Physical Interpretation of Complex-Energy Negative-Metric Theories*

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It has long been held that the divergences of field theory could be removed if the theory was formulated on a Hilbert space with an indefinite metric. The difficulty of this approach was that the theory then possessed channels with negative probability and therefore violated unitarity. In a recent paper, Lee and Wick have observed that this difficulty could be eliminated if negative-metric states had complex energy. In this paper, we produce a model which shows several of the difficulties associated with the Lee-Wick proposal. The existence of complex energies leads to interpretation difficulties and, in higher sectors, does not completely solve the unitarity problem. We use the model to show how the Lee-Wick conjecture can be extended to overcome the difficulties that are produced. We describe the physical meaning of "good" theories with an indefinite metric.

I. INTRODUCTION

ALTHOUGH the problem of the divergences in relativistic quantum field theory has been present for many years, the more recent successes of the approximation schemes based on S-matrix and bootstrap techniques have tended to mask the difficulties inherent in the underlying field structure. The problems of relativistic field theory have to be faced: The renormalization scheme, found so successful for quantum electrodynamics, is not the panacea in a field theory that is to be used for describing particle-physics phenomena, and not only because the couplings are considerably stronger. In particle physics we are interested in calculating mass shifts, comparison of effective coupling constants with the unrenormalized value, and identification of resonant levels. We must therefore look for a finite quantum field theory, not one where apparently infinite quantities are renormalized away.

In the past, the possibility of circumventing these difficulties by using an appropriate ultraviolet cutoff has been considered. This was accomplished either in the form of a nonlocal field theory with geometrical form factors, or by an invariant regularization procedure. We think it only fair to say that no one has yet succeeded in formulating a satisfactory nonlocal theory.[1] The precise mathematical formulation of the invariant regularization procedure shows that such a field theory lacks so many of the desirable features for a quantum field theory that it is of no particular interest. We are obliged to recognize that the invariant regularization method of Pauli and Villars and of Feynman does *not* provide the basic idea for a new theory of quantized fields.[2]

There is a related but distinct method of constructing a finite theory based on a possible generalization of the mathematical framework of quantum mechanics. This generalization was invented by Dirac[3] several decades ago and has been successfully used by Gupta and by Bleuler in giving a most elegant formulation of quantum electrodynamics.[4] The method utilizes a linear vector space with an indefinite bilinear form for the inner product of two vectors, and leads to the possibility of removal of the standard infinities of local relativistic field theory. A quantum field theory formulated in such a space is referred to as a "field theory with indefinite metric."[5] The major problem encountered in the formulation of such a theory is that we must be careful in defining the physical transition amplitudes since we want physical transition probabilities to be non-negative. The proper identification of physical amplitudes is part of the dynamical problem in an indefinite-metric quantum theory.

A careful formulation of quantum field theory in which these dynamical problems are treated adequately has been developed in the past decade. A finite quantum electrodynamics has been constructed on the basis of such a formulation. This theory is in complete accord with our empirical knowledge in the field.[6] The analytic structure of the transition amplitudes and other questions of relevance to dispersion theory have also been studied. A systematic discussion of the accomplishments and the problems of quantum field theories with

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[1] Examples of attempts at a nonlocal formulation of quantum field theory can be found in R. E. Peierls and H. McManus, Proc. Roy. Soc. (London) **A195**, 323 (1948); H. Yukawa, Phys. Rev. **76**, 300 (1949); **76**, 1731 (1949); **77**, 219 (1950); C. Bloch, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **27**, No. 8 (1952); M. Auretieu and R. E. Peierls, Proc. Roy. Soc. (London) **A223**, 968 (1954); T. Takabayash, in *Proceedings of the International Theoretical Physics Conference on Particles and Fields, Rochester, New York, 1967*, edited by C. R. Hagen *et al.* (Wiley-Interscience, Inc., New York, 1968), p. 413.

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indefinite metric is presented in a paper by Sudarshan.[7] In that paper it is pointed out that an essential prerequisite to a complete theory is the restriction of physical states to a subset of the vectors in the indefinite-metric vector space, the restriction in turn being defined by the eigenvectors of positive norm of the scattering operator. The subset is to be determined dynamically.

A further modification is possible. In a conventional theory, the masses of particles should be real; but an indefinite-metric theory permits the occurrence of complex masses, provided they occur in pairs. In a series of ingenious papers, Lee and Wick have explored the advantages offered by this freedom.[8] They have studied several models that can be solved exactly in their lower nontrivial scattering sectors. They show that in a model with complex (renormalized) masses for some of its particles, in a completely soluble sector, the scattering amplitude is unitary. They emphasize that this unitarity is maintained even though this sector is dynamically coupled to states with negative norm. Moreover, all the scattering states have positive norm in the sectors studied. No difficulties arise in the physical interpretation of the scattering process. Lee and Wick expressed the conviction that this removal of the difficulties could be maintained in any negative metric theory if they require that all the particles associated with a negative norm have complex mass. In the cases that they study, they require the free parameters of the model to be adjusted so that all single-particle negative-norm states have complex energy. As will be emphasized in the present paper, it is then the requirement of energy conservation for real energy scattering that decouples these negative-norm states from the unitarity requirements. In any sector with only one complex mass state of negative norm, the scattering via the negative metric is virtual and does not cause violations of unitarity.

In this paper we have produced a more complete indefinite-metric model which satisfies the requirements that all negative-norm particles have complex energy states. It is the purpose of this model to show that the requirement of complex energies for negative norm states is still a possibile mechanism for decoupling of the unitarity violations in the lower sectors, but that it is not realistic to expect that the higher sectors will also satisfy this condition.

A similar situation obtains in the Hamiltonian formulation of the charged scalar theory with restriction to one-meson (and no-meson) states.[9] We can recover the Lee-Serber scattering amplitude obeying unitarity in a formulation that contains negative-energy negative-probability meson states dynamically coupled to the physical states. But no physical transitions occur between positive- and negative-energy states. However, this theory encounters serious difficulties as soon as production channels are included: Energy conservation can no longer prevent mixing of positive- and negative-norm states.

We have reinvestigated the questions proposed and partially solved by Lee and Wick and subsequently discussed by Coleman and Glashow. We found it more convenient to construct a new model which incorporates negative-probability states associated with complex-mass particles, but in which the particle-production amplitudes can be explicitly and simply solved. We recover a result essentially equivalent to the Lee-Wick result whenever asymptotic energy conservation forbids the physical transition to states containing complex-mass particles. But we also show that such a circumstance does not always obtain: In this model there are production processes involving complex-mass particles (with real total energies), the amplitude for which does not vanish and which must be included in the complete set of states. Under suitable circumstances the production channel leads to a negative-probability state.

The omission of any of these states would yield an S matrix that would not be unitary. The availability of the exact solutions allows us to avoid the questions raised by the Coleman-Glashow–Lee-Wick debate. The states considered are not examined in a perturbation treatment of the interaction picture, or a graphical analysis, but in an exact treatment. The constraints imposed by the solvability of the model require that these production sectors be simple and that our mesons have a simple energy-momentum relationship. The resulting model is then only quasi-relativistic, but the complications intrinsic to this case are probably not removable by a more general relativistic treatment of scattering. Regardless, we find that we cannot arbitrarily drop states in our vector space and that the Lee-Wick proposal is inadequate and incorrect. If we aim to have a quantum theory of the scattering matrix where there is a vector space of states and where the S-matrix elements are related to the scalar product of "in" and "out" states, we are led to a program in which the physical interpretation of the theory has to involve the choice of physical states, which must be considered as part of the dynamical problem.[7]

The program of Lee and Wick can be interpreted by noting that in their indefinite-metric theory the dynamical restriction of the physical states to states of positive norm is brought about by using the non-Hermiticity of the Hamiltonian to decouple the unwanted states. In the present model, we show that this program is impossible to maintain in higher sectors even if it is imposed successfully in the lower sectors. The use of complex-energy eigenvalues, besides not being able to maintain the distinction between positive- and negative-

[7] E. C. G. Sudarshan, *Fundamental Problems in Elementary Particle Physics* (Wiley-Interscience, Inc., New York, 1968).

[8] T. D. Lee and G. C. Wick, Nucl. Phys. **B9**, 209 (1969); **B10**, 1 (1969); T. D. Lee, CERN Report No. Th. 914 (unpublished).

[9] T. D. Lee and R. Serber (unpublished); C. J. Goebel, Phys. Rev. **109**, 1946 (1958).

norm states in the higher sectors, introduces new complications of interpretation of the one- and two-particle complex-energy states.

It is our proposal that the use of any dynamical variable to identify the negative-norm states is probably doomed to fail in any state composed of direct products of the states that meet the identification requirements in the lower sectors. The only consistent solution appears to be restricting the physical states to the positive-norm steady states of the scattering operator. The question may naturally arise as to why one went to the larger space. This question has already been answered: We want to deal with the larger space because it is in this space that we have local relativistic fields. Restriction to the smaller ("physical") space destroys the locality of the field operators. In other words, an indefinite-metric theory may be the simplest and most elegant method of constructing a relativistic quantum theory that is equivalent to a nonlocal relativistic theory with a positive-definite metric. Manifest covariance for interacting fields makes an indefinite metric inevitable.

The plan of the present paper is as follows. In Sec. II we construct the model which contains fields associated with both positive and negative probability. In Sec. III we solve for the lower-lying sectors and we show that only one-particle states with complex mass have non-positive-norm components. We show that there are states with positive norm and real energies which can be composed of such particles. Some questions of physical measurability are also discussed here. The production amplitude is explicitly solved in Sec. IV. In Sec. V we derive and describe the S matrix for this model; but we also see that in general the complex-mass particle production amplitude should be explicitly included in the unitarity relations. Section VI describes the manner in which the dynamical restriction of physical states to positive-norm states can be accomplished. The last section contains a critical discussion.

II. MODEL

Since the purpose of this paper is to establish in a clear way some of the difficulties associated with indefinite-metric theories, we propose a model that will quickly and directly show the problems of interpretation and unitarity in these metric spaces. With this goal in mind, we carefully formulate the form of our Hamiltonian. There are four particles in this model. The source particle is a massive (static) nucleon (N) and provides the source of our scattering. The rest frame of this nonrelativistic nucleon will provide a convenient frame in which to describe our system. There are three distinct mesons (π_1,π_2,π_3); two mesons with positive norm, and one with negative norm. One of the positive-norm mesons is basically a beam meson and it has mass μ_1. The other positive- and negative-norm particles are mass-μ partners ($\mu_2=\mu_3=\mu$) which mix via a direct interaction. In order to separate the elastic and production thresholds, we assume that $\mu_1<2\mu$.

The Hamiltonian consists of three terms. The non-interacting or free Hamiltonian describes the kinetic energy of the freely propagating fields. These Hamiltonians are written directly in momentum space where $\omega_i(\mathbf{k})\equiv(\mu_i^2+\mathbf{k}^2)^{1/2}$ has the usual relativistic momentum-energy relationship. There are two types of terms in the interacting parts of the Hamiltonian. The first is the mixing of the pion partners with equal mass but opposite metric. The second type describes the direct production of meson pairs. The Hamiltonian is

$$H=H_0+H_{I1}+H_{I2}, \tag{2.1}$$

where

$$H_0=m_N\psi_N^\dagger\psi_N+\int\frac{d^3k}{2\omega_1(k)}\omega_1(k)a_1^\dagger(\mathbf{k})a_1(\mathbf{k})+\int\frac{d^3k}{2\omega_2(k)}\omega_2(k)a_2^\dagger(\mathbf{k})a_2(\mathbf{k})+\int\frac{d^3k}{2\omega_3(k)}\omega_3(k)a_3^\dagger(\mathbf{k})a_3(\mathbf{k}), \tag{2.2}$$

$$H_{I1}=g_1\mu^{-1}\int\frac{d^3k}{2\omega_2(k)}[a_2^\dagger(\mathbf{k})a_3(\mathbf{k})-a_3^\dagger(\mathbf{k})a_2(\mathbf{k})], \tag{2.3}$$

and

$$\begin{aligned}H_{I2}=\mu^{-3}\int\frac{d^3k_1}{2\omega_1(k_1)}\int\frac{d^3k_2}{2\omega_2(k_2)}\int\frac{d^3k_3}{2\omega_3(k_3)}\{\psi_N^\dagger a_1^\dagger(\mathbf{k}_1)\psi_N\{G_1(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)[a_2(\mathbf{k}_2)a_2(\mathbf{k}_3)+a_3(\mathbf{k}_2)a_3(\mathbf{k}_3)]\\+G_2(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)[a_2(\mathbf{k}_2)a_2(\mathbf{k}_3)-a_3(\mathbf{k}_2)a_3(\mathbf{k}_3)]+2G_3(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)a_2(\mathbf{k}_2)a_3(\mathbf{k}_3)+2G_4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)a_2(\mathbf{k}_2)a_3(\mathbf{k}_3)\}\\+\psi_N^\dagger\{G_1^*(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)[a_2^\dagger(\mathbf{k}_3)a_2^\dagger(\mathbf{k}_2)+a_3^\dagger(\mathbf{k}_3)a_3^\dagger(\mathbf{k}_2)]+G_2^*(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)[a_2^\dagger(\mathbf{k}_3)a_2^\dagger(\mathbf{k}_2)-a_3^\dagger(\mathbf{k}_3)a_3^\dagger(\mathbf{k}_2)]\\-2G_3^*(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)a_3^\dagger(\mathbf{k}_3)a_2^\dagger(\mathbf{k}_2)-2G_4^*(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)a_3^\dagger(\mathbf{k}_3)a_2^\dagger(\mathbf{k}_2)\}a_1(\mathbf{k}_1)\psi_N\},\end{aligned} \tag{2.4}$$

with the usual commutation relations. These are

$$[\psi_N,\psi_N^\dagger]=1,\quad [a_1(\mathbf{k}),a_1^\dagger(\mathbf{k}')]=2\omega_1(k)\delta^3(\mathbf{k}-\mathbf{k}'),\quad [a_2(\mathbf{k}),a_2^\dagger(\mathbf{k}')]=[a_3(\mathbf{k}),a_3^\dagger(\mathbf{k}')]=2\omega_2(k)\delta^3(\mathbf{k}-\mathbf{k}'), \tag{2.5}$$

with all others zero. The physical space is an indefinite-metric space with the metric operator

$$\eta=\exp\left(i\pi\int\frac{d^3k}{2\omega_3(k)}a_3^\dagger(\mathbf{k})a_3(\mathbf{k})\right). \tag{2.6}$$

The functions $G_i(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)$, $i=1, 2, 3$, are symmetric under the interchange of the $\mathbf{k}_2$, $\mathbf{k}_3$ arguments, while $G_4(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)$ is antisymmetric. The pseudo-Hermiticity condition

$$\eta H^\dagger \eta = H \tag{2.7}$$

is satisfied with g_1 real. For simplicity, we take $G_i(\mathbf{k}_1;\mathbf{k}_2,\mathbf{k}_3)$ to depend only on the magnitude of the momenta k_j.

With this assumption it is now easier to work in a spherical basis and we define

$$a_{ilm} \equiv \int d(\cos\theta_{\mathbf{k}}) d\phi_{\mathbf{k}} Y_{lm}^*(\theta_{\mathbf{k}},\phi_{\mathbf{k}}) a_i(\mathbf{k}). \tag{2.8}$$

In this form, the Hamiltonian becomes

$$H_0 = m_N \psi_N^\dagger \psi_N + \int_0^\infty \frac{k^2 dk}{2\omega_1(k)} \omega_1(k) \sum_{lm} a_{1lm}^\dagger(k) a_{1lm}(k) + \int_0^\infty \frac{k^2 dk}{2\omega_2(k)} \sum_{lm} \omega_2(k) [a_{2lm}^\dagger(k) a_{2lm}(k) + a_{3lm}^\dagger(k) a_{3lm}(k)], \tag{2.9}$$

$$H_{I1} = g_1 \mu^{-1} \int_0^\infty \frac{k^2 dk}{2\omega_2(k)} \sum_l [a_{2lm}^\dagger(k) a_{3lm}(k) - a_{3lm}^\dagger(k) a_{2lm}(k)], \tag{2.10}$$

and

$$\begin{aligned} H_{I2} = \mu^{-3} \int_0^\infty \frac{k_1^2 dk_1}{2\omega_1(k_1)} \int_0^\infty \frac{k_2^2 dk_2}{2\omega_2(k_2)} \int_0^\infty \frac{k_3^2 dk_3}{2\omega_3(k_3)} \{ & \psi_N^\dagger a_{100}^\dagger(k_1) \psi_N \{ G_1(k_1;k_2,k_3) [a_{200}(k_2) a_{200}(k_3) + a_{300}(k_2) a_{300}(k_3)] \\ & + G_2(k_1;k_2,k_3) [a_{200}(k_2) a_{200}(k_3) - a_{300}(k_2) a_{300}(k_3)] + 2G_3(k_1;k_2,k_3) a_{200}(k_2) a_{300}(k_3) \\ & + 2G_4(k_1;k_2,k_3) a_{200}(k_2) a_{300}(k_3) \} + \psi_N^\dagger \{ G_1^*(k_1;k_2,k_3) [a_{200}^\dagger(k_3) a_{200}^\dagger(k_2) + a_{300}^\dagger(k_3) a_{300}^\dagger(k_2)] \\ & + G_2^*(k_1;k_2,k_3) [a_{200}^\dagger(k_3) a_{200}^\dagger(k_2) - a_{300}^\dagger(k_3) a_{300}^\dagger(k_2)] - 2G_3^*(k_1;k_2,k_3) a_{300}^\dagger(k_3) a_{200}^\dagger(k_2) \\ & - 2G_4^*(k_1;k_2,k_3) a_{300}^\dagger(k_3) a_{200}^\dagger(k_2) \} a_{100}(k_1) \psi_N \}, \end{aligned} \tag{2.11}$$

and

$$[\psi_N,\psi_N^\dagger] = 1, \quad [a_{ilm}(k), a_{ilm}^\dagger(k')] = \delta_{ll'} \delta_{mm'} \delta_{ii'} \frac{2\omega_i(k)}{k^2} \delta(k-k') \tag{2.12}$$

with the metric

$$\eta = \exp\left(i\pi \int_0^\infty \frac{k^2 dk}{2\omega_2(k)} \sum_{lm} a_{3lm}^\dagger(k) a_{3lm}(k) \right). \tag{2.13}$$

We point out the $G_3(k_1,k_2,k_3)$ and $G_4(k_1,k_2,k_3)$ are symmetric and antisymmetric in k_2, k_3.

In order to investigate as completely as possible the unitarity of the model, we will have to use a solvable model. For this reason, when necessary, we will assume that $G_i(k_1;k_2,k_3)$ is separable,[10]

$$G_i(k_1;k_2,k_3) = B^i(k_1) A^i(k_2,k_3), \tag{2.14}$$

and that the form factor for the elastic channel is universal. This is the requirement that $B^i(k_1) = B(k_1)$. Although describing a model theory, and noting that this assumption will remove most of the unessential manipulative complications, this assumption has a very reasonable physical basis. Since the $B^i(k_1)$ describes the elastic-channel portion of the potential and this channel is common to all final states, it is not unreasonable to require it to factorize in a universal fashion.

[10] For a complete summary of the techniques of solution for separable potential theories see E. C. G. Sudarshan, in *Lectures in Theoretical Physics, Brandeis Summer Institute, 1961* (W. A. Benjamin, Inc., New York, 1962).

III. STATE VECTORS OF SINGLE-PARTICLE SECTORS

The state vectors of this model up to the production sector are quite simple and, since the Hamiltonian was chosen to minimize renormalization complications, most of the one-particle sectors are the same as the free Fock-space state vectors. The only renormalization required in the one-particle states is due to the direct interaction of the negative- and positive-metric mesons.

The energy eigenstates for this one-particle sector follow from the Hamiltonian

$$H|\pi_2,\mathbf{k}\rangle = \omega_2(\mathbf{k})|\pi_2,\mathbf{k}\rangle - g_1\mu^{-1}|\pi_3,k\rangle \tag{3.1}$$

and

$$H|\pi_3,\mathbf{k}\rangle = \omega_3(\mathbf{k})|\pi_3,\mathbf{k}\rangle + g_1\mu^{-1}|\pi_2,\mathbf{k}\rangle. \tag{3.2}$$

There are two complex-energy eigenvalues having the values

$$H|E_+,\mathbf{k}\rangle = [\omega_2(\mathbf{k}) + ig_1\mu^{-1}]|E_+\mathbf{k}\rangle \tag{3.3}$$

and

$$H|E_-,\mathbf{k}\rangle = [\omega_2(\mathbf{k}) - ig_1\mu^{-1}]|E_-,k\rangle, \tag{3.4}$$

where

$$|E_+\mathbf{k}\rangle = (\sqrt{\tfrac{1}{2}})(|\pi_2\mathbf{k}\rangle + i|\pi_3\mathbf{k}\rangle) \tag{3.5}$$

and

$$|E_-,\mathbf{k}\rangle = (\sqrt{\tfrac{1}{2}})(|\pi_2\mathbf{k}\rangle - i|\pi_3\mathbf{k}\rangle), \tag{3.6}$$

As is always the case in theories with a negative metric, the pseudo-Hermitian Hamiltonian admits complex-energy eigenvalues. As also follows from general arguments, these complex-energy states have norm zero:

$$\langle E_+\mathbf{k}|\eta|E_+\mathbf{k}'\rangle = 0 = \langle E_-\mathbf{k}|\eta|E_-\mathbf{k}'\rangle, \quad \langle E_+\mathbf{k}|\eta|E_-\mathbf{k}'\rangle = 2\omega_2(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}'). \tag{3.7}$$

All of these arguments are also applicable to the spherical states. These are

$$H|E_+,klm\rangle = [\omega_2(k)+ig_1\mu^{-1}]|E_+klm\rangle \tag{3.8}$$

and

$$H|E_-mklm\rangle = [\omega_2(k)-ig_1\mu^{-1}]|E_-,klm\rangle, \tag{3.9}$$

where

$$|E_\pm klm\rangle = (\sqrt{\tfrac{1}{2}})(|\pi_2klm\rangle \pm i|\pi_3klm\rangle), \tag{3.10}$$

with

$$\langle E_+klm|\eta|E_+k'l'm'\rangle = 0 = \langle E_-klm|\eta|E_-k'l'm\rangle, \quad \langle E_+klm|\eta|E_-k'l'm'\rangle = 2\omega_2(k)\frac{\delta(k-k')}{k^2}\delta_{ll'}\delta_{mm'}. \tag{3.11}$$

Before proceeding further, we make several modifications to our Hamiltonian. The operators

$$A_{2lm}(k) \equiv (\sqrt{\tfrac{1}{2}})[a_{2lm}(k)-ia_{3lm}(k)], \tag{3.12}$$

$$A_{3lm}(k) \equiv (\sqrt{\tfrac{1}{2}})[a_{2lm}(k)+ia_{3lm}(k)], \tag{3.13}$$

with their adjoints defined in the usual fashion, satisfy the usual commutation relations,

$$[A_{2lm}(k), A_{2l'm'}{}^\dagger(k')] = [A_{3lm}(k), A_{3l'm'}{}^\dagger(k')] = [2\omega_2(k)/k^2]\delta(k-k')\delta_{ll'}\delta_{mm'}, \tag{3.14}$$

with all others zero. The Hamiltonian in terms of these operators is

$$H_0 = m_N\psi_N{}^\dagger\psi_N + \int_0^\infty \frac{k^2dk}{2\omega_1(k)}\omega_1(k)\sum_{lm} a_{1lm}{}^\dagger(k)a_{1lm}(k) + \int_0^\infty \frac{k^2dk}{2\omega_2(k)}\omega_2(k)\sum_{lm}[A_{2lm}{}^\dagger(k)A_{2lm}(k)+A_{3lm}{}^\dagger(k)A_{3lm}(k)], \tag{3.15}$$

$$H_{I1} = ig_1\mu^{-1}\int_0^\infty \frac{k^2dk}{2\omega_2(k)}\sum_{lm}[A_{2lm}{}^\dagger(k)A_{2lm}(k)-A_{3lm}{}^\dagger(k)A_{3lm}(k)], \tag{3.16}$$

and

$$\begin{aligned}H_{I2} = \mu^{-3}\int_0^\infty \frac{k_1^2dk_1}{2\omega_1(k_1)}\int_0^\infty \frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty \frac{k_3^2dk_3}{2\omega_3(k_3)}\{&\psi_N{}^\dagger a_{100}{}^\dagger(k_1)\psi_N\{G_1(k_1;k_2,k_3)[A_{200}(k_2)A_{300}(k_3)+A_{300}(k_2)A_{200}(k_3)]\\
&+G_2(k_1;k_2,k_3)[A_{200}(k_2)A_{200}(k_3)+A_{300}(k_2)A_{300}(k_3)]+iG_3(k_1;k_2,k_3)[A_{200}(k_2)A_{200}(k_3)-A_{300}(k_2)A_{300}(k_3)]\\
&+iG_4(k_1;k_2,k_3)[A_{300}(k_2)A_{200}(k_3)-A_{200}(k_2)A_{300}(k_3)]\}+\psi_N{}^\dagger\{G_1{}^*(k_1;k_2,k_3)[A_{300}{}^\dagger(k_3)A_{200}{}^\dagger(k_2)\\
&+A_{200}{}^\dagger(k_3)A_{300}{}^\dagger(k_2)]+G_2{}^*(k_1;k_2,k_3)[A_{200}{}^\dagger(k_3)A_{200}{}^\dagger(k_2)+A_{300}{}^\dagger(k_3)A_{300}{}^\dagger(k_2)]+iG_3{}^*(k_1;k_2,k_3)\\
&\times[A_{200}{}^\dagger(k_3)A_{200}{}^\dagger(k_2)-A_{300}{}^\dagger(k_3)A_{300}{}^\dagger(k_2)]+iG_4{}^*(k_1;k_2,k_3)\\
&\times[A_{200}{}^\dagger(k_3)A_{300}{}^\dagger(k_2)-A_{300}{}^\dagger(k_3)A_{200}{}^\dagger(k_2)]\}a_{100}\psi_N\}.\end{aligned} \tag{3.17}$$

The metric operator in these new variables is

$$\eta = \exp\left\{\tfrac{1}{2}i\pi\int_0^\infty \frac{k^2dk}{2\omega_2(k)}\sum_{lm}[A_{2lm}{}^\dagger(k)A_{2lm}(k)-A_{3lm}{}^\dagger(k)A_{2lm}(k)-A_{2lm}{}^\dagger(k)A_{3lm}(k)+A_{3lm}{}^\dagger(k)A_{3lm}(k)]\right\}. \tag{3.18}$$

In this form it is apparent that H_0+H_{I1} form a new kinetic energy Hamiltonian with the variables associated with the metric operator having complex energy eigenvalues.

Splitting the metric operator into two commuting parts, we find

$$\eta = \eta_1\eta_2 = \exp\left\{\tfrac{1}{2}i\pi\int_0^\infty \frac{k^2dk}{2\omega_2(k)}\sum[A_{2lm}{}^\dagger(k)A_{2lm}(k)+A_{3lm}{}^\dagger(k)A_{3lm}(k)]\right\} \times\exp\left\{-\tfrac{1}{2}i\pi\int_0^\infty \frac{d^2dk}{2\omega_2(k)}\sum_{lm}[A_{3lm}{}^\dagger(k)A_{2lm}(k)+A_{2lm}{}^\dagger(k)A_{3lm}(k)]\right\}, \tag{3.19}$$

In this form the nondiagonal character of the metric is apparent. The η_1 portion of the metric is $(i)^{N(E+)+N(E-)}$. The i is necessary to recover the phase change in η_2. The metric η_2 is $\eta_2|E_+,klm\rangle = -i|E_-klm\rangle$. As before, the metric commutes with G_1 and G_2 parts of H_{I2} and anticommutes with the G_3 and G_4 parts.

Before leaving this sector, we note that the one-particle completeness relations in this sector require the presence of the complex-energy states. This may appear to cause difficulties in going to large positive and negative times. In order to interpret this theory, this difficulty must not exist in the physically relevant matrix elements of any operator. The time development of the expectation value of any operator in the two-dimensional space spanned by states with energies E_+ and E_- is given as

$$\langle A,t|\eta\mathcal{O}|B,t\rangle=a_+{}^*\mathcal{O}_{-+}b_+e^{-2\mathrm{Im}E+t}+a_+{}^*\mathcal{O}_{--}b_- +a_-{}^*\mathcal{O}_{++}b_++a_-{}^*\mathcal{O}_{+-}b_-e^{2\mathrm{Im}E+t}, \quad (3.20)$$

where

$$a_\pm=\langle E_\pm|A,0\rangle, \quad b_\pm\equiv\langle E_\pm|B,0\rangle, \quad (3.21)$$

and

$$\mathcal{O}_{\pm\pm}\equiv\langle E_\pm|\mathcal{O}|E_\pm\rangle. \quad (3.22)$$

Thus only operators with $\mathcal{O}_{+-}=\mathcal{O}_{-+}=0$ are asymptotically reasonable, and these are stationary. These are, as usual, the operators that are simultaneously diagonal with the Hamiltonian. The problem here is that they give zero for pure energy eigenstates. This is a direct result of the pseudo-Hermiticity of the negative-metric Hamiltonians. This would appear to imply the undetectability of these single-particle asymptotic-energy eigenstates in any normal context. From this nondetectability of the single-particle states, it does not follow that one cannot detect single-particle operators in the many-particle portions of the spectrum. In cases which will be produced by this Hamiltonian, a suitable combination of two complex-meson spectra will have states in which each single particle can be detected. This problem will be discussed in more detail in that section.

IV. SCATTERING STATES

The next higher sector of interest is the $N+\pi_1\to N+\pi_1$ scattering sector. The Hamiltonian was chosen to force this elastic channel to couple directly to the mixed-metric production sector. The assumption of the G_i depending only on k_i allows us to study only the scattering and production of S-wave mesons in the nucleon rest frame. The states in this sector are expanded in the spherical basis in the states

$$|N;k_1,0,0\rangle,\quad |N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_+\equiv|1,k_1,k_2\rangle,\quad |N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_-\equiv|4,k_2,k_3\rangle,$$
$$|N;E_2(k_2)0,0;E_3(k_3),0,0\rangle_+\equiv|2,k_2,k_3\rangle,\quad \text{and}\quad |N;E_2(k_2)0,0;E_3(k_3),0,0\rangle_-\equiv|3,k_2,k_3\rangle,$$

where

$$|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_\pm=\tfrac{1}{2}[|N,E_+(k_2),0,0;E_-(k_3),0,0\rangle\pm|N,E_+(k_3),0,0;E_-(k_2),0,0\rangle] \quad (4.1)$$

and

$$|N;E_2(k_2),0,0;E_3(k_3),0,0\rangle_\pm=(\sqrt{\tfrac{1}{2}})[|N,E_+(k_2),0,0;E_+(k_3),0,0\rangle\pm|N,E_-(k_3),0,0;E_-(k_2),0,0\rangle]. \quad (4.2)$$

This set of states are eigenstates of the metric operator. They satisfy

$$\eta|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_\pm=\pm|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_\pm \quad (4.3)$$

and

$$\eta|N;E_2(k_2),0,0;E_3(k_3),0,0\rangle_\pm=\pm|N;E_2(k_2),0,0;E_3(k_3),0,0\rangle_\pm, \quad (4.4)$$

which implies the norm

$$_\pm\langle N;E_+(k_2'),0,0;E_-(k_3'),0,0|\eta|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_\pm$$
$$=\pm\frac{1}{2}\left[\frac{2\omega_2(k_2)}{k_2^2}\delta(k_2-k_2')\frac{2\omega_3(k_3)}{k_3^2}\delta(k_3-k_3')\pm\frac{2\omega_2(k_2)}{k_2^2}\delta(k_2-k_3')\frac{2\omega_2(k_3)}{k_3^2}\delta(k_3-k_2')\right]$$
$$={}_\pm\langle N;E_2(k_2'),0,0;E_3(k_3'),0,0|\eta|N;E_2(k_2),0,0;E_3(k_3),0,0\rangle_\pm, \quad (4.5)$$

where the relative minus occurs only for $|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_-$ and with all other combinations zero. The state $|N;E_+(k_2),0,0;E_-(k_3),0,0\rangle_\pm$ has real-energy eigenvalues and positive and negative norm. The simplicity of the state $|N,E_+(k_2),0,0,E_-(k_3),0,0\rangle_-$ implies the existence of states that violate the Lee-Wick conjecture. In all the subsequent calculations, we shall suppress the redundant $l=0$, $m=0$ indices.

The scattering wave function with outgoing-wave boundary conditions is given by

$$|Nk_1\rangle^{\mathrm{out}}=\int_0^\infty\frac{k^2dk}{2\omega_2(k)}\phi_+(k_1,k)|Nk\rangle+\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}[\psi_+{}^1(k_1;k_2k_3)|N;E_+(k_2),E_-(k_3)\rangle_+$$
$$+\psi_+{}^2(k_1;k_2k_3)|N,E_2(k_2),E_3(k_3)\rangle_++\psi_+{}^3(k_1;k_2k_3)|N,E_2(k_2),E_3(k_3)\rangle_-$$
$$+\psi_+{}^4(k_1;k_2k_3)|N,E_+(k_2),E_-(k_3)\rangle_-], \quad (4.6)$$

where

$$\phi_+(k_1,k)=[2\omega_1(k)/k^2]\delta(k_1-k)+(\text{outgoing wave})$$
$$\equiv[2\omega_1(k)/k^2]\delta(k_1-k)+\eta_+(k_1,k) \quad (4.7)$$

and[11]

$$\psi_+{}^i(k_1;k_2,k_3)=(\text{outgoing wave}). \tag{4.8}$$

The Hamiltonian equation requires

$$[\omega_1(k_1)-\omega_1(k)]\phi_+(k_1,k)=\mu^{-3}\sqrt{2}\int_0^\infty\frac{k_2{}^2dk}{2\omega_2(k_2)}\int_0^\infty\frac{k_3{}^2dk_3}{2\omega_3(k_3)}[\psi_+{}^1(k_1;k_2k_3)G_1(k;k_2,k_3)+\psi_+{}^2(k_1;k_2,k_3)G_2(k;k_2k_3)$$
$$+i\psi_+{}^3(k_1;k_2k_3)G_3(k;k_2k_3)-i\psi_+{}^4(k_1;k_2k_3)G^4(k;k_2k_3)] \tag{4.9}$$

and

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]\psi_+{}^1(k_1;k_2k_2)=\mu^{-3}\int_0^\infty\frac{k^2dk}{2\omega_1(k)}\sqrt{2}G_1{}^*(k;k_2k_3)\phi_+(k_1,k), \tag{4.10}$$

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]\psi_+{}^2(k_1;k_2k_3)+2ig_1\mu^{-3}\psi_+{}^3(k_1;k_2k_3)=\mu^{-3}\int_0^\infty\frac{k^2dk}{2\omega_1(k)}\sqrt{2}G_2{}^*(k;k_2k_3)\phi_+(k_1,k), \tag{4.11}$$

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]\psi_+{}^3(k_1;k_2k_3)+2ig_1\mu^{-1}\psi_+{}^2(k_1;k_2k_3)=\mu^{-3}\int_0^\infty\frac{k^2dk}{2\omega_1(k)}i\sqrt{2}G_3{}^*(k;k_2k_3)\phi_+(k_1,k), \tag{4.12}$$

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]\psi_+{}^4(k_1;k_2k_3)=\mu^{-3}\int_0^\infty\frac{k^2dk}{2\omega_1(k)}(-i)\sqrt{2}G_4{}^*(k;k_2k_3)\phi_+(k_1,k). \tag{4.13}$$

Utilizing the separability assumption, we define

$$\gamma_+{}^j(k_1)\equiv\int_0^\infty\frac{k_2{}^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3{}^2dk_3}{2\omega_3(k_3)}A^j(k_2k_3)\psi_+{}^j(k_1,k_2k_3) \tag{4.14}$$

and

$$C_+(k_1)\equiv\int_0^\infty\frac{k^2dk}{2\omega_1(k)}B^*(k)\phi_+(k_1k). \tag{4.15}$$

The mixing of $\psi_+{}^2$ and $\psi_+{}^3$ is removed by studying

$$\psi_+{}^{(\pm)}(k_1;k_2,k_3)\equiv\psi_+{}^2(k_1;k_2,k_3)\pm\psi_+{}^3(k_1;k_2,k_3). \tag{4.16}$$

With substitution and some straightforward algebra, the middle two Hamiltonian equations are replaced by

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+2ig_1\mu^{-1}]\psi_+{}^{(+)}(k_1;k_2,k_3)=\mu^{-3}\sqrt{2}[A^{2*}(k_2k_3)+iA^{3*}(k_2k_3)]C_+(k_1) \tag{4.17}$$

and

$$[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)-2ig_1\mu^{-1}]\psi_+{}^{(-)}(k_1;k_2k_3)=\mu^{-3}\sqrt{2}[A^{2*}(k_2,k_3)-iA^{3*}(k_2,k_3)]C_+(k_1). \tag{4.18}$$

Since both coefficients of $\psi^{(\pm)}$ do not vanish for all physical k_1, k_2, and k_3, the solutions are directly given. The $\psi_+{}^1(k_1;k_2,k_3)$ and $\psi_+{}^4(k_1;k_2k_3)$ have to satisfy the outgoing-wave condition. This outgoing-wave condition is imposed on the entire state and is due to the fact that its energy denominator has a zero in the physical region. The method for developing the outgoing-wave condition is the classic one and is described by Dirac.[11] We note that this prescription is not modified by the presence of complex-mass states, since it is a condition on the total outgoing state and does not depend on the nature of the constituent particles. This is not the same as the $i\epsilon$ utilized in a perturbation expansion of an interaction-picture solution. The $i\epsilon$ used in that case is intrinsically related to the nature of the constituent particles. In this model which is exactly soluble, we avoid these possible ambiguities.

The solutions for the coefficients are

$$\phi_+(k_1,k)=\frac{2\omega_1(k)}{k^2}\delta(k-k_1)+\sum_j\frac{\mu^{-3}\sqrt{2}}{\omega_1(k_1)-\omega_1(k)+i\epsilon}\{\eta^jB(k)\gamma_+{}^j(k_1)\}, \tag{4.19}$$

where $\eta^j=1,1,i,-i$ for $j=1,2,3,4$, respectively, and

$$\psi_+{}^1(k_1;k_2k_3)=\frac{\mu^{-3}\sqrt{2}C_+(k_1)A^{1*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon}, \tag{4.20}$$

[11] The general procedure for the construction of outgoing and incoming waves is given in P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1958), 4th ed., pp. 195–198.

$$\psi_+{}^2(k_1;k_2k_3)=\tfrac{1}{2}[\psi_+{}^{(+)}(k_1;k_2k_3)+\psi_-{}^{(-)}(k_1;k_2,k_3)]$$
$$=\frac{\mu^{-3}\sqrt{2}}{2}\left[\frac{A^{2*}(k_2,k_3)+iA^{3*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+2ig_1\mu^{-1}}+\frac{A^{2*}(k_2,k_3)-iA^{3*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)-2ig_1\mu^{-1}}\right]C_+(k_1),\quad(4.21)$$

$$\psi_+{}^3(k_1;k_2k_3)=\tfrac{1}{2}[\psi_+{}^{(+)}(k_1;k_2,k_3)-\psi_+{}^{(-)}(k_1;k_2k_3)]$$
$$=\frac{\mu^{-3}\sqrt{2}}{2}\left[\frac{A^{2*}(k_2k_3)+iA^{3*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+2ig_1\mu^{-1}}-\frac{A^{2*}(k_2,k_3)-iA^{3*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)-2ig_1\mu^{-1}}\right]C_+(k_1),\quad(4.22)$$

$$\psi_+{}^4(k_1;k_2,k_3)=\mu^{-3}\sqrt{2}(-i)\frac{C_+(k_1)A^{4*}(k_2k_3)}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon}.\quad(4.23)$$

The γ separability coefficients are given by

$$\gamma_+{}^j(k_1)=\mu^{-3}\sqrt{2}\sum_{j'}\rho_+{}^{jj'}(k_1)\eta^{j'}C_+(k_1),\quad(4.24)$$

where

$$\rho_+{}^{11}=\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A^1(k_2k_3)|^2}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon},\quad(4.25)$$

$$\rho_+{}^{22}=\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A^2(k_2k_3)|^2\omega_1(k)-\omega_2(k_2)-\omega_3(k_3)}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}},\quad(4.26)$$

$$\rho_+{}^{23}=-i\int_0^\infty\frac{k_2^2dk^2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{A^2(k_2,k_3)A^{3*}(k_2k_3)2g_1\mu^{-1}}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}},\quad(4.27)$$

$$\rho_+{}^{32}=-i\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{A^3(k_2,k_3)A^{2*}(k_2,k_3)2g_1\mu^{-1}}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}},\quad(4.28)$$

$$\rho_+{}^{33}=\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A^3(k_2,k_3)|^2[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}},\quad(4.29)$$

$$\rho_+{}^{44}=\int_0^\infty\frac{k_2^2dk_2}{2\omega_2(k_2)}\int_0^\infty\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A^4(k_2,k_3)|^2}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon},\quad(4.30)$$

and all others zero.

The C_+ separability coefficient is the solution of

$$C_+(k_1)=B^*(k_1)+\mu^{-6}2\sum_{j'j''}\int_0^\infty\frac{k^2dk}{2\omega_1(k)}\frac{B^*(k)B(k)}{\omega_1(k_1)-\omega(k)+i\epsilon}\eta^{j''}\rho_+{}^{j''j'}(k_1)\eta^{j'}C_+(k).\quad(4.31)$$

The inversion of this system is expressed simply in terms of

$$h_+(\omega(k_1))\equiv1-\mu^{-6}2\int\frac{k^2dk}{2\omega_1(k)}\frac{|B(k)|^2}{\omega_1(k_1)-\omega_1(k)+i\epsilon}[\sum_{jj'}\eta^j\rho_+{}^{jj'}(k_1)\eta^{j'}],\quad(4.32)$$

with

$$\sum_{jj'}\eta^j\rho_+{}^{jj'}(k_1)\eta^{j'}=\left\{\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A^1(k_2k_3)|^2}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon}+\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}\right.$$
$$\times\frac{|A^2(k_2k_3)|^2\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}}+\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{A^3(k_2k_3)A^{2*}(k_2k_3)2g_1\mu^{-1}}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}}$$
$$-\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{A^2(k_2k_3)A^{3*}(k_2k_3)2g_1\mu^{-1}}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}}-\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}$$
$$\left.-\frac{|A^3(k_2k_3)|^2\omega_1(k_1)-\omega_2(k_2)-\psi_3(k_3)}{[\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)]^2+4g_1^2\mu^{-2}}-\int\frac{k_2^2dk_2}{2\omega_2(k_2)}\int\frac{k_3^2dk_3}{2\omega_3(k_3)}\frac{|A_4(k_2k_3)|^2}{\omega_1(k_1)-\omega_2(k_2)-\omega_3(k_3)+i\epsilon}\right\}.\quad(4.33)$$

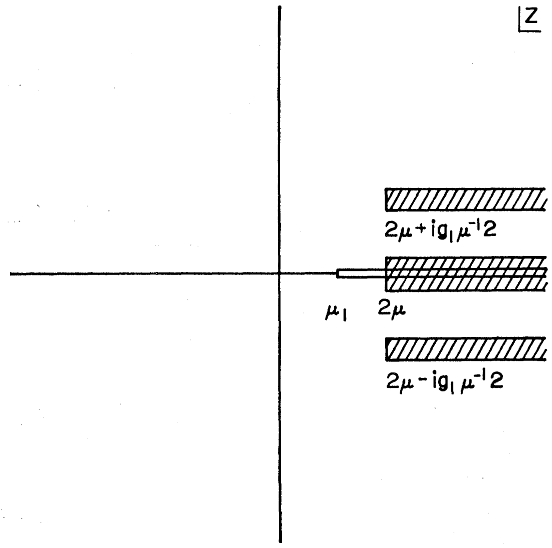

FIG. 1. Cut structure of $h(z)$.

The function $h_+(\omega(k_1))$ is the characteristic function of this sector. Most of our conclusions about this model can be seen directly in $h_+(\omega(k_1))$. The solution associated with zeros of $h_+(\omega_1(k_1))$ for $\omega_1(k_1)$ below any of the cuts of the spectral functions is a bound state. As is usual with negative-metric theories, $\text{Im}h(\omega_1(k_1))$ below the production sector is not of positive-definite value and, therefore, by suitable choice of strengths in G_1, G_2, G_3, and G_4, complex-mass bound states can be generated. For simplicity, we choose strengths that generate no new bound states. The analytic structure of the $h(z)$ which reduces to $h_+(\omega_1(k_1))$ when $z\to\omega_1(k_1)+i\epsilon$ determines the analytic structure of the S matrix. As is always the case with separable potentials, there are no left-hand cuts in z. The absence of bound states and complex poles removes all poles from the physical sheet. The cuts from the production terms drive the amplitude (see Fig. 1). Each term within the curly brackets contributes to both the elastic and the production cuts. We will discuss each term separately and in doing so will assume that the other nonrelevant terms are zero. The complete solution is displayed later and it can be verified that the omitted effects do not alter our conclusions about each part.

The first term in the curly brackets of $h(z)$ is due to $G_1(k_1k_2k_3)$. This term in the Hamiltonian is responsible for the direct production of a meson pair with positive metric. This can be seen from the Hamiltonian (2.4) and metric (2.6) or, equivalently, the Hamiltonian (3.7) and the metric (3.19). From the form of (3.17), those mesons are produced which have complex conjugate energies or a net real energy. This real energy of the mesons of this production sector is reflected in $h(z)$ by associating a production cut along the real axis with this term. To a large extent this term in the scattering states might be called the most normal. If the Hamiltonian contained only G_1 and no other production terms, the S matrix would be unitary, all cuts would be real-axis cuts, and only positive-metric states would exist. The unitarity of the resultant S matrix will be seen when all terms except G_1 are set to zero. We note here that the sign of this term in $h(z)$ is crucial to the unitarity of this portion of the solution. Where $\omega_1(k_1)$ is above the production threshold, $h(z)$ has a negative-imaginary part which is absorptive and reduces the S matrix. Imaginary parts of opposite sign violate unitarity. In addition, we see that for energy below the production threshold, the virtual production acts as an effective attractive potential. This does not mean that this portion of the scattering is normal. The unitarity is inelastic unitarity and some of the beam is going to production of meson pairs. But the mesons of these pairs are not observable as single-particle states. In other words, a normal detection of coincident pairs is possible, but no single-particle production (anticoincidence) detection of these particles would be possible. If we add a term to our Hamiltonian to produce just π_2 or π_3, this single particle would not be detectable in any apparatus. This problem was described earlier and when compared with this production result would certainly lead to an anomalous interpretation of the nature of this particle. The astute reader may now ask why, when we added π_2 or π_3 single production, no violation of unitarity resulted. The addition of this term will not affect the unitarity of this sector. This result is quite general and is due to energy conservation. Since this problem arises in the next term, we will describe it in more detail there.

The next term is another almost normal term and it owes all of its uniqueness to the presence of g_1. This driving term is due to $G_2(k_1;k_2k_3)$, which is also a contribution that produces two mesons in a positive-metric configuration. The presence of g_1 forces these mesons to have complex energies with imaginary parts $2g_1\mu^{-1}$. This term has two symmetric complex cuts as its production cuts. The only real-axis cut is due to elastic unitarity. Since unitarity sees only the real cuts and the elastic cut satisfies unitarity, there is no violation of unitarity in this term. Below production threshold, this term is an effective attractive potential. This is again due to the positive metric. This complex-energy state has the same interpretation difficulties described earlier.

Skipping the next two terms, the fifth term in the curly brackets is very similar in nature to the second term. It has complex-energy cuts and does not violate unitarity. In this case the complex-energy mesons have a negative metric. This is exactly the case of Lee and Wick. Here the energy conservation requires that the only cut contribution be the elastic-unitarity cut of this term. One manifestation of the negative metric of this effective repulsive potential due to this term. This is the only real difference between this and the second term discussed above. In the limit $g_1 \to 0$, the complex cut becomes real and this term would lead to a factor which above the production would have a typical negative-metric violation of unitarity. This would not be a violation that would disagree with the requirement of Lee and Wick since the theory would no longer have complex-energy particles. In this limit the second term does not violate unitarity.

The third and fourth terms are the cross terms of the previous two effects. They do not violate unitarity and vanish if $g_1 \to 0$.

The sixth term is the term that directly violates Lee and Wick's requirements. This factor is driven by G_4, which produces two mesons with total energy real. The mesons are in a negative-metric configuration. The negative metric is manifest by the sign of this term. The force is repulsive and below the production threshold the elastic unitarity holds. Above production threshold, the imaginary part of this term would tend to drive the amplitude above the unitarity limit.

V. ELASTIC TRANSITION MATRIX

The t matrix for the elastic scattering amplitude is easily constructed and we obtain

$$t(N+\pi_1(k_1) \to N+\pi_1(k_1')) \equiv t(\omega(k_1),\omega(k_1')), \quad (5.1)$$

with

$$t(\omega(k_1),\omega(k_1)) = \mu^{-6}2\{|B(k_1)|^2[\sum_{jj'} \eta^j \rho_+^{jj'}(k_1)\eta^{j'}]\}/h_+(\omega_1(k_1)). \quad (5.2)$$

Below the production threshold this matrix is unitary, since

$$t(\omega(k_1),\omega(k_1)) = \frac{\mathrm{Im}h_+(\omega_1(k_1))}{h_+(\omega_1(k_1))}. \quad (5.3)$$

This is directly due to the fact that $\rho_+^{jj'}$ has purely real contributions in this region.

If $\omega_1(k_1)$ increases above the production threshold, $\rho_+^{jj'}$ contributes an imaginary part. The imaginary part of $\sum_{jj'} \eta^j \rho_+^{jj'}(k_1)\eta^{j'}$ comes from the ρ_+^{11} and ρ_+^{44} terms. The sign of this factor is determined by $(\eta^j)^2$. For $j=1$, we get the usual type of absorption seen in normal production processes. The imaginary part is negative and reduces the modulus of the elastic S matrix. For $j=4$, we get a positive-imaginary part which in turn forces the elastic amplitude to increase out of the unitarity circle in the Argand diagram.

Another and more obvious problem with these models is, of course, the interpretation problem. Obviously, once $\omega_1(k_1)$ is above the production threshold, the S matrix is no longer "elastic unitary." If the channel is a positive-norm channel, one still has a normal absorption, but must be able to describe completely the strange two-meson final states. If one has strong negative-metric scattering, one will have both a strange S matrix and a strange state to interpret.

We emphasize once again why the simpler calculation of Lee and Wick yielded a positive result while we have a much more complex situation. The ρ^{22}, ρ^{23}, ρ^{32}, and ρ^{33} terms in the t matrix, although strange, do not possess a real-axis cut and therefore are decoupled from unitarity. These are the terms that scatter with complex energy, and the conservation of energy associated with crossing the real-axis cut removes the effects of these terms. We note that although they do not violate unitarity, they do act as driving terms and their virtual production is possible.

In order to clarify our results, we will interpret this model in terms of a much simpler two-body elastic scattering model. Assuming a model with only N and π_1 scattering, we can produce the above t matrix at a given $\omega_1(k_1)$ by using the Hamiltonian

$$H_{\omega_1(k_1)} = H_0 + H_I(\omega_1(k_1)), \quad (5.4)$$

where

$$H_0 = m\psi_N^+\psi_N + \int \frac{d^3k}{2\omega_1(k)}\omega_1(k)a_1^+(k_2)a(k_2) \quad (5.5)$$

and

$$H_I(\omega_1(k_1)) = g(\omega_1(k_1)) \int \frac{d^3k}{2\omega_1(k)} \int \frac{d^3k'}{2\omega_1(k')} \times \psi_N^+ a_1^+(k)B(k)B^*(k')\psi_N a_1(k'). \quad (5.6)$$

In the usual formulations of separable potential scattering, $g(\omega_1(k_1))$ is a constant and is designated the coupling strength. Here, by allowing g to depend on $\omega_1(k_1)$, we reproduce the elastic sector of the previous model at energy $\omega_1(k_1)$, making the identification

$$g(\omega(k_1)) = \sum_{jj'} \eta^j \rho_+^{jj'}(k)\eta^{j'}. \quad (5.7)$$

The function $g(z)$ for which

$$\lim_{z\to\omega_1(k)+i\epsilon} g(z) = g(\omega_1(k_1))$$

has very similar analytic structure to that of $h(z)$. The only significant difference is that $g(z)$ does not have an elastic-unitarity cut (see Fig. 2). We can now make all of the usual statements about the properties of g in potential scattering.[12] If $\omega_1(k_1)$ is below the production threshold, the g utilized is real. Real g has a unitary S matrix and therefore so does our model. We see also that the sign of g determines whether or not a potential is attractive or repulsive. The positive-metric contributions are an effective attractive potential and the negative-metric contributions, by virtue of having negative g's, are repulsive. This result is actually the usual perturbation-theory result that any second-order perturbation lowers the energy of the lowest bound state. The production terms enter the elastic channel in

[12] Several examples of simple indefinite-metric models are described in E. C. G. Sudarshan, Phys. Rev. **123**, 2183 (1961); H. J. Schnitzer and E. C. G. Sudarshan, *ibid.* **123**, 2193 (1961).

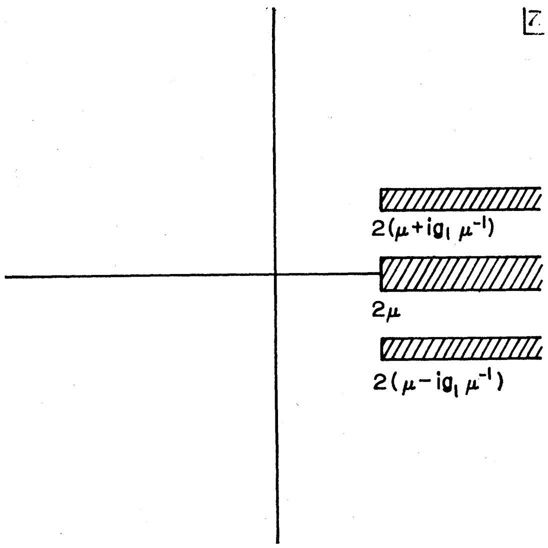

FIG. 2. Cut structure of $g(z)$.

the second order. If these production potentials were all Hermitian contributions, we would have only attraction. The negative-metric terms are anti-Hermitian or, in terms of potentials, pure imaginary. They are therefore repulsive in the second order.

Above the production threshold, $g(\omega_1(k_1))$ becomes complex from the A^1 and A^4 contributions. The simple model now looks like a typical optical model with complex potentials. In that case the potentials corresponding to absorption have positive-imaginary parts. Potentials with negative-imaginary parts violate unitarity. The positive-metric A^1 contribution has a negative-imaginary part and looks like a typical absorptive optical potential. The negative-metric A^4 has a positive-imaginary part and violates inelastic unitarity. The A^2 and A^3 contributions, although production terms, do not have a real-axis cut and, if only these terms were present, the associated g would be real and all scattering would be unitary. We note that these two terms correspond to the case of Lee and Wick. The sign of g from pure A^2 or pure A^3 Hamiltonians again follows directly from their respective Hermitian or anti-Hermitian value.

VI. DYNAMICAL DEFINITION OF PHYSICAL STATES

With the exact solution of this simple model available, we would like to return to the problem of the physical interpretation of the states in a theory with an indefinite metric.[7] The physical basis of such an interpretation has already been discussed elsewhere and is based on the recognition that the identification of physical states is a *dynamical* problem.[13] Unlike the proposal of Lee and Wick,[8] it does not arbitrarily exclude the contribution from complex-mass particles and is therefore not subject to the problems we have demonstrated in the previous sections. To carry out this program for this simple model, we note that the primary problems to be resolved is associated with the fact that when $\omega_1(k_1)$ rises above the threshold of the production channel, we see that the elastic scattering amplitude by itself does not account for the conservation of probability. We ought to take account of the three-particle channels made up of pairs of complex-energy particles. If the metric for these states is positive, the elastic scattering amplitude will have an absolute magnitude too small to satisfy unitarity; it would be the standard situation in multichannel scattering but for the fact that this three-particle state cannot be physically identified with a state that is made up of a nucleon and two particles. We would have to get used to the idea of having a continuum of masses for physical states *without* the state having physical *constituent particles*. (The situation would be similar to the quark picture in which multiquark states are identified without being able to identify the quark states.) It would be desirable, if possible, to have a physical interpretation in which all continuum states are analyzable into particles with well-defined discrete masses.

In the case in which the coupled three-particle sector has negative norm, we have no choice but to seek such a physical interpretation, since otherwise probability would not be conserved at all; the elastic amplitude in the model has an absolute magnitude too large to satisfy unitarity. We must either abandon the indefinite metric or develop a new physical interpretation.

We have already remarked that the identification of the physical particles is a dynamical problem. The identification is to be made so that probability would be conserved. Let us see to what extent such a choice of physical states can be made within our model.

For this purpose, it is necessary to construct the operator **R** which maps the "out" wave functions onto the "in" wave functions. This operator must include transitions from three-particle initial states into three-particle or two-particle final states. This construction is straightforward, but the resulting expressions look somewhat unwieldy, though basically they have a very simple structure. In order to simplify the resulting matrix, we will describe only real-energy scattering and thereby remove the external states associated with the G_2 and G_3 production elements. This *does not* remove the effects of the virtual production of these states. The virtual effects of G_2 and G_3 can be seen in the explicit form of the matrix below. We construct this operator by means of the usual defined **S** matrix. We

[13] This program was described originally in Ref. 12, applied to quantum electrodynamics in Ref. 6, and reviewed extensively in Ref. 7.

obtain for **S**

$$\langle\mathbf{S}\rangle\equiv\begin{pmatrix} {}^{\rm in}\langle N,k|\eta|N,k'\rangle^{\rm out} & {}^{\rm in}\langle N,k|\eta|1,k_2'k_3'\rangle^{\rm out} & {}^{\rm in}\langle Nk|\eta|4,k_2'k_3'\rangle^{\rm out} \\ {}^{\rm in}\langle 1,k_2k_3|\eta|Nk'\rangle^{\rm out} & {}^{\rm in}\langle 1,k_2k_3|\eta|1,k_2'k_3'\rangle^{\rm out} & {}^{\rm in}\langle 1,k_2k_3|\eta|4,k_2'k_3'\rangle^{\rm out} \\ {}^{\rm in}\langle 4,k_2k_3|\eta|Nk'\rangle^{\rm out} & {}^{\rm in}\langle 4,k_2k_3|\eta|1,k_2'k_3'\rangle^{\rm out} & {}^{\rm in}\langle 4,k_2k_3|\eta|4,k_2'k_3'\rangle^{\rm out}\end{pmatrix}, \tag{6.1}$$

with

$${}^{\rm in}\langle Nk|\eta|Nk'\rangle^{\rm out}=\frac{2\omega_1(k)}{k^2}\delta(k-k')-2\pi i\delta(\omega_1(k)-\omega_1(k'))\mu^{-6}2\frac{B^*(k)B(k')}{h_+(\omega_1(k))}[\sum_{j,'}\eta^j\rho_+{}^{jj'}(k)\eta^{j'}], \tag{6.2}$$

$${}^{\rm in}\langle 1,k_2k_3|\eta|1,k_2'k_3'\rangle^{\rm out}=\left[\frac{2\omega_2(k_2)}{k_2{}^2}\delta(k_2-k_2')\frac{2\omega_3(k_3)}{k_3{}^2}\delta(k_3-k_3')+\frac{2\omega_2(k_2)}{k_2{}^2}\delta(k_2-k_3')\frac{2\omega_2(k_2)}{k_3{}^2}\delta(k_2-k_2')\right]$$
$$-2\pi i\delta(\omega_2(k_2)+\omega_3(k_3)-\omega_2(k_2')-\omega_3(k_3'))\mu^{-6}\frac{2A^{1*}(k_2k_3)}{h_+(\omega_2(k_2)+\omega_3(k_3))}A^1(k_2'k_3')$$
$$\times\int\frac{k^2dk}{2\omega_1(k)}\frac{|B(k)|^2}{\omega_1(k_2)+\omega_3(k_3)-\omega_1(k)+i\epsilon}, \tag{6.3}$$

$${}^{\rm in}\langle 4,k_2k_3|\eta|4k_2'k_3'\rangle^{\rm out}=-\left[\frac{2\omega_2(k_2)}{k_2{}^2}\delta(k_2-k_2')\frac{2\omega_3(k_3)}{k_3{}^2}\delta(k_3-k_3')-\frac{2\omega_2(k_2)}{k_2{}^2}\delta(k_2-k_3')\frac{2\omega_3(k_3)}{k_3{}^2}\delta(k_3-k_2')\right]$$
$$-2\pi i\delta(\omega_2(k_2)+\omega_3(k_3)-\omega_2(k_2')-\omega_3(k_3'))\mu^{-6}\frac{2A^{4*}(k_2k_3)}{h_+(\omega_2(k_2)+\omega_3(k_3))}A^4(k_2'k_3')$$
$$\times\int\frac{k^2dk}{2\omega_1(k)}\frac{|B(k)|^2}{\omega_2(k_2)+\omega_3(k_3)-\omega_1(k)+i\epsilon}, \tag{6.4}$$

$${}^{\rm in}\langle Nk|\eta|1,k_2',k_3'\rangle^{\rm out}=-2\pi i\delta(\omega_1(k)-\omega_2(k_2')-\omega_3(k_3'))\frac{B(k)A^1(k_2'k_3')}{h_+(\omega_1(k_1))}\mu^{-3}\sqrt{2}, \tag{6.5}$$

$${}^{\rm in}\langle Nk|\eta|4,k_2'k_3'\rangle^{\rm out}=-2\pi\delta(\omega_1(k)-\omega_2(k_2')-\omega_3(k_3'))\frac{B(k)A^4(k_2'k_3')}{h_+(\omega_1(k))}\mu^{-3}\sqrt{2}, \tag{6.6}$$

$${}^{\rm in}\langle 1;k_2k_3|\eta|4,k_2'k_3'\rangle^{\rm out}=-2\pi\delta(\omega_2(k_2)+\omega_3(k_3)-\omega_2(k_2')-\omega_3(k_3'))$$
$$\times\mu^{-6}2\frac{A^{1*}(k_2k_3)A^4(k_2'k_3')}{h_+(\omega_2(k_2)+\omega_3(k_3))}\int\frac{k^2dk}{2\omega_1(k)}\frac{|B(k)|^2}{\omega_2(k_2)+\omega_3(k_3)-\omega_1(k)+i\epsilon}, \tag{6.7}$$

with all the others following from the pseudo-Hermiticity of the Hamiltonian. Because of this pseudo-Hermiticity, **S** does not satisfy unitarity.

The external channels associated with complex energies can be further simplified. To exhibit this, we form a quasi-two-particle state of energy $E=\omega_4(k)=(4\mu^2+k^2)^{1/2}$ from the three-particle states which is parallel to the produced state; in other words,

$$|N,k;3\rangle_{\rm II}{}^{\rm out}=|\Delta(\omega_4(k))|^{-1}\int\frac{k_2{}^2dk_2}{2\omega_2(k_2)}\int\frac{k_3{}^2dk_3}{2\omega_3(k_3)}\delta(\omega_4(k)-\omega_2(k_2)-\omega_3(k_3))$$
$$\times[A^{1*}(k_2,k_3)|1;k_2k_3\rangle^{\rm out}-iA^{4*}(k_2k_3)|4;k_2k_3\rangle^{\rm out}], \tag{6.8}$$

where

$$\Delta^2(\omega_4)=\tfrac{1}{2}k\int\frac{k_2{}^2dk_2}{2\omega_2(k_2)}\int\frac{k_3{}^2dk_3}{2\omega_3(k_3)}\delta(\omega_4(k)-\omega_2(k_2)-\omega_3(k_3))[|A^1(k_2k_3)|^2-|A^4(k_2k_3)|^2]. \tag{6.9}$$

The state orthogonal to $|Nk;3\rangle_{\text{II}}{}^{\text{out}}$ is formed with the interchange $A^{1*}\leftrightarrow iA^4$. We emphasize that these states are normalized in the physical indefinite metric. With these states, the **S** matrix is block-diagonalized to

$$\langle \mathbf{S}'\rangle=\begin{bmatrix} {}^{\text{in}}\langle Nk|\eta|Nk'\rangle^{\text{out}} & {}^{\text{in}}\langle Nk|\eta|Nk';3\rangle_{\text{II}}{}^{\text{out}} & 0 \\ {}^{\text{in}}\langle Nk;3|\eta|Nk'\rangle_{\text{II}}{}^{\text{out}} & {}^{\text{in}}\langle Nk;3|\eta|Nk';3\rangle_{\text{II}}{}^{\text{out}} & 0 \\ 0 & 0 & {}^{\text{in}}{}_{\perp}\langle Nk;3|\eta|Nk;3\rangle_{\perp}{}^{\text{out}} \end{bmatrix}, \tag{6.10}$$

where

$${}^{\text{in}}\langle Nk|\eta|Nk';3\rangle_{\text{II}}{}^{\text{out}}=-2\pi i\frac{B(k)}{h_+(\omega_1(k))}\delta(\omega_4(k)-\omega_4(k'))\frac{2}{k}\frac{\Delta^2(\omega_4(k'))\mu^{-3}}{|\Delta(\omega_4(k'))|}\sqrt{2}, \tag{6.11}$$

$${}^{\text{in}}{}_{\text{II}}\langle Nk;3|\eta|Nk';3\rangle_{\text{II}}{}^{\text{out}}=\frac{2}{k}\frac{\Delta^2(\omega_4(k))}{|\Delta(\omega_4(k))|^2}\delta(\omega_4(k)-\omega_4(k'))-2\pi i\delta(\omega_4(k)-\omega_4(k'))\mu^{-6}$$
$$\times 2\int\frac{k''^2dk''}{2\omega_1(k'')}\frac{|B(k'')|^2}{\omega_4(k)-\omega_1(k'')+i\epsilon}\left(\frac{2}{k}\right)^2\frac{\Delta^4(\omega_4(k))}{h_+(\omega_4(k))|\Delta(\omega_4(k))|^2}, \tag{6.12}$$

and a similar form for ${}^{\text{in}}{}_{\text{II}}\langle Nk;3|\eta|Nk';3\rangle_{\text{II}}{}^{\text{out}}$.

This last term is now decoupled from the beam channel and is therefore irrelevant to physically accessible scattering. We emphasize again that its effects, like those of the complex-energy three-body states, are not neglected. They appear virtually in the driving forces but they are not produced physically. We see that, for this very complex four-channel problem with which we started, the physically accessible scattering is restricted to two channels: the beam and the produced state. All other channels are necessary to drive the model virtually but are not observable. It may appear that, since all these states are not observable, there is no reason for incorporating them in the original Hamiltonian. As is apparent in this formulation, they have the effect of nonlocal potentials. In other words, by means of a simple "local" Hamiltonian, we have constructed a system with nonlocal effects. In addition, the state $|N,k;3\rangle_{\text{II}}{}^{\text{out}}$ is a state that, although composed of two-particle-plus-nucleon contributions in the local fields, appears as one particle. This interpretation correlates with the difficulty of interpreting these complex-energy states one at a time. To draw an analogy from the present "style" of physics, in the quark model, many physicists believe that, although the baryon and meson resonances are composed of multiquark states, there will never be a single quark or continuum of quark states discovered. In our case, the local fields cannot be isolated by physical experiments. They are necessary to completely describe with local fields the complex physical processes observed.

We have now isolated all physical problems to a two-by-two **S**-matrix problem. It is at this point that a generalization of the Lee-Wick program must be applied. This two-by-two process of one real particle and one quasiparticle (two complex-energy particles) must be physically interpretable. The physical norm of the quasiparticle state is

$$\xi\equiv\frac{\Delta^2(\omega_4(k))}{|\Delta(\omega_4(k))|^2}, \tag{6.13}$$

or the sign of the norm is the sign of $\Delta^2(\omega_1(k))$. We see therefore that this problem reduces to a simple two-by-two scattering problem with the second channel as either a positive- or negative-norm channel, depending on the relative strength of A^4 to A^1. In this model, the channel associated with the beam meson has several direct nonlocal potentials associated with the virtual states that have been decoupled. The coupling potential between the channels is $B(k)$ times the $\Delta(\omega_4(k'))$.

The case of A^4 dominating A^1 produces a pure imaginary effective coupling potential. The legitimacy of this result is brought about by the fact that this case obtains only when the quasiparticle has negative norm.

With the other sign of the norm, the quasiparticle in the quasiparticle nucleon state of $\omega_4(k)=(4\mu^2+k^2)^{1/2}$ does not have an associated single-particle contribution to which it can be related. It is our program to analyze the dynamical system and discover what superposition of particle and quasiparticle plus nucleon will be stationary (i.e., map "in" to "out" with only an over-all phase change) and have positive norm. To carry out this program in the remaining two-by-two subspace, we define the scattering operator **R**:

$$\mathbf{R}\equiv\mathbf{NS},$$

with

$$\mathbf{N}=\begin{pmatrix}1 & 0\\ 0 & \xi\end{pmatrix}.$$

It is this operator which generates the transformation of the scattering from the "in" wave function to the "out" wave function. This phase shift is the observable consequence of scattering, and we will require that this theory must generate an observable physical state with positive norm and unimodular phase shifts. This sentence describes both the conditions of physical states (positive norm and unimodular phase shift), and potential strengths admissible in interpretable theories (such that the model possesses a physical state). It can be shown that positive-norm eigenchannels of the scattering operator possess a real phase shift. In other words, we will analyze the two possible decoupled eigenchannels and define the physical state as that eigenchannel which possess a unimodular phase shift and positive norm as a one-channel process. This program is philosophically more complicated than it appears on the surface. We emphasize that the π_1 single-particle sector is the physical π_1 state only when it does not appear with a nucleon. The physical π_1-plus-nucleon state is the scattering eigenchannel that possesses a unitary scattering matrix and positive norm. In other words, the higher sectors of the scattering matrix are not formed from simple direct products of the lower-sector particle states; but, when quasi-particle states appear in the **S** matrix, the channel must be diagonalized further and the resulting physical states decoupled.

We now investigate in our simple case what conditions will obtain an interpretable theory. These conditions are directly related to the non-Hermitian terms of the Hamiltonian, and these are the off-diagonal elements.

In the Appendix we investigate the conditions under which a two-by-two matrix possesses unimodular eigenvalues. In terms of our new **R** matrix, these conditions can be applied directly. The reality condition follows directly from the form of **R**. The other conditions become

$$(1/h_+h_-)[(\alpha_+\beta_--\alpha_-\beta_+)^2+4\xi(\alpha_+-\alpha_-)(\beta_+-\beta_-)]<0,$$

where

$$\xi\alpha_+=\sum_{jj'}\eta^j\rho_+{}^{jj'}\eta^{j'},$$

$$\beta_+=\int\frac{|B(k')|^2dk'}{\omega_1(k)-\omega_1(k')+i\epsilon}.$$

The case of ξ positive, and therefore A^1 dominating A^4, obviously satisfies the criteria. The case of ξ negative is satisfied for broad classes of potential strengths. Physically, the results for $\xi=-1$ can be interpreted directly by noting that they reduce to the statement that diagonal elements must dominate. Theories characterized by production or decay (i.e., strong off-diagonal elements into negative-metric channels) will violate unitarity. Theories characterized by strong elastic processes of either metric will have unimodular eigenphases and be satisfactory. In the case of this model, the many virtual channels and any direct elastic potentials could be used to enable the eigenphases to become unimodular.

The norm of the channel that is open below the production threshold is positive. This channel connects smoothly to the positive-norm eigenchannel above threshold for production. The same condition that assured us that the eigenphases were real implies that they are distinct. This ensures that the norms of these two eigenchannels are steady independently of the energy. This enables us to make a unique identification of the physical scattering state.

VII. CONCLUSIONS

In this paper we have attempted to produce a model that is sufficiently complex to be interesting but still exactly solvable. In this model we have produced an amplitude that contains the properties of a possible "good" theory by the requirements of Lee and Wick in a simple sector, but which in higher sectors has all the usual difficulties of negative-metric theories. This problem is compounded by the interpretation problem for the complex-energy states that we incorporated to cure the sickness caused by the negative metric.

We feel that this model is sufficiently realistic to allow us to draw the following inferences:

(i) The scattering matrix in a theory involving an indefinite metric is, in general, pseudounitary even if we arrange to have all stable particle states to be of positive squared length.

(ii) We must therefore deal with a scattering matrix involving transitions to negative-norm states, and solve the dynamical problems of identifying the proper physical states and of defining physical-particle operators.

(iii) We must construct states that have properties dictated by the **R** matrix in that sector. The dynamical problem is the selection of those states with a unimodular phase shift. In order to accomplish this result, one must construct states of unobservable constituent particles. Particle physicists may find a similarity between this situation and the quark picture in which multiquark and quark-antiquark states are identified without being able to identify quark states.

While these are the main results of our study, the model itself may be of interest since it illustrates much of the kinematics of production amplitudes. This model may provide a valuable tool for the investigation of the structures that arise when production channels are present. In a very real sense, it has all the features that unitarity in the production region can provide.

APPENDIX

In this appendix we describe the conditions under which a two-by-two matrix will have unimodular eigenvalues. The necessary conditions are established directly. We designate the two eigenvalues λ_1 and λ_2. If λ_1 and λ_2 are unimodular, they can be written

$\lambda_1 = e^{i\delta}e^{i\chi}$ and $\lambda_2 = e^{i\delta}e^{-i\chi}$ with real δ and χ. It then follows directly that $|\lambda_1\lambda_2| = 1$, $|\lambda_1+\lambda_2| < 2$, and $(\lambda_1+\lambda_2)/(\lambda_1\lambda_2)^{1/2}$ is a real number.

The sufficiency is more difficult to establish. If $|\lambda_1\lambda_2| = 1$, then $\lambda_1 = e^{i\delta}\mu$ and $\lambda_2 = e^{i\delta}\mu^{-1}$. If $(\lambda_1+\lambda_2)/(\lambda_1\lambda_2)^{1/2}$ is a real number, μ is either real or unimodular. This follows directly from

$$(\lambda_1+\lambda_2)/(\lambda_1\lambda_2)^{1/2} = \mu + 1/\mu \tag{A1}$$

and, if $\mu = \rho e^{i\theta}$, the imaginary part of $\mu + 1/\mu$ is $(\rho - 1/\rho) \times \sin\theta$. The case μ real $\neq 1$ violates $|\lambda_1+\lambda_2| < 2$.

These conditions can be restated directly in terms of the usual matrix operations. Calling the matrix $\mathbf{R}$, we have as our conditions

$$|\det\mathbf{R}| = 1\,, \tag{A2}$$

$$|\mathrm{tr}\mathbf{R}| < 2\,, \tag{A3}$$

and

$$\frac{\mathrm{tr}\mathbf{R}}{\sqrt{(\det\mathbf{R})}} = \left(\frac{\mathrm{tr}\mathbf{R}}{\sqrt{(\det\mathbf{R})}}\right)^*. \tag{A4}$$

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Higher-Dimensional Field Theory*

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An attempt to construct a theory that links the macroscopic and microscopic properties of matter is presented. This is done using a space of more than four dimensions. Equations for microscopic-particle fields are investigated in a fixed six-dimensional metric space. The topology of the space is solely responsible for the quantization of mass and charge. The metric contains terms which transform like the Yang-Mills B field. These terms appear appropriately in all particle equations. Without the B field, the symmetry group of the theory is P×SU(2)/Z(2). The presence of the B field lowers this symmetry. Particle mass spectra are presented for six-dimensional scalar, spinor, and vector fields, and a coupling-constant ratio is predicted. The later part of the paper deals with the cosmological implications of the microscopic model presented in the first part of the paper. It is shown that Einstein's equations for the metric are consistent if a massless cosmological vector field is introduced. A critique of previous higher-dimensional field theories along with a summary of the results of an eight-dimensional theory is given. Since all symmetries dealt with result as approximations to the equations of the model, the no-go theorems are not applicable. Nevertheless, the six- and eight-dimensional models contain the shadows of the $P\times SU(2)/Z(2)$ and $P\times SU(3)/Z(3)$ symmetries in all particle-field representations.

I. INTRODUCTION

In this paper we shall report on the results of an attempt to construct a theory that links together the cosmological and elementary-particle properties of matter. The idea that a connection should exist between these two properties is, of course, not new. Mach's principle[1] suggests that the local inertial properties of matter depend upon the cosmological distribution of distant matter. In his *Fundamental Theory*, Eddington[2] attempted to relate the so-called cosmological numbers to parameters characterizing the then-known elementary particles. Perhaps the best example of such a "unified" theory is Einstein's general theory of relativity. In this theory, the same dynamical laws that govern the local gravitational interactions of matter also govern the cosmological structure of the universe. Since the advent of general relativity, numerous proposals have been put forth to include other interactions (e.g., electromagnetic, mesonic) into a unified theory with varying degrees of success.

One can of course always include these nongravitational interactions within the framework of general relativity. Their inclusion, however, does not in general affect the cosmological consequences of the theory. Only by unifying the space-time and internal properties of matter in some nontrivial way can one hope to obtain a theory that would allow us to deduce properties of the cosmos from a knowledge of elementary-particle interactions and vice versa. The

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[2] A. S. Eddington, *Fundamental Theory* (Cambridge University Press, New York, 1948).