Covariant Approach to Kinematic Constraint Relations for Helicity Amplitudes*

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With the aid of a covariant spin formalism, the kinematic constraint equations for helicity amplitudes are studied in a systematic way for all mass configurations, including the case of zero-mass particles. The complete set of constraints at thresholds and pseudothresholds is given in a form convenient for calculation; that is, the coefficients of the helicity amplitudes are simple numerical ones (as opposed, for instance, to D functions). For nonzero-mass particles, the reduction of the constraints in terms of total spin amplitudes is shown to follow. Amplitudes with different values of total spin are not related at thresholds or pseudothresholds.

I. INTRODUCTION

Many papers have appeared in the literature during the past few years dealing with the kinematic properties of helicity amplitudes.[1–10] The nature of the singular behavior of these amplitudes at special values of s and t (i.e., thresholds, pseudothresholds, and boundaries of the physical region) are now well understood. Nevertheless, from either a pedagogical or a practical standpoint, these earlier treatments have been, in our opinion, somewhat complicated and often incomplete. One of our purposes here is to present a simpler approach based on a covariant description of the scattering process. Within this framework the treatment of massless particles poses no additional difficulty, in contrast to conventional treatments.[5,10] The method used is based on the covariant spin formalism developed by Feldman and Matthews,[11] King and Feldman,[12] and King.[7]

A second purpose is to give the threshold and pseudothreshold constraint equations for the helicity amplitudes in a form which is convenient to calculate for particles of arbitrary spin. Since any phenomenological model of the scattering process ought to satisfy all kinematic constraints, these constraint relations with simple numerical coefficients may prove more practical than the relations given previously[4] in terms of the rotation matrix elements $D_{mm'}^{s}(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$ [there does not seem to exist a simple explicit expression for the function $d_{m'm}^{j}(\frac{1}{2}\pi)$].

We mention here a few approaches that have been used in the past. With an intermediate step based on spinor amplitudes, the helicity amplitudes can be expanded in terms of invariant amplitudes free of kinematic singularities and constraints[4]; all the singularities are contained in the coefficients of the expansion. In principle, the problem of finding the kinematic singularities is easily understood, but in practice the method is quite tedious. The constraint equations may then be found by transforming to transversity amplitudes[13] and studying the kinematic properties of the latter. This is most easily done using the crossing matrix for transversity amplitudes, which implies that we must know the crossing matrix for the helicity amplitudes.

Other approaches have been based on the crossing matrix for the helicity amplitudes,[3] but crossing alone does not provide enough information; one must also have some knowledge of the kinematic structure of the helicity amplitudes. Trueman[6] has given an approach independent of spinor amplitudes (and therefore invariant amplitudes) and of crossing. It is assumed the kinematic singularities arise because of the singular definitions of the helicity states. He does not, however, give the general form of the constraint equations, and his prescription for finding them in the case of arbitrary spin is, albeit straightforward, rather tedious.[14] We emphasize that our method is also independent of spinor amplitudes and of crossing.

In Sec. II, the essential features of the covariant approach are reviewed and the basic assumptions stated. Section III contains the derivation of constraint relations for the general mass configuration, i.e., the case $m_A \pm m_B \neq m_C \pm m_D$, $m_A \neq m_B$, $m_C \neq m_D$, and masses being nonzero. All other cases are treated in Sec. IV, which also contains discussion and summary. We also show in Sec. III that the constraint relations can be reduced in the sense that a group representation can be reduced. For instance, the constraint relations are completely reduced in terms of the transversity amplitudes

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[1] A partial list of recent works would include Refs. 2–10, other works can be traced from them.

[2] Y. Hara, Phys. Rev. **136**, B507 (1964).

[3] L. L. Wang, Phys. Rev. **124**, 1187 (1966).

[4] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1969).

[5] J. P. Ader, M. Capdeville, and H. Navelet, Nuovo Cimento **56A**, 315 (1968).

[6] T. L. Trueman, Phys. Rev. **173**, 1684 (1968).

[7] M. King, Nuovo Cimento **61**, 273 (1969).

[8] J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968).

[9] J. Franklin, Phys. Rev. **170**, 1606 (1968).

[10] S. R. Cosslett, Phys. Rev. **176**, 1783 (1968).

[11] G. Feldman and P. T. Matthews, Phys. Rev. **168**, 1587 (1968).

[12] M. King and G. Feldman, Nuovo Cimento **60**, 86 (1969).

[13] A. Kotanski, Acta Phys. Polon. **29**, 699 (1966); **30** 629 (1966).

[14] A third approach which is based on the partial-wave expansion (see, e.g., Refs. 2, 8, and 9) will not be discussed here.

because each constraint relation contains only one transversity amplitude. However, the helicity amplitudes have the advantage of a simpler partial-wave expansion. Our criterion for reduction is to retain all the advantages of helicity amplitudes. It will be shown that the amplitudes which emerge from our method of partial reduction are the "total spin" amplitudes used by Freedman and Wang[15] and by Franklin.[9] Since the total spin amplitudes are linear combinations of helicity amplitudes with the same values of initial and final total spin projections, they enjoy all the properties of the helicity amplitudes that depend only on total spin projections; this includes the structure of kinematic singularities and partial-wave expansions.

The Appendix contains some useful information on the rotation functions and the proof for a lemma used in Secs. III and IV.

II. COVARIANT SPIN FORMALISM

Trueman[6] has shown how the kinematic singularities of the helicity amplitudes arise from the singular definitions of the helicity states. We take a similar approach using a covariant description of the two-body process, in which the single-particle states implicit in the scattering amplitudes are eigenstates of covariant spin operators. For particles of nonzero mass, these amplitudes may be the covariant helicity amplitudes discussed in Refs. 7, 11, and 12, or the transversity amplitudes introduced by Kotanski.[13] The kinematic singularities of the covariant helicity amplitudes (which become the usual Jacob and Wick[16] helicity amplitudes in appropriate center-of-mass frames) have previously been discussed from this point of view.[7] In this section we review the arguments and extend them to include particles of zero mass.

We recall that the states for a single particle form a basis for an irreducible representation of the Poincaré group. These irreducible representations are divided into four different classes of eigenvalues of P^2, where P is the four-momentum of the particle:

$$\begin{aligned}&\text{Class I:} && P^2=m^2>0\,,\\ &\text{(a) } P_0>0, && \text{(b) } P_0<0\,,\\ &\text{Class II:} && P^2=m^2<0\,,\\ &\text{Class III:} && P^2=0\,,\\ &\text{Class IV:} && P_\mu=0\,.\end{aligned}$$

The particle states may be specified in terms of the eigenvalues of the Casimir operators P^2, W^2, the three-momentum $\mathbf{P}$, and some generator of the little group of transformations which leaves the four-momentum P_μ-invariant.[17] For each class the little group has a different structure.

Physical particles of nonzero mass are classified by I(a). In the "standard" frame $p=(m,0,0,0)$, the little group is the rotation group generated by J_1, J_2, and J_3. To define the little group for an arbitrary frame, Feldman and Matthews[11] have made use of the fact that two-particle states are products of single-particle states, and have given a covariant description of the little-group generators for single-particle states in terms of the momenta of both particles. In their covariant description, the single-particle states are eigenstates of the following complete set of commuting operators:

$$(W^{(i)})^2,\ (P^{(i)})^2;\ \mathbf{P}^{(i)},\ W_\mu{}^{(i)}p^{(j)\mu}\,, \tag{2.1}$$

where (i) and (j) denote particles i and j, respectively, and $p^{(j)}$ is the four-momentum of the second particle. Choices of (j) corresponding to two different particles partaking in the reaction give rise to the s- or t-channel covariant helicities, which we will define and discuss shortly. In the appropriate reference frames, the covariant helicities reduce to conventional helicities. If, instead of $p_\mu{}^{(j)}$ in (2.1), we use the vector $\epsilon_{\mu\nu\lambda\rho}p_j{}^\nu p_k{}^\lambda p_l{}^\rho$, where j, k, and l refer to the other three particles in the process, we are led to the transversity operator.

Since $W_\mu{}^{(i)}P^{(i)\mu}=0$, $W_\mu{}^{(i)}$ has only three independent components. The s-channel and t-channel covariant helicity and the transversity operators correspond to three independent ways of specifying the spin of a particle with mass. Because the definitions of these covariant spin operators involve the momenta of other particles, these operators become ill-defined at certain values of the Mandelstam invariants s and t, independent of the frame of reference. For the covariant helicities, these values correspond to thresholds and pseudothresholds of the s and t channels of the reaction, as we shall see presently. Physically we can understand this by remembering that at a threshold (or pseudothreshold), two particles are at rest relative to each other, so that the spin component of one along the direction of momentum of the second is not defined.

Further kinematic singularities are present in the covariant helicity amplitudes, namely, at those values of s and t corresponding to the boundary of the physical region. These arise because the total spin projections in the initial and final states are along different axes. To see this most simply, let us consider the over-all center-of-mass system of the reaction $1+2\rightarrow 3+4$. The initial state is an eigenstate of

$$\mathbf{J}\cdot\mathbf{p}^{(1)}=(\mathbf{J}^{(1)}+\mathbf{J}^{(2)})\cdot\mathbf{p}^{(1)}\,, \tag{2.2}$$

that is, the total spin projection is taken along the axis in the direction $\mathbf{p}_1$. Similarly, the final state is an eigenstate of

$$\mathbf{J}\cdot\mathbf{p}^{(3)}=(\mathbf{J}^{(3)}+\mathbf{J}^{(4)})\cdot\mathbf{p}^{(3)}\,, \tag{2.3}$$

or the total spin is projected in the direction of $\mathbf{p}^{(3)}$. Thus, the S matrix for the reaction involves a rotation of the spin axis through an angle θ, which is known as

[15] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).
[16] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).
[17] E. P. Wigner, Ann. Math. **40**, 149 (1939).

the center-of-mass scattering angle. This little-group operation enables us to write the little-group decomposition, or partial-wave expansion, of the scattering amplitude in terms of the rotation functions $d_{\lambda\mu}{}^j(\theta)$. The singularities at forward and backward scattering are explicitly shown and can be factored out of the scattering amplitude.

Massless particles are contained in Class III, for which $p^2=0$. In the standard frame $p=(\omega,0,0,\omega)$, the little group is the noncompact group $E(2)$, i.e., the Euclidean group in two dimensions. All the irreducible representations of this group are infinite-dimensional except for the trivial one-dimensional representation. The physical massless particles correspond to this trivial representation, for which $W^2=0$ and W^μ is proportional to P^μ.[18] Thus there is only one independent component of W^μ, or only one independent way to specify the particle spin. The little group reduces to the group of transformations generated by the operator J_3. In an arbitrary reference frame, the little group consists of all rotations about the axis $\mathbf{p}$, and is generated by the operator $\mathbf{J}\cdot\hat{p}$. In the trivial representation, $\mathbf{J}\cdot\hat{p}$ is a Lorentz invariant, and can be considered as a Casimir operator. The single-particle states may be specified by the eigenvalues of the complete set of operators

$$\mathbf{P},\ \mathbf{J}\cdot\mathbf{P} \tag{2.4}$$

($W^2=P^2=0$ in this representation). Since the momentum of no other particle is present in this covariant description,[19] there is no reason to expect the covariant spin operator to become ill-defined at thresholds or pseudothresholds. We therefore assume that the massless particle does not induce any corresponding singularities in the scattering amplitude.

If the parity transformation is included among the allowed transformations, we have a doubling of states. Since $\mathbf{J}$ is even and $\mathbf{P}$ is odd under parity, the helicity changes sign under inversion.

First we shall discuss the covariant helicity amplitudes for particles of nonzero mass. We consider the two-body scattering of particles of arbitrary mass and spin, and define the s- and t-channel processes and corresponding momenta as

$$\begin{array}{cccc} A+B & \to & C+D & (s\text{ channel}), \\ p_A\ \ p_B & & p_C\ \ p_D & \end{array} \tag{2.5}$$

and

$$\bar{D}+B\to C+\bar{A}\quad (t\text{ channel}). \tag{2.6}$$

The variables s and t are defined by

$$s=(p_A+p_B)^2=(p_C+p_D)^2,$$
$$t=(p_A-p_C)^2=(p_B-p_D)^2.$$

For brevity, we limit the discussion to the case of unequal masses and the threshold and pseudothreshold of the initial state only. The method is easily extended to the other mass cases and final-state threshold and pseudothreshold, using the equations of Ref. 7. The results are summarized in Sec. IV.

The scattering process may be described by s-channel covariant helicity amplitudes $f_{cdab}{}^s(s,t)$, or, as a second choice, by t-channel covariant helicity amplitudes $f_{cdab}{}^t(s,t)$.[7,12] The single-particle states implicit in these scattering amplitudes are eigenstates of the s- or t-channel covariant helicity operators, respectively. For the s-channel process, the s-channel covariant helicity operators corresponding to particles A and B are defined by the relations[7,12]

$$F_A{}^s=\frac{2W_\mu{}^{(A)}p_B{}^\mu}{\Delta(s,A,B)},\quad F_B{}^s=\frac{2W_\mu{}^{(B)}p_A{}^\mu}{\Delta(s,A,B)}, \tag{2.7}$$

where

$$\Delta(s,A,B)=\{[s-(m_A+m_B)^2][s-(m_A-m_B)^2]\}^{1/2}. \tag{2.8}$$

In the s-channel center of mass these operators reduce to the familiar helicity operator.

Similarly, the t-channel covariant helicity operators for particles A and B are defined by

$$F_A{}^t=\frac{2W_\mu{}^{(A)}p_C{}^\mu}{\Delta(t,A,C)},\quad F_B{}^t=-\frac{2W_\mu{}^{(B)}p_D{}^\mu}{\Delta(t,B,D)}. \tag{2.9}$$

Although we are defining these spin operators for the s-channel process, they reduce, in fact, in the t-channel center of mass to helicity operators.

The transformations between the eigenstates of the two sets of operators are given by[12]

$$\psi^s(p_{(r)},\lambda)=\sum_{\lambda'} d_{\lambda'\lambda}{}^{S_r}(\chi_r)\psi^t(p_{(r)},\lambda'), \tag{2.10}$$

where λ and λ' are eigenvalues of the operators $F_{(r)}{}^s$ and $F_{(r)}{}^t$, respectively, and $r=A$ or B. The angle χ_A is the angle between $-\mathbf{p}_B$ and $-\mathbf{p}_C$ in the rest frame of particle A ($\mathbf{p}_A=0$), while χ_B is the angle between $-\mathbf{p}_A$ and $\mathbf{p}_D$ in the rest frame of particle B ($\mathbf{p}_B=0$). We always consider the coordinate axes to be such that $0\leq\chi_r\leq\pi$. In terms of the invariants s and t, the angles χ_r are given by the relations

$$\cos\chi_A=\frac{-(s+m_A{}^2-m_B{}^2)(t+m_A{}^2-m_C{}^2)-2m_A{}^2M}{\Delta(s,A,B)\Delta(t,A,C)}, \tag{2.11a}$$

$$\cos\chi_B=\frac{(s+m_B{}^2-m_A{}^2)(t+m_B{}^2-m_D{}^2)-2m_B{}^2M}{\Delta(s,A,B)\Delta(t,B,D)}, \tag{2.11b}$$

[18] See, e. g., S. Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1967), p. 72.

[19] One might expect to be able to use the four-momentum of some other particle in the process to specify the projection of W_μ; for instance, $\lambda=-W^\mu P^{(i)}{}_\mu/P_\mu P^{(i)\mu}$. However, using the Poincaré algebra, one may show that λ reduces to the helicity $\mathbf{J}\cdot\mathbf{P}$ in any frame of reference.

where

$$M = m_C^2 - m_A^2 + m_B^2 - m_D^2, \quad (2.12)$$

$$\sin\chi_A = \frac{2m_A[\lambda(s,t)]^{1/2}}{\Delta(s,A,B)\Delta(t,A,C)}, \quad (2.13a)$$

$$\sin\chi_B = \frac{2m_B[\lambda(s,t)]^{1/2}}{\Delta(s,AB,)\Delta(t,B,D)}, \quad (2.13b)$$

and

$$\lambda(s,t) = st(\textstyle\sum_i m_i^2 - s - t) - s(m_B^2 - m_D^2)(m_A^2 - m_C^2) - t(m_A^2 - m_B^2)(m_C^2 - m_D^2) + M(m_A^2 m_D^2 - m_B^2 m_C^2). \quad (2.14)$$

We see from Eqs. (2.7) that F_A^s and F_B^s are badly defined at the threshold $s=(m_A+m_B)^2$ and pseudothreshold $s=(m_A-m_B)^2$.[20] The covariant helicity amplitudes f_{cdab}^s are singular at these values of s. Introducing a new set of amplitudes $f_{cdab}^{st}(s,t)$ in which the states for particles C and D remain unchanged but the states for A and B are now eigenstates of F_A^t and F_B^t, we find we can write

$$f_{cdab}^s(s,t) = \sum_{a'b'} d_{a'a}^{S_A}(\chi_A) d_{b'b}^{S_B}(\chi_B) f_{cda'b'}^{st}(s,t). \quad (2.15)$$

The singularities at $s=(m_A \pm m_B)^2$ are now isolated in the d functions, and $f_{cda'b'}^{st}$ is regular at these points.

In the case that particle A has zero mass ($m_A=0$), the threshold and pseudothreshold coincide at $s=m_B^2$. The covariant helicity for particle A, which now becomes helicity in every frame, is still well-defined at $s=m_B^2$, and it is only necessary to transform away from the eigenstates of F_B^s to the eigenstates of, say, F_B^t. Thus, we can write the s-channel amplitude as

$$f_{cdab}^s(s,t) = \sum_{b'} d_{b'b}^{S_B}(\chi_B) B_{cdab'}^{st}(s,t), \quad (2.16)$$

where $B_{cdab'}^{st}$ is regular at $s=m_B^2$. The particle states for A, C, and D in the amplitude $B_{cdab'}^{st}$ are s-channel center-of-mass helicity states, while the state for particle B is an eigenstate of F_B^t with eigenvalue b'. Note that (2.16) can also be obtained by letting $m_A \to 0$ in Eqs. (2.11) and (2.15), keeping in mind that only two values of a (and a') are allowed.

Equations (2.15) and (2.16) are convenient starting points for the derivation of the various constraints. The unequal-mass case is discussed in Sec. III, while the zero-mass results are summarized in Sec. IV, along with the other mass cases.

[20] In all cases, we shall assume that the threshold or pseudothreshold is approached in such a way that the square roots appearing in the denominators of $\cos\chi_A$, $\cos\chi_B$, etc., are real and positive. Also we assume that we have already analytically continued these functions to a fixed large positive value of t (therefore outside of s-channel physical region). Finally, we shall always take $s^{1/2}$ to be positive and the functions $\cos\chi_A$, $\cos\chi_B$, etc., to be real.

III. CONSTRAINT RELATIONS

Our general method is to study the behavior of the $d(\chi)$'s in (2.15) and (2.16) as the threshold and pseudothreshold values are approached. For simplicity, we first treat the general case, i.e., the case where no equalities exist among the following quantities: m_A, m_B, m_C, m_D, $m_A \pm m_B$, $m_C \pm m_D$, and zero. All special cases will be summarized in Sec. IV.

It has been noted by Edmonds[21] and more recently by Kotanski[13] that a d matrix can be diagonalized by a unitary matrix $D^j(\frac{1}{2}\pi, \frac{1}{2}\pi, -\frac{1}{2}\pi)$,[22] i.e.,

$$d_{m'm}^j(\chi) = \sum_{m''} e^{-i(m'-m)\pi/2} \Delta_{m'm''}^j \Delta_{mm''}^j e^{-im''\chi}, \quad (3.1)$$

where

$$\Delta_{m'm}^j \equiv d_{m'm}^j(\tfrac{1}{2}\pi). \quad (3.2)$$

Because the $d_{m'm}^j$'s are real, the complex conjugate of (3.1) is also true:

$$d_{m'm}^j(\chi) = \sum_{m''} e^{i(m'-m)\pi/2} \Delta_{m'm''}^j \Delta_{mm''}^j e^{im''\chi}. \quad (3.3)$$

These relations are particularly convenient for studying the asymptotic behavior of $d(\chi)$. If $e^{-i\chi} \to \pm\infty$ (this corresponds to $\cos\chi \to \pm\infty$ and $\sin\chi \to \pm i\infty$), then the leading term of $d(\chi)$ is gotten by setting $m''=j$ in (3.1):

$$d_{m'm}^j(\chi) \xrightarrow[e^{-i\chi}\to\pm\infty]{} \times \frac{e^{-i(m'-m)\pi/2}(e^{-i\chi})^j (2j)!\, 2^{-2j}}{[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}}, \quad (3.4)$$

where we have used

$$\Delta_{mj}^j = \left[\frac{(2j)!}{(j+m')!(j-m')!}\right]^{1/2} 2^{-j}, \quad (3.5)$$

which follows immediately from Eq. (A16) of the Appendix. If $e^{i\chi} \to \pm\infty$ ($\cos\chi \to \pm\infty$ and $\sin\chi \to \mp i\infty$), the leading term is given by setting $m''=j$ in (3.3):

$$d_{m'm}^j(\chi) \xrightarrow[e^{i\chi}\to\pm\infty]{} \times \frac{e^{i(m'-m)\pi/2}(e^{i\chi})^j (2j)!\, 2^{-2j}}{[(j+m')!(j-m')!(j+m)!(j-m)!]^{1/2}}. \quad (3.6)$$

With the aid of the relations (3.1) and (3.3), we may convert Eq. (2.15) to a form suitable for deriving the constraint relations. Before doing this, we study the

[21] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N. J., 1957).

[22] We remind the reader that the definition of the d functions in Edmonds (Ref. 21) differs from that of Rose (Ref. 23) by a sign of the argument. We follow the convention of the latter.

behavior of χ_A and χ_B near threshold and pseudothreshold. Because $\sin\chi_{A,B}$ and $\cos\chi_{A,B}$ have branch point singularities at threshold and pseudothreshold, their relative phases become crucial. By direct computation, we see that

$$\sin(\chi_A \pm \chi_B) = \frac{2[m_A(t+m_B^2-m_D^2) \mp m_B(t+m_A^2-m_C^2)]}{[s-(m_A\mp m_B)^2]\Delta(t,A,C)\Delta(t,B,D)} \times[\lambda(s,t)]^{1/2}, \quad (3.7)$$

$$\cos(\chi_A \pm \chi_B) = \frac{-2[m_A(t+m_B^2-m_D^2) \mp m_B(t+m_A^2-m_C^2)]^2}{[s-(m_A\mp m_B)^2]\Delta(t,A,C)\Delta(t,B,D)} - \frac{(t+m_A^2-m_C^2)(t+m_B^2-m_D^2)\mp 4m_Am_Bt}{\Delta(t,A,C)\Delta(t,B,D)}. \quad (3.8)$$

It follows that $e^{i(\chi_A+\chi_B)}$ ($e^{i(\chi_A-\chi_B)}$) has neither a pole nor a zero at threshold (pseudothreshold). This determines the relative behavior of $e^{i\chi_A}$ and $e^{i\chi_B}$. However, $e^{i(\chi_A-\chi_B)}$ ($e^{i(\chi_A+\chi_B)}$) is expected to have either a pole or a zero at threshold (pseudothreshold) depending on the value of t. For large and positive values of t,[4] we see that

$$e^{-i(\chi_A-\chi_B)} \propto s-(m_A+m_B)^2 \quad \text{near threshold}, \quad (3.9)$$

so that

$$e^{-i\chi_A} \propto e^{i\chi_B} \propto [s-(m_A+m_B)^2]^{1/2} \quad \text{near threshold}. \quad (3.10)$$

Similarly,

$$e^{i\epsilon(\chi_A+\chi_B)} \propto s-(m_A-m_B)^2 \quad \text{near pseudothreshold}, \quad (3.11)$$

so that

$$e^{i\epsilon\chi_A} \propto e^{i\epsilon\chi_B} \propto [s-(m_A-m_B)^2]^{1/2} \quad \text{near pseudothreshold}, \quad (3.12)$$

where ϵ is the sign of m_B-m_A, i.e.,

$$\epsilon=(m_B-m_A)/|m_B-m_A|. \quad (3.13)$$

We are now ready to derive the constraint equations. Near threshold we substitute (3.1) for $d(\chi_A)$ and (3.3) for $d(\chi_B)$ into (2.15), getting

$$f_{cdab}{}^s(s,t)=\sum_{a'',b''} e^{i(a-b)\pi/2}e^{-i(a''\chi_A-b''\chi_B)}\Delta_{aa''}{}^{S_A} \times\Delta_{bb''}{}^{S_B}g_{cda''b''}, \quad (3.14)$$

where

$$g_{cda''b''}\equiv\sum_{a',b'} e^{-i(a'-b')\pi/2}\Delta_{a'a''}{}^{S_A} \times\Delta_{b'b''}{}^{S_B}f_{cda'b'}{}^{st}(s,t), \quad (3.15)$$

and is regular at $s=(m_A\pm m_B)^2$. The Δ matrices in (3.14) can be easily inverted to give

$$\sum_{a,b}\Delta_{aa'}{}^{S_A}\Delta_{bb'}{}^{S_B}e^{-i(a-b)\pi/2}f_{cdab}{}^s = e^{-i(a'\chi_A-b'\chi_B)}g_{cda'b'}. \quad (3.16)$$

Near pseudothreshold, we use (3.1) or (3.3) for both $d(\chi_A)$ and $d(\chi_B)$ depending on the value of ϵ. A relation similar to the above is obtained:

$$\sum_{a,b}\Delta_{aa'}{}^{S_A}\Delta_{bb'}{}^{S_B}e^{i\epsilon(a+b)\pi/2}f_{cdab}{}^s = e^{i\epsilon(a'\chi_A+b'\chi_B)}g'_{cda'b'}, \quad (3.17)$$

with

$$g'_{cda''b''}\equiv\sum_{a',b'} e^{i(a'+b')\pi/2}\Delta_{a'a''}{}^{S_A}\Delta_{b'b''}{}^{S_B}f_{cda'b'}{}^{st}, \quad (3.18)$$

which is regular at $s=(m_A\pm m_B)^2$.

At this point our constraint relations (3.16) and (3.17) are equivalent to those of Ref. 4 because the transformations involved are precisely those which bring the spin basis of particles A and B from helicity to transversity. The equivalence between the two approaches mentioned in Sec. I is demonstrated. To simplify these constraint equations we need more detailed knowledge of the Δ's, which is provided by the following lemma proved in the Appendix:

Lemma: If we define $F_m{}^j(n)$ by

$$\Delta_{mn}{}^j\equiv(-1)^{j-n}(2j)! \times[(j+m)!(j-m)!(j+n)!(j-n)!]^{-1/2}F_m{}^j(n),$$

then, for $2j=$ integer and $m, n=j, j-1, \ldots, -j$,

(i) $F_m{}^j(n)=F_n{}^j(m)$, i.e., $F_m{}^j(n)$ is symmetric in m and n;
(ii) $F_m{}^j(n)$ is a polynomial in n of order $j-m$.

We first set $a'=S_A$, $b'=S_B$ in Eqs. (3.16) and (3.17) and use the lemma, along with Eqs. (3.10) and (3.12), to obtain

$$\sum_{a,b} e^{-i(a-b)\pi/2}f_{cdab}{}^sX_{ab}{}^{-1} \propto[s-(m_A+m_B)^2]^{(S_A+S_B)/2}, \quad (3.19)$$

$$\sum_{a,b} e^{i\epsilon(a+b)\pi/2}f_{cdab}{}^sX_{ab}{}^{-1} \propto[s-(m_A-m_B)^2]^{(S_A+S_B)/2}, \quad (3.20)$$

where

$$X_{ab}\equiv[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!]^{1/2}. \quad (3.21)$$

Next we set $a'=S_A-1$, $b'=S_B$, and find

$$\sum_{a,b} ae^{-i(a-b)\pi/2}f_{cdab}{}^sX_{ab}{}^{-1} \propto[s-(m_A+m_B)^2]^{(S_A+S_B-1)/2}, \quad (3.22)$$

$$\sum_{a,b} ae^{i\epsilon(a+b)\pi/2}f_{cdab}{}^sX_{ab}{}^{-1} \propto[s-(m_A-m_B)^2]^{(S_A+S_B-1)/2}, \quad (3.23)$$

where equations (3.19) and (3.20) have been used to eliminate terms in the sum which do not contain a factor of a. The generalization of this procedure is very simple. For $a'=S_A-m$, $F_{S_A-m}{}^{S_A}(a)$ is a polynomial in a of order m. Only the term with the highest power of a need be retained in the sum. All lower-power terms can be eliminated by using constraint relations having lower m values. The general constraint equations are

$$\sum_{a,b}\frac{a^m b^n e^{-i(a-b)\pi/2}f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!]^{1/2}} \propto [s-(m_A+m_B)^2]^{(S_A+S_B-m-n)/2} \quad (3.24)$$

at threshold,

$$\sum_{a,b}\frac{a^m b^n e^{i\epsilon(a+b)\pi/2}f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!]^{1/2}} \propto [s-(m_A-m_B)^2]^{(S_A+S_B-m-n)/2} \quad (3.25)$$

at pseudothreshold, with $m=0, 1, 2\ldots(2S_A)$; $n=0, 1, 2\ldots(2S_B)$; and $\epsilon=(m_B-m_A)/|m_B-m_A|$.

The reduction of the constraint equations follows if we substitute into (3.16) and (3.17) the following slight variation of the Clebsch-Gordan series[21,23]:

$$\begin{aligned}\Delta_{aa'}{}^{S_A}\Delta_{bb'}{}^{S_B} &=(-1)^{S_B-b'}\Delta_{aa'}{}^{S_A}\Delta_{-bb'}{}^{S_B}\\ &=(-1)^{S_B-b'}\textstyle\sum_S (S_A, a; S_B, -b|S, a-b)\Delta_{a-b,a'+b'}{}^S\\ &\quad\times(S, a'+b'|S_A,a'; S_Bb'). \quad (3.26)\end{aligned}$$

We also make the substitution $b=a-\lambda$, $b'=\lambda'-a'$, and make use of the orthogonality property of the Clebsch-Gordan coefficients to bring the constraint relations (3.16) and (3.17) to the form

$$\textstyle\sum_\lambda \Delta_{\lambda\lambda'}{}^S e^{-i\lambda\pi/2}\sum_a (S_A, a; S_B, \lambda-a|S, \lambda)f_{cda\,a-\lambda}{}^s = e^{i\lambda'\chi_B}G_{cdS\lambda'}, \quad (3.27)$$

$$\textstyle\sum_\lambda \Delta_{\lambda\lambda'}{}^S e^{-i\epsilon\lambda\pi/2}\sum_a (S_A, a; S_B, \lambda-a|S,\lambda)e^{i\epsilon a\pi}f_{cda\,a-\lambda}{}^s = e^{i\lambda'\chi_B}G'_{cdS\lambda'}, \quad (3.28)$$

where

$$G_{cdS\lambda'}\equiv\sum_{a'} (S_Aa', S_B; a'-\lambda'|S\lambda')\times(-1)^{S_B-\lambda'+a'}e^{-ia'(\chi_A+\chi_B)}g_{cda'\,\lambda'-a'}, \quad (3.29)$$

$$G'_{cdS\lambda'}\equiv\sum_{a'} (S_A, a'; S_B, a'-\lambda'|S,\lambda')\times(-1)^{S_B-\lambda'+a'}e^{i\epsilon a'(\chi_A-\chi_B)}g'_{cda'\,\lambda'-a'}, \quad (3.30)$$

and $G_{cds\lambda'}$ ($G'_{cds\lambda'}$) is regular at threshold (pseudothreshold) because $e^{i(\chi_A+\chi_B)}$ ($e^{i(\chi_A-\chi_B)}$) is. We now see that the constraint relations are reduced in terms of the total spin amplitudes at threshold,

$$f_{cd(S\lambda)}{}^{s(\text{th})}\equiv\textstyle\sum_a (S_A, a; S_B, \lambda-a|S,\lambda)f_{cda\,a-\lambda}{}^s, \quad (3.31)$$

and the total pseudospin amplitudes at pseudothreshold,

$$f_{cd(S\lambda)}{}^{s(ps)}\equiv\textstyle\sum_a (S_A, a; S_B, \lambda-a|S,\lambda)\times(-1)^{S_A-a}f_{cda\,a-\lambda}{}^s. \quad (3.32)$$

The argument based on the lemma can be applied again, to yield constraint relations in a very compact form:

$$\sum_\lambda\frac{\lambda^m e^{-i\lambda\pi/2}f_{cd(S\lambda)}{}^{s(\text{th})}}{[(S+\lambda)!(S-\lambda)!]^{1/2}}\propto[s-(m_A+m_B)^2]^{(S-m)/2} \quad \text{near threshold}, \quad (3.33)$$

and

$$\sum_\lambda\frac{\lambda^m e^{-i\epsilon\lambda\pi/2}f_{cd(S\lambda)}{}^{s(\text{ps})}}{[(S+\lambda)!(S-\lambda)!]^{1/2}}\propto[s-(m_A-m_B)^2]^{(S-m)/2} \quad \text{near pseudothreshold}, \quad (3.34)$$

where $m=0, 1, \ldots, 2S$; $S=S_A+S_B, S_A+S_B-1, \ldots, |S_A-S_B|$; and $\epsilon=(m_B-m_A)/|m_B-m_A|$.

IV. SUMMARY OF RESULTS

In the following, we list all the constraint relations for initial and final thresholds and pseudothresholds for all mass cases not discussed in Sec. III.

The final-state threshold and pseudothreshold behavior of the covariant helicity amplitudes in the case of unequal masses is obtained in a manner completely analogous to that of Sec. III. The s- and t-channel covariant helicity operators for particles C and D have been defined in Refs. 7 and 12. The amplitudes $f_{cdab}{}^s$ behave as $[s-(m_C+m_D)^2]^{-(S_C+S_D)/2}$ and

$$[s-(m_C-m_D)^2]^{-(S_C+S_D)/2}$$

near the final-state threshold and pseudothreshold, respectively, and we find the constraint relations

$$\sum_{c,d}\frac{c^m d^n e^{i(c-d)\pi/2}f_{cdab}{}^s}{[(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto [s-(m_C+m_D)^2]^{(S_C+S_D-m-n)/2} \quad (4.1)$$

near threshold,

$$\sum_{c,d}\frac{c^m d^n e^{i\epsilon'(c+d)\pi/2}f_{cdab}{}^s}{[(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto [s-(m_C-m_D)^2]^{(S_C+S_D-m-n)/2} \quad (4.2)$$

near pseudothreshold, where ϵ' is the sign of m_C-m_D, or

$$\epsilon'=(m_C-m_D)/|m_C-m_D|, \quad (4.3)$$

and $m=0, 1\ldots(2S_C)$; $n=0, 1\ldots(2S_D)$. The reduction

[23] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

of these amplitudes may be carried out to yield

$$\sum_\mu \frac{\mu^m e^{i\mu\pi/2} f_{(S'\mu)ab}{}^{s(\mathrm{th})}}{[(S'+\mu)!(S'-\mu)!]^{1/2}} \propto [s-(m_C+m_D)^2]^{(S'-m)/2} \quad (4.4)$$

near threshold, and

$$\sum_\mu \frac{\mu^m e^{-i\epsilon'\mu\pi/2} f_{(S'\mu)ab}{}^{s(\mathrm{ps})}}{[(S'+\mu)!(S'-\mu)!]^{1/2}} \propto [s-(m_C-m_D)^2]^{(S'-m)/2} \quad (4.5)$$

near pseudothreshold, with

$$f_{(S'\mu)ab}{}^{s(\mathrm{th})} \equiv \sum_c (S_C, c; S_D, \mu-c|S',\mu) f_{c\,c-\mu\,ab}{}^s, \quad (4.6)$$

$$f_{(S'\mu)ab}{}^{s(\mathrm{ps})} \equiv \sum_c (-1)^{S_C-c} (S_C, c; S_D \mu-c|S',\mu) \times f_{c\,c-\mu\,ab}{}^s, \quad (4.7)$$

and $m=0, 1\ldots(2S')$; $S'=S_C+S_D$, $S_C+S_D-1\ldots |S_C-S_D|$. It is possible, of course, to define amplitudes characterized by the total spin or pseudospin in the initial and final states and to combine the results of Eqs. (3.33), (3.34), (4.4), and (4.5). Total spin amplitudes of this sort have been defined by Franklin,[9] who has derived (3.33), (3.34), (4.4), and (4.5) for $m=0$, 1 with a different method.[3]

For the case $m_A=m_B$, but $(m_A\pm m_B)^2\neq(m_C\pm m_D)^2$, the threshold constraints are the same as for the unequal-mass case. The pseudothreshold in the initial state is now at $s=0$, and the behaviors of χ_A and χ_B depend on the relative magnitudes of the masses of particles C and D, with the result that

$$\sum_{a,b} \frac{a^m b^n e^{-i\epsilon'(a+b)\pi/2} f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!]^{1/2}} \propto (s)^{(S_A+S_B-m-n)/2}. \quad (4.8)$$

In the case of elastic scattering, $m_A=m_C$, $m_B=m_D$, the method of Sec. III must be modified in order to derive the constraints. Since the initial and final thresholds (and pseudothresholds) coincide, a transformation to the t-channel covariant helicity operators for all four particles must be made in order to remove the singularities. This is equivalent to using the methods of Secs. II and III as applied to the full crossing relation between the s- and t-channel center-of-mass helicity amplitudes. It turns out that the final result may also be obtained by combining equations (3.24) [(3.25)] and (4.1) [(4.2)] and then allowing $m_A\to m_C$, $m_B\to m_D$. We obtain the relations

$$\sum_{abcd} \frac{a^m b^n c^p d^q e^{-i(a-b-c+d)\pi/2} f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto [s-(m_A+m_B)^2]^{(S_A+S_B+S_C+S_D-m-n-p-q)/2} \quad (4.9)$$

near threshold, and

$$\sum_{abcd} \frac{a^m b^n c^p d^q e^{i\epsilon(a+b-c-d)\pi/2} f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto [s-(m_A-m_B)^2]^{(S_A+S_B+S_C+S_D-m-n-p-q)/2} \quad (4.10)$$

near pseudothreshold, with $\epsilon=m_B-m_A/|m_B-m_A|$, where $m=0, 1\cdots(2S_A)$; $n=0, 1\ldots(2S_B)$; and $p=0, 1, \ldots (2S_C)$, $q=0, 1\ldots(2S_D)$.

For the case of equal-mass–to–equal-mass scattering, i.e., $m_A-m_B=m_C-m_D=0$, the threshold equations are given by (3.24) when $m_A\neq m_C$. At pseudothreshold, the method of Secs. II and III must be modified because $s=0$ is now on the boundary of the physical region. The constraint relations at $s=0$ have also been derived by other methods[4,6,8] in the form

$$\sum_{abcd} d_{a'a}{}^{S_A}(\tfrac{1}{2}\pi) d_{b'b}{}^{S_B}(\tfrac{1}{2}\pi) d_{c'c}{}^{S_C}(\tfrac{1}{2}\pi) d_{d'd}{}^{S_D}(\tfrac{1}{2}\pi) f_{cdab}{}^s \propto (s)^{|a'-b'-c'+d'|/2}. \quad (4.11)$$

Application of the lemma of Sec. III yields two equations:

$$\sum_{abcd} \frac{(-1)^{a+d} a^m b^n c^p d^q f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto (s)^{(S_A+S_B+S_C+S_D-m-n-p-q)/2}, \quad (4.12)$$

and

$$\sum_{abcd} \frac{(-1)^{b+c} a^m b^n c^p d^q f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!(S_B+b)!(S_B-b)!(S_C+c)!(S_C-c)!(S_D+d)!(S_D-d)!]^{1/2}} \propto (s)^{(S_A+S_B+S_C+S_D-m-n-p-q)/2}, \quad (4.13)$$

with $m+n+p+q\leq S_A+S_B+S_C+S_D$. Equations (4.12) and (4.13) are the complete set of constraints at $s=0$.

Finally, we include the case of zero-mass particles. If $m_A=0$, we may apply the method of Sec. III to obtain the result[5,10]

$$\sum_b \frac{b^m e^{ib\pi/2} f_{cdab}{}^s}{[(S_B+b)!(S_B-b)!]^{1/2}} \propto [s-m_B{}^2]^{(S_B-m)/2} \quad (4.14)$$

near $s=m_B{}^2$, where $m=0, 1, \ldots, 2S_B$. If $m_B=0$, we get

$$\sum_a \frac{a^n e^{-ia\pi/2} f_{cdab}{}^s}{[(S_A+a)!(S_A-a)!]^{1/2}} \propto [s-m_A{}^2]^{(S_A-n)/2} \quad (4.15)$$

near $s=m_A{}^2$, and $n=0, 1, \ldots, 2S_A$.

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APPENDIX

The d function is defined in terms of a hypergeometric function by

$$d_{mn}{}^j(\beta)=\left(\frac{(j+m)!(j-n)!}{(j-m)!(j+n)!}\right)^{1/2}\frac{1}{(m-n)!} \times(\cos\tfrac{1}{2}\beta)^{m+n}(-\sin\tfrac{1}{2}\beta)^{m-n} \times F(-j+m,\, j+m+1;\, m-n+1;\, \sin^2(\tfrac{1}{2}\beta)). \quad (A1)$$

The notation of Rose is used here. It is generally stated[21,23] that the above definition applies only to cases when

$$m+n\geq 0 \quad\text{and}\quad m-n\geq 0. \quad (A2)$$

In all other cases the symmetry relations of the d function should be used:

$$d_{mn}{}^j(\beta)=d_{-n-m}{}^j(\beta), \quad (A3)$$

$$d_{mn}{}^j(\beta)=(-1)^{m-n}d_{nm}{}^j(\beta). \quad (A4)$$

We shall show that the definition (A1) in fact satisfies relations (A3) and (A4); therefore, the restrictions (A2) are unnecessary.

We see, for instance, that the hypergeometric function in (A1) has poles at negative integral values of $m-n$. However, these poles are canceled by the zeros of the factor $1/(m-n)!$ in the coefficient, so that (A1) remains finite when $m<n$. A similar thing happens when $m+n<0$. To see this in detail, we use the identity

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,\, 1-c+a;\, 1-b+a;\, 1/z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} \times F(b,\, 1-c+b;\, 1-a+b;\, 1/z) \quad (A5)$$

to transform (A1) into a more convenient form:

$$d_{mn}{}^j(\beta) =(-1)^{j-m}\left[\binom{2j}{j-m}\binom{2j}{j-n}\right]^{1/2} \times(\cos\tfrac{1}{2}\beta)^{m+n}(-\sin\tfrac{1}{2}\beta)^{2j-m-n} \times F(-j+m,\, -j+n;\, -2j;\, \csc^2(\tfrac{1}{2}\beta)) +(-1)^{-j-m-1}\left[\binom{-2j-2}{-j-m-1}\binom{-2j-2}{-j-n-1}\right]^{1/2} \times(\cos\tfrac{1}{2}\beta)^{m+n}(-\sin\tfrac{1}{2}\beta)^{-2j-2-m-n} \times F(j+m+1,\, j+n+1;\, 2j+2;\, \csc^2(\tfrac{1}{2}\beta)). \quad (A6)$$

The second term above can be obtained simply from the first one by replacing $j\to -j-1$. Calling the first term $d_{mn}{}^{j(1)}$, we see that it is symmetric in m, n except for the factor $(-1)^{j-m}$, which makes it satisfy (A4). By using the relation

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,\, c-b;\, c;\, z), \quad (A7)$$

one can find that (A3) is also satisfied. The same is, of course, also true for $d_{mn}{}^{-j-1}(\beta)$; consequently, (A1) defines $d_{mn}{}^j(\beta)$ without the restrictions (A2).

Lemma: If $F_m{}^j(n)$ is defined by

$$\Delta_{mn}{}^j\equiv d_{mn}{}^j(\tfrac{1}{2}\pi) =(-1)^{j-n}\left[\binom{2j}{j-m}\binom{2j}{j-n}\right]^{1/2}F_m{}^j(n), \quad (A8)$$

then, for $2j=$ integer and $m, n=j, j-1, \ldots, -j$,

(i) $F_m{}^j(n)=F_n{}^j(m)$;
(ii) $F_m{}^j(n)$ is a polynomial in n, of order $j-m$.

Proof: Statement (i) follows immediately from (A4). By setting $\beta=\frac{1}{2}\pi$ in (A6), we obtain, for $j-m$ a nonnegative integer,

$$F_m{}^j(n)=2^{-j}F(-j+m,\, -j+n;\, -2j;\, 2). \quad (A9)$$

The second term in (A6) does not contribute because of a factor $1/[\Gamma(-j+m)]^{1/2}$ in the coefficient. We first give the proof for the case $m \geq n$. In this case

$$F_m{}^j(n) = 2^{-j} \sum_{r=0}^{j-m} \binom{j-m}{r} \times \frac{(j-n)(j-n-1)\cdots(j-n-r+1)}{(2j)(2j-1)\cdots(2j-r+1)}(-2)^r = \frac{(j+m)!2^{-m}}{(2j)!} n^{j-m} + \cdots, \quad \text{(A10)}$$

so that statement (ii) is true. For $m < n$ the proof is by induction. We recall the relation between contiguous hypergeometric functions

$$[2a-c-(a-b)z]F(a) + (c-a)F(a-1) + a(z-1)F(a+1) = 0, \quad \text{(A11)}$$

where $F(a)$ is shorthand for $F(a,b;c;z)$.

This gives

$$F_m{}^j(n+1) = [1/(j-n)] \times [2mF_m{}^j(n) - (j+n)F_m{}^j(n-1)]. \quad \text{(A12)}$$

The induction proof is completed if the polynomial in the above square bracket is divisible by $j-n$. This can be shown explicitly by expanding the hypergeometric functions. The only terms that are not divisible by $j-n$ are the first term of $F_m{}^j(n)$ and the first and second terms of $F_m{}^j(n-1)$. They give

$$2m-(j+n)\left(1-\frac{(j+m)(j-n+1)}{j}\right) = (j-n)\frac{(j+n)(j-m)+m}{j}. \quad \text{(A13)}$$

This completes the proof.

Finally, we list some properties of $\Delta_{mn}{}^j$. From

$$d_{mn}{}^j(\pi-\beta) = (-1)^{j-n} d_{-mn}{}^j(\beta) \quad \text{(A14)}$$

and from (A3) and (A4), we deduce

$$\Delta_{mn}{}^j = (-1)^{j-n}\Delta_{-m,n}{}^j = (-1)^{j+m}\Delta_{m,-n}{}^j. \quad \text{(A15)}$$

For $n = j,\ j-1$, we have

$$\Delta_{mj}{}^j = 2^{-j}\binom{2j}{j-m}^{1/2}, \quad \text{(A16)}$$

$$\Delta_{m,j-1}{}^j = -2^{-j}\frac{m}{j}\left[\binom{2j}{j-m}2j\right]^{1/2}. \quad \text{(A17)}$$

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Two-Body Unitarity and the Asymptotic Duality Series

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The discontinuity of the Regge trajectory obtained recently from a duality series by Kikkawa, Sakita, and Virasoro is investigated. The trajectory is shown to possess all the thresholds corresponding to two-particle intermediate states, i.e., Regge recurrences or daughters. A correction to a linear trajectory is also shown to arise by requiring the Born term (the Veneziano amplitude) to be modified to satisfy two-body partial-wave unitarity near the leading pole in the J plane. This discontinuity is evaluated up to the third two-particle threshold and found to be identical to that obtained from the duality series. Above higher thresholds, such agreement is dependent on the function used to ensure factorization at internal vertices in each term of the duality series.

1. INTRODUCTION

RECENTLY, Kikkawa, Sakita, and Virasoro[1] (called KSV hereafter) have proposed a series representation of the scattering amplitude in which each term is crossing symmetric and possesses duality for all internal lines in the sense that simultaneous poles in "overlapping" variables are not permitted. However, the series does not incorporate unitarity explicitly. Fubini and Veneziano[2] have discovered how the general Born term of the duality series factorizes, necessitating a certain level structure. From this, KSV should be modified to incorporate factorization of all levels at internal vertices, a minimal requirement of unitarity. Although this gives rise to divergent integrals, the discontinuities remain finite.

In this paper, we show that the asymptotic amplitude obtained by KSV satisfies partial-wave unitarity with two-particle intermediate states in the weak-coupling limit, at least up to the third two-particle threshold. In

[1] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. 184, 1701 (1969).

[2] S. Fubini and G. Veneziano, MIT Report No. CTP 81, 1969 (unpublished); D. K. Campbell, D. I. Olive, and W. J. Zakrzewski, University of Cambridge Report No. DAMTP 69/26, 1969 (unpublished).