit when $C_n=0$). The general analysis of the amplitude given in Secs. 2 and 3 is immediately applicable, provided only that $G_{(n)}$ is allowed to depend also on the nth cluster variables.

Finally, we briefly sketch the formalism for particles of arbitrary spin for the amplitude which, for simplicity, we take to be that of Secs. 2 and 3 (that is without clusters).

The general n-particle state was defined in Sec. 2, Eq. (2.15). The implication of that definition is that the spin components of all n particles is measured in some fixed frame. It is convenient for the later expansions to take the spin components such that the spins can be combined consecutively in a given order. That is, we shall specify the spin components of particles "0" and "1" to be the helicities in the (0,1) c.m. frame, the spin component of particle "2" to be the helicity in the (0,1,2) c.m. frame etc. To this end, we define the nth-particle state in the nth frame thus[25]:

$$|p_n,j_n,\lambda_n\rangle \equiv O(p_n)U_{(n)}{}^{[n]}(p_n)\,|\sigma_n,j_n,\lambda_n\rangle_{(n)}\,, \tag{5.2}$$

where the index $[n]$ implies that the boost U is a function of the operators $J_{\mu\nu}{}^{[n]}$ which operate only in the space of the nth particle; and the suffix (n) implies that the boost is the nth-frame boost [see Eq. (3.20)]. In (5.2) the label σ_n is an abbreviation for the pair of labels M_n, σ_n which specify a representation of the homogeneous Lorentz group.

The argument then goes essentially as before, provided we note that in the expression [see Eq. (3.22)]

$$O(q_n) = U_{(n)}(q_n)O(q_n{}^{(r)})U_{(n)}{}^{-1}(q_n)\,, \tag{5.3}$$

only those operators $J^{[i]}$ are involved for which[26]

$$i \leqslant n\,. \tag{5.4}$$

The combination of factors appearing in (3.24) which go to make up $D(\psi_n)$ [defined by equation (3.29)] also involve these same $J^{[i]}$ operators times an additional factor

$$[U_{(n+1)}{}^{[n+1]}(q_{n+1})]^{-1}\,. \tag{5.5}$$

This latter operator depends only on the $J_{\mu\nu}{}^{[n+1]}$ and so commutes with all the B_i and $O(q_i)$ operators to its right. In can then be combined with the operator $U_{(n+1)}{}^{[n+1]}(p_{n+1})$ in (5.2) for the $(n+1)$th particle to form a spinor. Thus, for the nth particle, we define

$$\langle\sigma_n,j_n',\lambda_n'|v_n|\sigma_n,j_n,\lambda_n\rangle \equiv \langle\sigma_n,j_n',\lambda_n'|\,U_{(n)}{}^{[n]}(q_n)^{-1}U_{(n)}{}^{[n]}(p_n)\,|\sigma_n,j_n,\lambda_n\rangle\,. \tag{5.6}$$

The final result for the amplitude is then[27] of the general

[23] For processes of the type (4.27) the ϕ dependence can in general be somewhat more difficult to extract in the "multi-Toller" region, i.e., where s_1 and s_2 are both large and t_1, t_2 both small. In this region the ϕ dependence can be governed by M_1, M_2 or a combination of M_1 and M_2, depending on the relative (s_1,s_2) and (t_1,t_2) values. An analysis of events of type (4.27) in the high s_1, s_2 region has been made by Chan Hong-Mo, K. Kajantie, and G. Ranft, Nuovo Cimento **49A**, 157 (1967). They assume that $\langle T\rangle$ is a slowly varying function of ϕ. See also I. T. Drummond, Phys. Rev. **176**, 2003 (1968).

[24] Our coordinate frames differ from those of Bali *et al.* (Ref. 1) in that they are defined in terms of "scattering planes" rather than "cluster frames." The advantage of our choice of frames is that there is no difficulty in passing to the "no-cluster" limit (i.e., $C_n=0$, all n). Also the dependence on ϕ_n is always explicitly displayed.

[25] The λ_n are thus eigenvalues of the "covariant helicity" operators $W_\mu{}^{[n]}\nu_n{}^\mu$, where $W_\mu{}^{[n]}$ are the Pauli-Lubanski operators defined by (2.3). Note that $W_\mu{}^{[0]}\nu_0{}^\mu=0$. For this particle we specify $W_\mu{}^{[0]}\nu_1{}^\mu$. Thus, we are specifying the spin components of particles "0" and "1" to be the helicities in the (0,1) c.m. frame. Also $W_\mu{}^{[N+1]}\nu_N{}^\mu=0$. Since the process under discussion is the decay of particle $N+1$, we can choose its spin component to be the z component of $J_{\mu\nu}{}^{[N+1]}$, where $\gamma_N{}^\mu$ is taken as the z axis. See G. Feldman and P. T. Matthews [Phys. Rev. **168**, 1587 (1968)] for the properties of "covariant helicity."

[26] This is immediately obvious from the definition of the operators Q_n given by (3.10) and (3.9).

[27] A possible choice for q_n is p_n [see Eq. (3.11)]. This would be a great simplification by reducing all but two of the v_n to unity. Thus, in the initial state, we would have remaining $v_0=U_{(1)}{}^{[0]}(p_1)^{-1}\times U_{(1)}{}^{[0]}(p_0)$, and in the final state a factor $v_N=U_{(N)}{}^{[N+1]}(p_N)$.

form

$$\langle T\rangle = \langle \sigma_{N+1},j_{N+1},\lambda_{N+1}| FG_{(N)}v_N \times D(\psi_{N-1})G_{(N-1)}v_{N-1}D(\psi_{N-2})\cdots G_{(2)}v_2 \times D(\psi_1)G_{(1)}v_1v_0|\sigma_0,j_0,\lambda_0,\ldots,\sigma_N,j_N,\lambda_N\rangle, \quad (5.7)$$

where $v_0 = U_{(1)}{}^{[0]}(q_1)^{-1}U_{(1)}{}^{[0]}(p_0)$.

The matrices $G_{(n)}$ and $D(\psi_n)$ are diagonal in the space of all particles i for which $i>n$. The nth-particle labels enter the matrix product in (5.7) from the right in a nontrivial way through the spinor v_n, and are active in the matrix product in factors which lie to the left of v_n.

The spinor factors are well behaved in the $s\to\infty$, $t\to 0$ limit, so the multi-Toller approximation goes through as before.

6. SUMMARY

Using the projection-operator technique developed previously,[4] we have obtained a multiple expansion of the $N+2$ particle amplitude into $O(3)$ and $O(3,1)$ functions. This amplitude may be regarded as a series of pseudoscattering processes. The Toller variables[1,2] (in the no-cluster limit—only three lines at each vertex) are just the angles that appear in the transformation from one scattering frame to the next. In particular, the "Toller angles" ϕ_n are just the angles between the perpendiculars to two successive "scattering" planes. The use of frames of reference defined in terms of "scattering" processes, rather than the cluster frames used by Bali *et al.*,[1] leads to a minor simplification in the phase factors, which appear in the final formula (3.39).

By suitable analytic continuation of the $O(3)$ expansion into the high-partial-energy regions ($s_n\to\infty$), we obtain the multi-Regge expansions previously derived by Bali *et al.*[1]

The expansion in $O(3,1)$ functions shows that problems of analyticity in the multi-Regge region for small momentum transfers ($t_n\to 0$), can be removed in a manner similar to that for scattering processes. In addition, we see that the value of the M quantum number of a Toller pole can be found by measuring the distribution in the Toller angle for large energies and small momentum transfer. In a suitable region of the other variables, the Toller angle is just the Treiman-Yang angle. Further, if we assume that we can interchange high-s limits with summations over $O(3,1)$ amplitudes, it is possible to show that the dependence of the production amplitude on ϕ (in the small-t region) is a Fourier series with the maximum frequency given by the sum of the spins at the t vertex.

APPENDIX

In this appendix, we define the nth frame and obtain the Lorentz transformation which takes the nth frame into the $(n+1)$th frame. The nth frame is defined by the orthogonal tetrad[28]

$$(\nu_n{}^\mu,\alpha_n{}^\mu,\beta_n{}^\mu,\gamma_n{}^\mu),$$

where

$$\nu_n{}^\mu = \hat{k}_n{}^\mu,$$

$$k_n{}^\mu = \sum_{i=0}^{n} p_i{}^\mu, \quad (A1)$$

$$\gamma_n{}^\mu = \frac{\nu_n{}^\mu(\nu_n\cdot\nu_{n-1}) - \nu_{n-1}{}^\mu}{[(\nu_n\cdot\nu_{n-1})^2-1]^{1/2}}, \quad (A2)$$

$$(\beta_n)_\mu = \epsilon_{\mu\lambda\pi\rho}k_{n-1}{}^\lambda k_n{}^\pi k_{n+1}{}^\rho/[\Phi(k_{n-1},k_n,k_{n+1})]^{1/2}, \quad (A3)$$

and

$$(\alpha_n)_\mu = \epsilon_{\mu\lambda\pi\rho}\nu_n{}^\lambda\beta_n{}^\pi\gamma_n{}^\rho, \quad (A4)$$

and $\Phi(a,b,c)$ is a Kibble function. (See I, Appendix A.) Thus,

$$-4\Phi(k_{n-1},k_n,k_{n+1}) = 4t_{n-1}t_nt_{n+1} + (t_n+t_{n-1}-p_n{}^2) \times(t_{n+1}+t_n-p_{n+1}{}^2)(t_{n+1}+t_{n-1}-s_n) - (t_n+t_{n-1}-p_n{}^2)^2t_{n+1} - (t_{n+1}+t_n-p_{n+1}{}^2)^2t_{n-1} - (t_{n+1}+t_{n-1}-s_n)^2t_n. \quad (A5)$$

For the multidecay process (3.1a), all the $k_n{}^\mu$ are positive timelike and thus

$$\nu_n{}^2 = -\alpha_n{}^2 = -\beta_n{}^2 = -\gamma_n{}^2 = 1. \quad (A6)$$

Note also that successive frames are such that

$$\nu_{n+1}\cdot\beta_n = \gamma_{n+1}\cdot\beta_n = \beta_{n+1}\cdot\nu_n = \alpha_{n+1}\cdot\nu_n = 0. \quad (A7)$$

In each frame n, we define the set of Lorentz generators that implies that the unit tetrad $(\nu_n{}^\mu,\alpha_n{}^\mu,\beta_n{}^\mu,\gamma_n{}^\mu)$ is taken to be the coordinate frame (t,x,y,z). Thus we shall have, e.g.,

$$J_3{}^{(n)} \equiv J_{12}{}^{(n)} \equiv \alpha_n{}^\mu J_{\mu\lambda}\beta_n{}^\lambda, \quad (A8)$$

$$J_2{}^{(n)} \equiv J_{31}{}^{(n)} \equiv \gamma_n{}^\mu J_{\mu\lambda}\alpha_n{}^\lambda, \quad (A9)$$

$$K_3{}^{(n)} \equiv J_{03}{}^{(n)} \equiv \nu_n{}^\mu J_{\mu\lambda}\gamma_n{}^\lambda, \quad (A10)$$

and

$$(\mathbf{J}^{(n)})^2 = (\beta_n{}^\mu J_{\mu\lambda}\gamma_n{}^\lambda)^2 + (\gamma_n{}^\mu J_{\mu\lambda}\alpha_n{}^\lambda)^2 + (\alpha_n{}^\mu J_{\mu\lambda}\beta_n{}^\lambda)^2. \quad (A11)$$

By operating on the unit vectors in the (t,x,y,z) directions, respectively, it is easy to see that the Lorentz transformation which takes the nth frame into the $(n+1)$th frame is $B_n(\chi_n)$, where

$$B_n(\chi_n) \equiv B_n(\epsilon_n,\theta_n,\phi_n) = \exp(-i\theta_nJ_2{}^{(n)}) \times\exp(-i\epsilon_nK_3{}^{(n)})\exp(-i\phi_nJ_3{}^{(n)}), \quad (A12)$$

[28] The nth frame has been chosen to be the c.m. frame for the pseudoscattering process (3.19) (i.e., $\nu_n{}^\mu$ is the time axis). The z direction $\gamma_n{}^\mu$ is the direction of the incoming "beam" particle k_{n-1}, in this c.m. frame. The y direction $\beta_n{}^\mu$ is the normal to the "scattering" plane.

where

$$-\cos\phi_n=\beta_{n+1}\cdot\beta_n=\tfrac{1}{8}[(t_n+t_{n-1}-p_n^2)(t_{n+1}+t_n-p_{n+1}^2)(t_{n+2}+t_{n+1}-p_{n+2}^2)+(t_{n+1}+t_{n-1}-s_n)(t_{n+2}+t_n-s_{n+1})$$
$$\times(t_{n+1}+t_n-p_{n+1}^2)+4t_nt_{n+1}(t_{n+2}+t_{n-1}-S_n)-2t_n(t_{n+1}+t_{n-1}-s_n)(t_{n+2}+t_{n+1}-p_{n+2}^2)$$
$$-(t_{n+1}+t_n-p_{n+1}^2)^2(t_{n+2}+t_{n-1}-S_n)-2t_{n+1}(t_{n+2}+t_n-s_{n+1})(t_n+t_{n-1}-p_n^2)]$$
$$\times[\Phi(k_{n-1},k_n,k_{n+1})\Phi(k_n,k_{n+1},k_{n+2})]^{-1/2},\quad n=1,2,\ldots,N-2 \quad \text{(A13)}$$

$$\cosh\epsilon_n=\nu_{n+1}\cdot\nu_n=\hat{k}_{n+1}\cdot\hat{k}_n=k_{n+1}\cdot k_n/(k_{n+1}^2k_n^2)^{1/2}=(t_n+t_{n+1}-p_{n+1}^2)/2(t_nt_{n+1})^{1/2},\quad n=1,2,\ldots,N-1 \quad \text{(A14)}$$

and

$$\begin{aligned}\cos\theta_n&=\cos\theta(\gamma_n,\gamma_{n+1};\nu_n)\\&=\cos\theta(q_n,q_{n+1};\nu_n)\\&=\cos\theta(p_n,p_{n+1};\nu_n)\\&=\cos\theta(k_{n-1},k_{n+1};\nu_n)\\&=-\gamma_n\cdot\gamma_{n+1}/\nu_n\cdot\nu_{n+1}\end{aligned}\quad \text{(A15)}$$

$$=\frac{2t_n(t_{n+1}+t_{n-1}-s_n)-(t_n+t_{n-1}-p_n^2)(t_{n+1}+t_n-p_{n+1}^2)}{\Delta(t_n,t_{n+1},p_{n+1}^2)\Delta(t_{n-1},t_n,p_n^2)},\quad n=1,2,\ldots,N-1 \quad \text{(A16)}$$

where

$$\Delta^2(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx\,, \quad \text{(A17)}$$

and

$$\cos\theta(a,b;c)=\hat{a}_c\cdot\hat{b}_c\,, \quad \text{(A18)}$$

where $\hat{a}_c$ is the unit three-vector in the direction $\mathbf{a}$ in the frame in which $\mathbf{c}=0$. The various equalities in (A15) follow from the definitions (3.4) and (3.11) and the fact that all the vectors in (A15) lie in one plane.

Note that the factors which appear in $B_n(\chi_n)$, defined through (A13)–(A15), have very simple physical interpretations in terms of the nth and $(n+1)$th "scattering" subprocesses (3.19). The rotation through ϕ_n aligns the scattering planes, that through θ_n aligns the "beams," and the boost ϵ_n relates the two total momentum four-vectors. Of course θ_n is also the c.m. scattering angle in the nth subprocess.

If we take the tetrad $(\nu_n{}^\mu,\alpha_n{}^\mu,\beta_n{}^\mu,\gamma_n{}^\mu)$ in the (t,x,y,z) directions, respectively, then we obtain

$$\begin{aligned}\nu_{n+1}{}^\mu&=(\cosh\epsilon_n,\ \sinh\epsilon_n\sin\theta_n,0,\ \sinh\epsilon_n\cos\theta_n)\,,\\ \alpha_{n+1}{}^\mu&=(0,\ \cos\theta_n\cos\phi_n,\ \sin\phi_n,\ -\sin\theta_n\cos\phi_n)\,,\\ \beta_{n+1}{}^\mu&=(0,\ -\cos\theta_n\sin\phi_n,\ \cos\phi_n,\ \sin\theta_n\sin\phi_n)\,,\\ \gamma_{n+1}{}^\mu&=(\sinh\epsilon_n,\ \cosh\epsilon_n\sin\theta_n,0,\ \cosh\epsilon_n\cos\theta_n)\,.\end{aligned}\quad \text{(A19)}$$

From the definitions of $B_n(\chi_n)$ and $J^{(n)}$, we have

$$J^{(n+1)}=B_n(\chi_n)J^{(n)}B_n^{-1}(\chi_n)\,. \quad \text{(A20)}$$

We introduce "nth frame states," which are representations of the homogeneous Lorentz group, by defining them as follows:

$$(\mathbf{J}^2-\mathbf{K}^2)|M,\sigma,j,\lambda\rangle_{(n)}=(M^2+\sigma^2-1)|M,\sigma,j,\lambda\rangle_{(n)}\,, \quad \text{(A21)}$$

$$\mathbf{J}\cdot\mathbf{K}|M,\sigma,j,\lambda\rangle_{(n)}=-iM\sigma|M,\sigma,j,\lambda\rangle_{(n)}\,, \quad \text{(A22)}$$

$$(\mathbf{J}^{(n)})^2|M,\sigma,j,\lambda\rangle_{(n)}=j(j+1)|M,\sigma,j,\lambda\rangle_{(n)}\,, \quad \text{(A23)}$$

$$J_3{}^{(n)}|M,\sigma,j,\lambda\rangle_{(n)}=\lambda|M,\sigma,j,\lambda\rangle_{(n)}\,, \quad \text{(A24)}$$

with

$$M\leqslant j\,.$$

From (A20) it follows that

$$|M,\sigma,j,\lambda\rangle_{(n+1)}=B_n(\chi_n)|M,\sigma,j,\lambda\rangle_{(n)}\,. \quad \text{(A25)}$$

Thus, substituting for $B_n(\chi_n)$ from (A12), we obtain

$$\begin{aligned}&{}_{(n)}\langle M',\sigma',j',\lambda'|M,\sigma,j,\lambda\rangle_{(n+1)}\\&=\delta_{MM'}\delta_{\sigma\sigma'}\ {}_{(n)}\langle j'\lambda'|\exp(-i\theta_nJ_2{}^{(n)})|j'\lambda\rangle_{(n)}\\&\quad\times{}_{(n)}\langle M,\sigma,j',\lambda|\exp(-i\epsilon_nK_3{}^{(n)})|M,\sigma,j,\lambda\rangle_{(n)}\\&\quad\times{}_{(n)}\langle j\lambda|\exp(-i\phi_nJ_3{}^{(n)})|j\lambda\rangle_{(n)}\\&\equiv\delta_{MM'}\delta_{\sigma\sigma'}d_{\lambda'\lambda}{}^{j'}(\theta_n)d_{j'\lambda j}{}^{M\sigma}(\epsilon_n)\exp(-i\lambda\phi_n)\,,\end{aligned}\quad \text{(A26)}$$

where the $d^j(\theta)$ and $d^{M\sigma}(\epsilon)$ are the conventional $O(3)$ and $O(3,1)$ functions, respectively.

A useful identity which helps to transform $O(3)$ expansions into $O(3,1)$ expansions is (see I, Appendix B)

$$\begin{aligned}&\exp[iK_3\epsilon(a,b)]\exp[iJ_2\theta(b,c;a)]\exp[-iK_3\epsilon(c,a)]\\&\quad\equiv\exp[-iJ_2\theta(a,c;b)]\exp[iK_3\epsilon(c,b)]\\&\quad\quad\times\exp[iJ_2\theta(b,a;c)]\,,\end{aligned}\quad \text{(A27)}$$

where a, b, and c are any three timelike four-vectors and

$$\cosh\epsilon(a,b)=\hat{a}\cdot\hat{b}\,, \quad \text{(A28)}$$

and $\cos\theta(b,c;a)$, etc., are defined in (A18). For the special case of a, b, and c in the same plane,

$$\exp[iK_3\epsilon(a,b)]\exp[iK_3\epsilon(c,a)]=\exp[iK_3\epsilon(c,b)]\,. \quad \text{(A29)}$$

We may also write (see I, Appendix A)

$$\cos\theta(a,b;c)=\frac{4[(a\cdot c)(b\cdot c)-(a\cdot b)c^2]}{\Delta(a^2,c^2,(a\pm c)^2)\Delta(b^2,c^2,(b\pm c)^2)}\,, \quad \text{(A30)}$$

and

$$\sin\theta(a,b;c)=\frac{4[c^2\Phi(a,b,c)]^{1/2}}{\Delta(a^2,c^2,(a\pm c)^2)\Delta(b^2,c^2,(b\pm c)^2)}. \quad \text{(A31)}$$

If for example we take a, b, and c to be k_n, p_n, and p_{n+1}, respectively, one finds that as $s_n\to\infty$,

$$\theta(k_n,p_n;p_{n+1})\sim t_n^{1/2}, \quad \text{for } t_n \text{ small.} \quad \text{(A32)}$$

Choosing q_n and q_{n+1} as in (3.11), we find also that as $s_n\to\infty$,

$$\theta(k_n,q_n;q_{n+1})\sim t_n^{1/2}, \quad \text{(A33)}$$

and

$$\theta(k_n,q_{n+1};q_n)\sim t_n^{1/2}. \quad \text{(A34)}$$

Similarly,

$$\cosh\epsilon(p_n,p_{n+1})=\frac{s_n-p_n^2-p_{n+1}^2}{2(p_n^2p_{n+1}^2)^{1/2}} \sim s_n \quad \text{as} \quad s_n\to\infty. \quad \text{(A35)}$$

Similarly,

$$\cosh\epsilon_n^c\equiv\cosh\epsilon(q_n,q_{n+1}) \sim s_n \quad \text{as } s_n\to\infty. \quad \text{(A36)}$$

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Simply Factorizable n-Point Amplitude

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(Received 16 September 1969)

A general n-point amplitude based on tree diagrams which has Regge asymptotic behavior and poles in all channels, but which is not dual, is explicitly factorized, and the coupling of three or more Reggeons is computed. The degeneracy of levels of the factorization of the n-point function is n^2, the same as in the four-point function. Group-theoretical implications are briefly discussed.

I. INTRODUCTION

IT is of interest to have an n-point amplitude containing infinite multiplets of states that is *explicitly* factorizable in every channel. One can then define arbitrary amplitudes with external particles of arbitrary spins through the factorization, in particular, the coupling of three or more Regge poles or composite particles.

For the n-point functions[1] based on the Veneziano-type four-point function, the factorization problem is very complicated and exhibits a high degree of degeneracy of the type $e^{\sqrt{n}}$.[2]

Here we present a model based on the four-point and n-point representations given by one of us[3] which allows a very simple factorization with the same n^2 degeneracy as the four-point function. The model has crossing symmetry and Regge asymptotic behavior in all channels. It does not have the so-called "duality" in the sense of the Veneziano model that the sum of the s-channel poles alone also contains an infinite number of poles in the t channel. On the phenomenological level, all the experimental successes of the Veneziano model can be reproduced in an equally if not more satisfactory way with this model.[3–5] On the theoretical side, whether there is a duality in the strict sense of the Veneziano model is a matter of dispute. What we want to show is that a reasonable model exists in which the coupling of all higher spin states is explicitly obtained from the amplitude and to exhibit general amplitudes with arbitrary spins. This information is crucial for a group-theoretical understanding of the multiplet, which we also briefly discuss. If we are working with a definite composite system, which has a certain degeneracy, we can put a physical requirement on the n-point amplitude to the effect that the factorization of an n-point amplitude should give the same degeneracy as the four-point amplitude. This is the case for the present model.

In Sec. II, we discuss the prescription for constructing the "pole" diagrams for the amplitudes described by the multiperipheral diagrams (Fig. 1) and other

* Supported in part by the U. S. Air Force Office of Scientific Research under Contract No. AFSOR-30-67.

† Supported in part by the National Research Council of Canada.

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