Realization of Current-Algebra Commutation Relations on the Space of Solutions of Finite- or Infinite-Component Wave Equations

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The relation between the algebra of current densities and the finite- or infinite-component wave equations is critically investigated. It is found that, at any arbitrary momentum, the charge-current density commutation relations can be satisfied by the solutions of the wave equation, but only in a trivial sense, viz., if $J_0(0)$ is taken to be unity, in which case the content of the current commutators is essentially $1\cdot J_\mu(k)=J_\mu(k)$, i.e., empty. Furthermore, it is shown explicitly in an example that, starting from the covariant wave equation, this condition $J_0(0)=1$ can be satisfied only if it is made true by definition. The precise connection between the current algebra and infinite-component wave equations is discussed by the introduction of translation operators in momentum space.

I. INTRODUCTION

A NUMBER of authors have discussed the realization of the current-algebra commutation relations[1,2] on the Hilbert space of the complete set of solutions of a wave equation.[3] Presumably the general program of the current-algebra techniques requires a complete set of all strong particle states including many-particle states of the real world, i.e., an unknown complete set of solutions of an unknown strong-interaction Hamiltonian. In order to simplify the problem, the infinite-momentum limit of the matrix elements of the commutator was introduced, in which case there is a chance that only one-particle states might contribute.[4] The problem then reduces to finding a complete set of one-particle states on which the algebra of current commutators is represented. For this set of one-particle states the solutions of a wave equation have been used. This is then a c-number approximation to the realization of the current algebra. In this way, the current-algebra approach would be connected to another approach where the infinite-component wave equations are used to obtain a concrete set of states of strongly interacting particles and where the matrix elements of the associated currents are used to obtain weak, electromagnetic, and strong form factors.[5,6] Now the solutions of the wave equation are c-number wave functions describing the various (excited) states of the system which is thought to be composite (first-quantized theory). For processes where only particles and resonant states are involved (i.e., where there is no pair production of nucleons, for example), we can use the model provided by the wave equation and the currents which result from the wave equation.

We want, therefore, to study the inverse question to the saturation of current algebra, namely, the validity of the current algebra in the Hilbert space given by the solutions of a wave equation, or a dynamical group (rest frame and boosted states). More specifically, the question is: What are the commutation relations of currents for a given wave equation? From a practical point of view, the given model already contains all matrix elements of the currents, no sum rules are needed, and the current algebra cannot bring anything new. *The problem is to state precisely the exact connection between the wave equations and the current algebra.* We study this problem critically and find that only the charge-current density algebra can be satisfied at arbitrary momentum, and this in a trivial sense, when $\tilde{J}_0(0)$ is unity. In this result the currents are taken to be tensor operators with respect to the internal quantum numbers, which is a little more general than the factorization approach and the standard octet form of the currents. Given a model based on a wave equation, all the commutation relations can be explicitly evaluated.

II. COMPARISON OF THE TWO METHODS

Table I shows the framework of the two approaches and where the postulates are made. In the second

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[2] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 340 (1964).
[3] (a) The problem of the general solution of the commutation relations is discussed in Ref. 2 and by the following: M. Gell-Mann, D. Horn, and J. Weyers, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (North-Holland Publishing Co., Amsterdam, 1968), p. 479; W. Rühl, Nucl. Phys. **B3**, 637 (1967); H. Bebié and H. Leutwyler, Phys. Rev. Letters **19**, 618 (1967); S. J. Chang and L. O'Raifeartaigh, Phys. Rev. **170**, 1316 (1968); H. D. I. Abarbanel and Y. Frishman, *ibid.* **171**, 1442 (1968); S. J. Chang, R. F. Dashen, and L. O'Raifeartaigh, Phys. Rev. Letters **21**, 1026 (1968); **21**, 1507(E) (1968). (b) We are considering in particular the solutions obtained by using wave equations, for which see also the following: H. Leutwyler, *ibid.* **20**, 561 (1968); H. Bebié, F. Ghilmetti, V. Gorgé, and H. Leutwyler, Phys. Rev. **177**, 2133 (1969); **177**, 2146 (1969); B. Hamprecht and H. Kleinert, *ibid.* **180**, 1410 (1969); D. Corrigan, B. Hamprecht, and H. Kleinert, in *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1969), Vol. XIA; G. Cocho, C. Fronsdal, and R. White, Phys. Rev. **180**, 1547 (1969); C. Fronsdal, *ibid.* **182**, 1564 (1969).
[4] S. Fubini and G. Furlan, Physics **1**, 229 (1964); F. Coester and G. Roeppstorf, Phys. Rev. **155**, 1583 (1967).
[5] For references see the review articles of A. O. Barut, in *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1968), Vol. XB, p. 377; Acta Phys. Acad. Sci. Hung. **26**, 1 (1969).
[6] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. **167**, 1527 (1968).

TABLE I. Comparison of methods based (a) on current algebra and (b) on wave equations of dynamical groups.

	Current algebra	Models based on wave equations or dynamical groups
(A) Complete set of states of strong interactions	not specified (unknown)	specified by the solutions of the (postulated) wave equation
(B) Explicit form of currents	not specified (unknown)	conserved currents specified from the wave equation
(C) Weak and electromagnetic transition amplitudes	matrix elements of currents	matrix elements of currents[a]
(D) Commutation relations of currents	specified (postulated), e.g., $[j_0^\alpha(x),j_\mu^\beta(y)]=if^{\alpha\beta\gamma}j_\mu^\gamma(x)\delta^3(\mathbf{x}-\mathbf{y})$	to be evaluated from the explicit form of currents given above (B) (in general, different from current algebra)
(E) Comparison with experiments	via sum rules obtained from matrix elements of (D)	directly via the matrix elements in (C)

[a] From a practical point of view, one can quite well use an incomplete c-number field theory in which (A) and (B) are not completely specified, but in which well-defined postulated vertex functions take the place of (C). (D) now means a relation between vertex functions, which may be checked to see if it holds or not.

approach, we have in mind, besides the usual finite-dimensional wave equations, also the infinite-dimensional equations such as the Majorana equation $(\Gamma^\mu P_\mu-\kappa)\psi=0$, its generalization $(\alpha_1\Gamma^\mu P_\mu+\alpha_2 P_\mu{}^2-\kappa)\cdot\psi=0$, or the $O(4,2)$ equation $(\alpha_1\Gamma^\mu P_\mu+\alpha_2 P_\mu P^\mu+\alpha_3 P_\mu P^\mu S+\beta S+\gamma)\tilde{\psi}=0$, and others of the same type.

III. EVALUATION OF CURRENT COMMUTATORS FOR WAVE EQUATIONS

We consider a c-number theory. There are, in general, infinitely many mass states for a given momentum. We shall evaluate explicitly the commutation relations of currents. We first set forth the kinematical framework.

States

The complete set of states obtained from the wave equation is of the form $|p_\mu\rangle|\sigma\rangle$, where p_μ are the eigenvalues of the total momentum P_μ and σ is a set of spinor indices, in general of infinite range. We denote the normalized so-called boosted states by[7]

$$|p;\sigma\boldsymbol{\xi}\rangle=1/N_\sigma|p\rangle e^{i\boldsymbol{\xi}\cdot\mathbf{M}}|\sigma\rangle, \tag{1}$$

where $e^{i\boldsymbol{\xi}\cdot\mathbf{M}}$ are the representations of the pure Lorentz transformations acting on the spinor space $|\sigma\boldsymbol{\xi}\rangle\equiv e^{i\boldsymbol{\xi}\cdot\mathbf{M}}|\sigma\rangle=\sum_{\sigma'}D_{\sigma\sigma'}(\boldsymbol{\xi})|\sigma'\rangle$. We shall take a general framework and admit timelike, lightlike, and spacelike momenta as well. The nature of the σ indices depends on whether p_μ is timelike, lightlike, or spacelike; for example, in the Majorana equation, for p_μ timelike, $|\sigma\rangle$ is an eigenstate of Γ_0; for p_μ spacelike, $|\sigma\rangle$ can be taken to be an eigenstate of Γ_3. For simplicity we restrict ourselves to the case of a discrete mass spectrum.

Operators

We have two kinds of operators: the algebraic ones, ike $e^{i\boldsymbol{\xi}\cdot\mathbf{M}}$ and Γ_μ, which act only on the σ indices, and the momentum-dependent operators which act on $|p\rangle$, according to $P_\mu|p;\sigma\boldsymbol{\xi}\rangle=p_\mu|p;\sigma\boldsymbol{\xi}\rangle$.

[7] In general, the physical states contain, besides the boosting operation, another, the so-called tilting operation (see Ref. 5) of the form $e^{i\theta T}$. In this case what we are going to say in the following is valid for these tilted states.

Important Remark

There is no mass-shell condition on the p_μ because we are considering a system which can have many mass states, discrete or continuous. That is why both p_μ and $\boldsymbol{\xi}$ labels are needed in the labeling of states $|p;\sigma\boldsymbol{\xi}\rangle$. Moreover, p_μ can even be lightlike or spacelike; the range of σ is then determined accordingly. Thus we have the complete four-dimensional Minkowski space as the range of p_μ in (1). The label $\boldsymbol{\xi}$ is related to p_μ by $p_\mu=m(\cosh|\boldsymbol{\xi}|,\ \hat{\boldsymbol{\xi}}\sinh|\boldsymbol{\xi}|)$.

Momentum Translation Operator $T(k)$

In the full Minkowski space of the momentum, we define the following operator (more precisely, operator-valued distribution):

$$\langle p'|T(k)|p\rangle=\delta^{(4)}(p'-p-k)f(p'p), \tag{2}$$

where $f(p'p)$ expresses the restrictions on the possible values of k. These operators have the same properties as the translation operators in x space; in fact, $T(k)$ together with the Lorentz group defines a group isomorphic to the Poincaré group whose invariants are the t-channel mass and spin. This operator is useful in the proper definition of the current, as we shall see.

Physical Interpretation of the Matrix Elements of Currents

If $S(k)$, $J_{\rm em}{}^\mu(k)$, $J_{\rm weak}{}^\mu(k)$, $\cdots$, are specified operators acting on the Hilbert space $\mathcal{H}$ of states (1), the amplitude for a transition with momentum transfer k_μ under a scalar, electromagnetic, weak, $\cdots$, interaction, for example, is proportional to their matrix elements, e.g.,

$$F_\mu(k)=\langle p';\sigma'\boldsymbol{\xi}'|J_{\rm em}{}^\mu(k)|p;\sigma\boldsymbol{\xi}\rangle. \tag{3}$$

Rule for Matrix Elements of P_μ

For realistic wave equations with increasing mass spectrum, the current operator in general consists of algebraic and momentum-dependent parts. We write a typical current operator linear in the momenta as

$$J_\mu{}^{(k)}=a_\mu T(k)+b\{P_\mu,T(k)\}\,, \qquad (4)$$

where a_μ and b are algebraic operators acting on the spinor indices σ.

From (2) and (4) we obtain

$$\begin{aligned}\langle p';\sigma'\xi'|J_\mu(k)|p;\sigma\xi\rangle &=\langle\sigma'\xi'|[a_\mu+(p'+p)_\mu b]|\sigma\xi\rangle\langle p'|T(k)|p\rangle\\ &=\langle\sigma'\xi'|j_\mu(p',p)|\sigma\xi\rangle f(p'p)\delta^{(4)}(p'-p-k)\,. \qquad (5)\end{aligned}$$

Here $j_\mu(p',p)$ is the current operator in the spinor space depending now on c-number eigenvalues p' and p of P_μ. Terms of higher order in P_μ can be handled similarly. The form factors are determined by the matrix elements of $j_\mu(p',p)$.

Normalization of States

At zero momentum transfer, $k_\mu=0$, we shall write the orthonormality conditions with respect to the metric $J_0(0)$. From (5) we have first, with $j_0(p_0)\equiv j_0(p_0,p_0)$,

$$\langle p';\sigma'\xi'|J_0(0)|p;\sigma\xi\rangle =\langle\sigma'\xi'|j_0(p_0)|\sigma\xi\rangle f(p'p)\delta^{(4)}(p'-p)\,.$$

Now the spinor part can be evaluated from the first part of Eq. (5) using the states (1). By a proper choice of N_σ, we set, for $\xi'=\xi$,

$$\langle\sigma'\xi|j_0(p_0)|\sigma\xi\rangle=N(p_0)\delta_{\sigma'\sigma}\equiv N_0(\xi)\delta_{\sigma'\sigma}\,. \qquad (6)$$

For timelike momentum, for example, $N(p_0)=\cosh\xi=p_0/m$. A factor $N(p_0)$ transforming like the zero component of a vector is clearly necessary from convariance (see remark in Sec. IV). With this normalization the spinor part can now be evaluated for arbitrary $\xi'\neq\xi$. In the Appendix we show how to define another current operator $\tilde{J}_\mu(k)$ whose 0 component is the charge, for which we have

$$\langle p';\sigma'\xi'|\tilde{J}_0(0)|p;\sigma\xi\rangle=\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta_{\sigma'\sigma}N_0(\xi)\,. \qquad (7)$$

An expression of this form is necessary for comparison with the current algebra, which has a certain noncovariant character. The orthogonality relation (5) is in agreement with the condition of current conservation which, in operator form, reads

$$[J_\mu(k),P^\mu]=0\,, \qquad (8a)$$

and which gives

$$(p'-p)_\mu\langle p';\sigma'\xi'|J_\mu(k)|p;\sigma\xi\rangle=0\,, \qquad (8b)$$

and in spinor space

$$(p'-p)_\mu\langle\sigma'\xi'|j_\mu(p',p)|\sigma\xi\rangle=0\,,\quad p'=p+k\,. \qquad (8c)$$

The current used in (6) is a conserved current and (7) implies that the probability density of all states is normalized to 1. In the case when $J_\mu{}^{\mathrm{em}}$ is proportional to J_μ, all states $|\sigma\rangle$ have the same unit charge. Equation (8c) can be used to obtain the mass spectrum.[5,6]

Completeness

We assume that the spinor states $|\sigma\xi\rangle$ satisfy the following completeness relations:

$$\sum_\sigma|\sigma\xi\rangle\langle\sigma\xi|j_0(p_0,p_0)=N_0(\xi)\,, \qquad (9a)$$

and for the states $|p;\sigma\xi\rangle$

$$\int\sum_\sigma\frac{d^3p}{(2\pi)^3}\frac{|p;\sigma\xi\rangle\langle\sigma\xi;p|\tilde{J}_0(0)}{N_0(\xi)}=1\,. \qquad (9b)$$

Internal Quantum Numbers

The wave equation is first written down with fixed internal quantum numbers. We now take the coefficients in the wave equation to be tensor operators with respect to the internal quantum numbers. This means that the current is also a tensor operator in the internal quantum numbers, so we may consider in the usual way a set of current operators $J_\mu{}^\alpha(k)$, where α is an internal group index, e.g., $SU(3)$ or $SU(3)\times SU(3)$.

Evaluation of Current Commutators

We are now in a position to evaluate explicitly a commutator

$$C_\mu{}^{\alpha\beta}(k,k')=[\tilde{J}_0{}^\alpha(k),J_\mu{}^\beta(k')]$$

in our Hilbert space. Using the completeness (9b), we find

$$\begin{aligned}C=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\frac{1}{N_0}\{&\langle p',\sigma'\xi'|\tilde{J}_0{}^\alpha(k)\tilde{J}_0(0)|p,\sigma\xi\rangle\\ &\times\langle p,\sigma\xi|J_\mu{}^\beta(k')|p'',\sigma''\xi''\rangle-\langle p',\sigma'\xi'|J_\mu{}^\beta(k')|p,\sigma\xi\rangle\\ &\times\langle p,\sigma\xi|\tilde{J}_0(0)\tilde{J}_0{}^\alpha(k)|p'',\sigma''\xi''\rangle\}\,, \qquad (10)\end{aligned}$$

where σ now includes the internal quantum numbers of the states. If $k^\mu=0$ and

$$\tilde{J}_0{}^\alpha(0)\tilde{J}_0(0)=\tilde{J}_0{}^\alpha(0)\,,\quad\text{i.e.,}\quad\tilde{J}_0(0)=1\,, \qquad (11)$$

then, using the Wigner-Eckart theorem for the internal indices assuming the intermediate states belong to an octet, we obtain

$$[\tilde{J}_0{}^\alpha(0),J_\mu{}^\beta(k)]=if^{\alpha\beta\gamma}J_\mu{}^\gamma(k)\,. \qquad (12)$$

The existence of a physically meaningful operator with the properties (11) has still to be shown (see below). Given a current with $\tilde{J}_0(0)$ satisfying (11), one can, of course, calculate all other commutators. In the simple case in which one has, in addition to (11),

$$[\tilde{J}_0(k),J_\mu(k')]=0\,, \qquad (13)$$

e.g., when $\tilde{J}_\mu(k)=\tilde{J}_\mu(0)\tilde{T}(k)$, one finds even

$$[\tilde{J}_0{}^\alpha(k),J_\mu{}^\beta(k')]=if^{\alpha\beta\gamma}J_\mu{}^\gamma(k+k'). \qquad (14)$$

If one takes a current of the form (4), for which we have

$$\tilde{J}_\mu(k)=\tfrac{1}{2}\{\tilde{J}_\mu(0),\tilde{T}(k)\}, \qquad (15)$$

we again derive Eq. (14).

We remark for clarity that the condition (11) now implies that together with Eq. (7) the following equation has to be simultaneously satisfied:

$$\langle p';\sigma'\xi'|p;\sigma\xi\rangle=\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{\sigma'\sigma}N_0(\xi). \qquad (16)$$

Hence we see that the condition $\tilde{J}_0(0)=1$ cannot be simply achieved by a normalization of the states, unless the scalar product is suitably defined (see Example I).

In the case where there are states with lightlike and spacelike momenta, the condition $\tilde{J}_0(0)=1$ must hold simultaneously for all such states in order for the current algebra to be valid. We shall come back later to the question of the existence of a physically meaningful $\tilde{J}_0(0)$.

So far we have started from the covariant wave equation and determined what are the commutation relations of the current in the Hilbert space $\mathcal{H}$ of the complete set of solutions. Conversely, let us start from the matrix elements of the postulated current commutators of current algebra in some Hilbert space $\mathcal{H}'$. We label the states in *this* Hilbert space $\mathcal{H}'$ by momentum, a set of quantum numbers N labeling one-particle states, and relative quantum numbers λ for many-particle states. A particular matrix element is then given by

$$\sum_{N\lambda}\frac{1}{(2\pi)^3}\int\frac{d^3p}{p^0/m} \times[\langle p'N'|\tilde{J}_0{}^\alpha(k)|pN\lambda\rangle\langle pN\lambda|J_\mu{}^\beta(k')|p''N''\rangle - \langle p'N'|J_\mu{}^\beta(k')|pN\lambda\rangle\langle pN\lambda|\tilde{J}_0{}^\alpha(k)|p''N''\rangle] =if^{\alpha\beta\gamma}\langle p'N'|J_\mu{}^\gamma(k+k')|p''N''\rangle, \qquad (17)$$

and we ask if we can identify the states $|pN\lambda\rangle$ occurring here with the states (1) of the Hilbert space $\mathcal{H}$ obtained from the wave equation, for then we shall have a representation of the current algebra. Clearly, we must first make an approximation and put $\lambda=0$ in the intermediate states (∞-momentum limit, for example). Then we put $k=0$ and use the completeness (9b), *after* the $SU(3)$ parts of the matrix elements have been taken out. We see then that Eq. (17) is indeed satisfied, and this was the observation made in Ref. 3(b).

However, in writing down Eq. (17) the assumed completeness of the states $|pN\lambda\rangle$ was

$$\sum_{N\lambda}\int\frac{1}{(2\pi)^3}\frac{d^3p}{p^0/m}|pN\lambda\rangle\langle pN\lambda|=1, \qquad (18)$$

which is consistent with (9b) only if $\tilde{J}_0(0)\equiv1$, and we arrive at the same conclusion.

We now come back to the question of the existence of a physically meaningful $\tilde{J}_0(0)=1$. This problem is not simply solvable. We study two models, however.

Example I

Infinite-Dimensional Unitary SL(2,C) Equation

There are infinitely many mass values. The metric (3′) or (4) is positive definite and to guarantee an increasing mass spectrum we may take the current (4) with $a^\mu=\alpha\Gamma^\mu$ and $b=\beta\cdot I$. Clearly the current $\alpha\Gamma^\mu T(k)+\beta[P^\mu,T(k)]$ is not equal to unity for all states for fixed α and β because the matrix elements of Γ^μ and P^μ depend on the states. In practice, however, one only uses the postulate that

$$\langle\sigma\xi|[\alpha\Gamma^\mu+\beta(p+p')^\mu]|\sigma'\xi'\rangle \qquad (19)$$

gives the form factors of the electromagnetic current. Now, as in Ref. 6, one can normalize the states by a number depending on the quantum numbers σ of the states, so that

$$\langle\sigma\xi|[\alpha\Gamma^0+2\beta p^0]|\sigma'\xi\rangle=(p^0/m)\delta_{\sigma\sigma'}. \qquad (20)$$

Thus the scalar product must be defined according to (A7) and (16) as

$$\langle p\sigma\xi|\sigma'\xi'p'\rangle=(p^0/m)\delta_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}'-\mathbf{p}). \qquad (21)$$

Example II

Dirac Case (Quark Model)

Here too we have, not a well-defined operator $J^\mu(k)$, but instead the postulate that

$$\bar{u}(p)\gamma^\mu u(p')$$

describes a vertex. Since the current γ^μ has the property $\gamma^0\gamma^0=1$, we find that in this case indeed $j^0(p,p)=1$, i.e., the vertex function is $\bar{u}(p)\gamma^0u(p')$, whereas the normalization is given by $\bar{u}(p)\gamma^0u(p)=p^0/m$ and hence current algebra,

$$j_0(pp')j_\mu(p'p'')\equiv\bar{u}(p)\gamma_0u(p')(m/p'^0) \times\bar{u}(p')\gamma_\mu u(p'')=\bar{u}(p)\gamma^\mu u(p''),$$

holds.[8] In other words, in our notation, we have by definition the scalar product $\langle p|p'\rangle\equiv\langle\sigma|\gamma_0|\sigma\rangle\delta^{(3)}(\mathbf{p}'-\mathbf{p})$ and $\langle p|\tilde{J}(0)|p'\rangle=\langle\sigma|\gamma_0|\sigma\rangle\delta^{(3)}(\mathbf{p}'-\mathbf{p})$.

IV. CONCLUSIONS

We have seen that both in the framework of the infinite-component wave equations and that of the current algebra, the matrix elements of currents are identified with the physical vertex functions, and shown that, starting from the covariant-wave-equation for-

[8] The assumptions needed to pass from the commutators to the simple product j_0j_μ are given in Ref. 3 (e.g., Leutwyler).

malism, the charge-density algebra can be satisfied only if $\tilde{J}_0(0)=1$. We can normalize the states so that $\langle|\tilde{J}_0|\rangle=1$, and make $\tilde{J}_0(0)$ be the identity operator if the scalar product is suitably defined. There is another formal way (see Fronsdal[3]) of saturating the current algebra starting from the wave equation. Here one uses a Lagrangian formalism, the corresponding conjugate moment π to the fields φ, and canonical commutation relations between π and φ. One can then define the currents $J_\mu(k)$, more precisely its matrix elements—the currents as operators are not defined—in such a way that the current algebra is satisfied if one takes positive- and *negative*-energy solutions of the wave equation. In particular, the matrix elements of the charge is given by the inner product, which is, of course, equivalent to having $\tilde{J}_0(0)=1$. However, the negative-energy solutions do not lead to any physically interpretable form factors.

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APPENDIX: TWO KINDS OF CURRENT OPERATORS AND THE SCALAR PRODUCT

Let $I_\mu(x)$ be the current operator in the x space. Its Fourier transform is the first kind of current operator:

$$J_\mu(k)=(2\pi)^{-4}\int d^4x\, e^{ikx}I_\mu(x). \qquad \text{(A1)}$$

We have then

$$\langle p|J_\mu(k)|p'\rangle=\langle p|I_\mu(0)|p'\rangle\delta^{(4)}(p-p'+k). \qquad \text{(A2)}$$

We identify this with our Eq. (5) obtained from the wave equation. The second kind of current is defined at $t=0$:

$$\tilde{J}_\mu(k)=(2\pi)^{-3}\int d^3x\, e^{ikx}I_\mu(x)\bigg|_{t=0}, \qquad \text{(A3)}$$

and we find, similarly [note also that $\tilde{J}_\mu(k)=\int dk^0 J_\mu(k)$],

$$\langle p|\tilde{J}_\mu(k)|p'\rangle=\langle p|I_\mu(0)|p'\rangle\delta^{(3)}(\mathbf{p}'-\mathbf{p}+\mathbf{k}). \qquad \text{(A4)}$$

From Eqs. (2) and (5), in the text, we determine

$$\langle p|I_\mu(0)|p'\rangle=\langle\sigma'\xi'|j_\mu(p'p)|\sigma\xi\rangle f(p'p), \qquad \text{(A5)}$$

and insert this in Eq. (A4),

$$\langle p|\tilde{J}_\mu(k)|p'\rangle=\langle\sigma'\xi'|j_\mu(p'p)|\sigma\xi\rangle \times f(p'p)\delta^{(3)}(\mathbf{p}'-\mathbf{p}+\mathbf{k}). \qquad \text{(A6)}$$

The zero component of this equation at $k=0$ [in which case we can choose $f(p'p)=1$], combined with (6), leads to the final result (7) in the text.

From the condition $\tilde{J}_0(0)=1$, we have observed the relation (16) between the basis vectors. Thus, for a consistent realization of $\tilde{J}_0(0)=1$, we must define (16) in the spinor space, exactly like Eq. (A6), i.e.,

$$\langle p';\sigma'\xi'|p,\sigma\xi\rangle\equiv\langle\sigma'\xi'|j_0(p'p)|\sigma\xi\rangle\delta^{(3)}(\mathbf{p}'-\mathbf{p}+\mathbf{k}). \qquad \text{(A7)}$$

With this definition the charge-current commutation relations are always satisfied. If we now expand a general element of the Hilbert space as

$$\chi=\sum_\sigma\int\frac{d^3p}{(2\pi)^3}\frac{1}{N_0}\chi(p\sigma)|p,\sigma\rangle, \qquad \text{(A8)}$$

we find for the scalar product

$$(\chi,\chi)=\sum_\sigma\int\frac{d^3p}{(2\pi)^3N_0}|\chi(p,\sigma)|^2. \qquad \text{(A9)}$$

Finally, we define a $\tilde{T}(\mathbf{k})$ similar to $\tilde{J}_\mu(k)$ such that

$$\langle p'|\tilde{T}(\mathbf{k})|p\rangle=\delta^{(3)}(\mathbf{p}'-\mathbf{p}-\mathbf{k})f(p'p) \qquad \text{(A10)}$$

and $\tilde{J}_\mu(k)=a_\mu\tilde{T}(k)+b\{P_\mu,\tilde{T}(k)\}$, with

$$\tilde{T}(\mathbf{k})\tilde{T}(\mathbf{k}')=\tilde{T}(\mathbf{k}+\mathbf{k}'),\quad \tilde{T}(0)=1. \qquad \text{(A11)}$$