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Coherence of the Radiation from the Superradiant State

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Coherence properties of the single-mode electromagnetic radiation emitted by a system of identical two-level atoms which is initially in the superradiant state are determined approximately. The approximation is valid only for short times when the state of the atomic system does not differ appreciably from the superradiant state. The electromagnetic field radiated spontaneously possesses full coherence in the sense of Glauber's definition, its density operator having the form of that for the ideal laser radiation. This coherence disappears when, because of external radiation, stimulated and spontaneous emission both take place.

I. INTRODUCTION

THE purpose of this paper is to study the coherence properties of the quantized electromagnetic field emitted by a system of identical atoms or molecules in the highly correlated state, called the superradiant state by Dicke.[1] Following Dicke, many authors have referred to this radiation as a coherent spontaneous radiation since its intensity is proportional to N^2, N being the number of oscillators. On the other hand, Senitzky[2] claimed on the basis of perturbation theory that this radiation is incoherent because the expectation value of the radiation field emitted in the superradiant state vanishes. Today, owing to Glauber's[3] work, a complete theory of coherence for a quantized electromagnetic field is available. This enables us to decide the type of coherence of the spontaneous and the stimulated radiation from the superradiant state. It seems to us that the problem is worth studying since radiation of this type has been generated and observed experimentally in several regions of the electromagnetic spectrum, from radio frequencies in the early experiments of Hahn[4] on nuclear paramagnetic spin resonance to optical frequencies in the photon echo experiments.[5] Coherence properties of the observed radiation have not yet been measured but such measurements are feasible with modern photoelectric coincidence-counting techniques.[6] These measurements would provide additional information on the structure of the radiating system.

II. MODEL

We study a simplified model of N identical two-level atoms coupled in the dipole approximation with a single mode of the electromagnetic field. The motion of the atoms is neglected and they are treated as distinguishable (space wave functions do not overlap). We assume that the level spacing for each atom is equal to the mode frequency ($\hbar=1$). In this paper, we treat only the case when the radiation wavelength is large compared to the linear dimensions of the radiating system, but an extension to the small-wavelength region can be made with the use of Dicke's[1] method. The Hamiltonian of the system with the neglect of nonresonant terms has the form[7]

$$\begin{aligned} H&=H_0+H_I\,,\\ H_0&=\omega a^\dagger a+\omega R_3\,,\\ H_I&=-\kappa^* aR_+-\kappa a^\dagger R_-\,, \end{aligned}\tag{1}$$

where κ is the effective (complex) coupling constant. For the reader's convenience we list below the relevant properties of the R operators and of unperturbed states of the atomic system:

$$\begin{aligned} &[R_3,R_\pm]_-=\pm R_\pm\,,\quad [R_+,R_-]_-=2R_3\,,\\ &R^2\equiv R_3{}^2+\tfrac{1}{2}(R_+R_-+R_-R_+)\,,\\ &R_3|r,m\rangle=\sum_{j=1}^N R_{j3}|r,m\rangle=m|r,m\rangle\,,\\ &R^2|r,m\rangle=r(r+1)|r,m\rangle\,,\\ &R_\pm|r,m\rangle=\sum_{j=1}^N R_{j\pm}|r,m\rangle\\ &\qquad=[r(r+1)-m(m\pm1)]^{1/2}|r,m\pm1\rangle\,. \end{aligned}\tag{2}$$

III. COHERENCE OF SPONTANEOUS RADIATION

At the initial moment ($t=0$) the atomic system is assumed to be in the superradiant state $|r=\frac{1}{2}N,\, m=0\rangle$ and the electromagnetic field in the vacuum state. The time evolution of the state vector of the total system is described by the unitary operator $U(t)$:

$$|\psi(t)\rangle=U(t)|\psi(0)\rangle=e^{-itH}|r,0\rangle|0\rangle\,.\tag{3}$$

All information concerning the state of the electromagnetic field at time t is contained in the reduced density operator $\rho(t)$,

$$\rho(t)\equiv\mathrm{Tr}_A\{|\psi(t)\rangle\langle\psi(t)|\}\,,\tag{4}$$

[1] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[2] I. R. Senitzky, Phys. Rev. **111**, 3 (1958).
[3] R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).
[4] E. L. Hahn, Phys. Rev. **77**, 297 (1950); **80**, 580 (1950).
[5] N. A. Kurnit, I. D. Abella, and S. R. Hartmann, Phys. Rev. Letters **13**, 567 (1964); I. D. Abella, N. A. Kurnit, and S. R. Hartmann, Phys. Rev. **141**, 391 (1966).
[6] Such measurement were started by R. Hanbury Brown and R. Q. Twiss [Proc. Roy. Soc. (London) **A242**, 300 (1957)] and they were used by F. Davidson and L. Mandel [Phys. Letters **27A**, 579 (1968)] to determine the six-point correlation function.
[7] The form of the Hamiltonian and the notation are taken from the paper by M. Tavis and F. W. Cummings [Phys. Rev. **170**, 379 (1968)].

where Tr_A denotes the trace with respect to atomic states only. It is convenient to represent this density operator in the diagonal representation[3] in terms of coherent states $|\alpha\rangle$:

$$\rho(t)=\int d^2\alpha\, P(\alpha,t)|\alpha\rangle\langle\alpha|\,.$$

To compute the weight function $P(\alpha,t)$, we follow the procedure described in the recent book by Klauder and Sudarshan.[8] Since we are interested only in the radiation from the highly unstable[9] superradiant state, we must confine ourselves to times t much shorter than the time $1/|\kappa|r^{1/2}$. In this time interval, the state of the atomic system does not differ appreciably from the superradiant state and we can neglect R_3 everywhere as compared to R. As a consequence, we can neglect the commutator of R_+ and R_- as compared with their anticommutator. Since H_0 and H_I commute and the initial state $|\psi(0)\rangle$ is the eigenstate of H_0 belonging to the eigenvalue zero, we can write

$$|\psi(t)\rangle=\exp(it\kappa a^\dagger R_- + it\kappa^* aR_+)|\psi(0)\rangle\,. \tag{5}$$

On account of the approximate equality

$$[a^\dagger R_-,aR_+]_-\simeq -R^2 \tag{6}$$

and with the use of the Baker-Hausdorff formula, we obtain

$$|\psi(t)\rangle=\exp(-\tfrac{1}{2}t^2|\kappa|^2r^2)\exp(it\kappa a^\dagger R_-)\times\exp(it\kappa^* aR_+)|\psi(0)\rangle\,. \tag{7}$$

The coherent-state expectation value,

$$T(\alpha,t)\equiv\langle\alpha|\rho(t)|\alpha\rangle\,,$$

of the density operator, as evaluated in the Appendix for $t\ll 1/|\kappa|r^{1/2}$, is given by the formula

$$T(\alpha,t)=\exp(-t^2|\kappa|^2r^2)\exp(-|\alpha|^2)I_0(2t|\kappa|\,|\alpha|r)\,. \tag{8}$$

Its Fourier transform $\tilde{T}(z)$ is

$$\tilde{T}(z)=\exp(-\tfrac{1}{2}|z|^2)J_0(\sqrt{2}t|\kappa|r|z|)\,. \tag{9}$$

The weight function $P(\alpha,t)$ is obtained as a Fourier transform of the product $\tilde{T}(z)\exp(\frac{1}{2}|z|^2)/\pi$ and takes on the form

$$P(\alpha,t)=(1/2\pi|\alpha|)\delta(|\alpha|-t|\kappa|r)\,. \tag{10}$$

The electromagnetic field emitted by the atomic system in the superradiant state has therefore the coherence properties of the so-called ideal laser radiation whose mean photon number is fixed and whose phase is completely unspecified:

$$\rho(t)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\,|t|\kappa|re^{i\varphi}\rangle\langle t|\kappa|re^{i\varphi}|\,. \tag{11}$$

The even-order correlation functions $G^{(n,n)}(t)$ are

$$G^{(n,n)}(t)\equiv\mathrm{Tr}\{\rho(t)(a^\dagger)^na^n\}=(t|\kappa|r)^{2n}=[G^{(1,1)}(t)]^n\,. \tag{12}$$

All mixed-order correlation functions $G^{(m,n)}(t)$, $m\neq n$, vanish for all t, which is a straightforward consequence of H_0 being a constant of the motion. For $m\neq n$, we have

$$G^{(m,n)}(t)=[1/(m-n)\omega]\times\langle\psi(t)|[H_0,(a^\dagger)^ma^n]_-|\psi(t)\rangle\,, \tag{13}$$

and this vanishes on account of the relation $H_0|\psi(t)\rangle=U(t)H_0|\psi(0)\rangle=0$. Thus the average field vanishes for all t, but the radiation field emitted by the atomic system in the superradiant state has the complete even-order coherence.

IV. COHERENCE OF STIMULATED RADIATION

The electromagnetic field is now assumed to be initially in the coherent state $|\beta\rangle$ with the atomic system again in the superradiant state, since we want to study the coherence properties of the radiation stimulated by the incoming external single-mode field of the resonant frequency. We assume that $t\ll 1/|\kappa|r^{1/2}$ and that the average number of photons in the external radiation is at most of the order of r, which enables us to use the approximate relation (6). The reduced density operator for the electromagnetic field can be written in the form

$$\rho(t)=\exp(-it\omega a^\dagger a)\rho_I(t)\exp(it\omega a^\dagger a)\,, \tag{14}$$

where

$$\rho_I(t)=\mathrm{Tr}_A\{\exp(-itH_I)|\psi(0)\rangle\langle\psi(0)|\exp(itH_I)\} \tag{15}$$

and

$$|\psi(0)\rangle=|r,0\rangle|\beta\rangle\,.$$

The weight function of $\rho_I(t)$ in the P representation can be computed explicitly in our approximation (see the Appendix). The expectation value $T(\alpha,t)$ or $\rho_I(t)$ is

$$T(\alpha,t)=\exp(-t^2|\kappa|^2r^2)\exp(-|\alpha-\beta|^2)\times I_0(2t|\kappa|r|\alpha-\beta|)\,, \tag{16}$$

and this leads to the formula

$$\rho(\beta,t)=D(\beta e^{-i\omega t})\rho(0,t)D^{-1}(\beta e^{-i\omega t})\,, \tag{17}$$

where $\rho(0,t)$ is the density operator (11) for the vacuum

[8] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968), p. 178.
[9] In reality, the instability of the superradiant state is mainly due to the relaxation processes and inhomogeneities of the atomic system.

initial state and $D(\gamma)$ is the displacement operator for coherent states. The even-order correlation functions in this state are

$$G^{(n,n)}(t)=\sum_{k=0}^{n}\binom{n}{k}^2(t|\kappa|r)^{2k}|\beta|^{2n-2k}. \qquad (18)$$

Since $G^{(n,n)}(t)>[G^{(1,1)}(t)]^n$ for $n>1$, the even-order coherence present in the previous case is now destroyed by the incoming radiation.

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APPENDIX

(1) Evaluation of Weight Function $P(\alpha,t)$ for Spontaneous Radiation

In the first step we evaluate $T(\alpha,t)$.

$$\begin{aligned}T(\alpha,t)&=\sum_{\psi_A}\langle\alpha|\langle\psi_A|\psi(t)\rangle\langle\psi(t)|\psi_A\rangle|\alpha\rangle\\&=|\langle\alpha|0\rangle|^2\exp(-t^2|\kappa|^2r^2)\sum_{\psi_A}\langle r,0|\exp(-i\kappa^*t\alpha R_+)|\psi_A\rangle\langle\psi_A|\exp(i\kappa t\alpha^*R_-)|r,0\rangle\\&=\exp(-|\alpha|^2)\exp(-t^2|\kappa|^2r^2)\|\exp(i\kappa t\alpha^*R_-)|r,0\rangle\|^2\\&\simeq\exp(-|\alpha|^2)\exp(-t^2|\kappa|^2r^2)[1+t^2|\kappa|^2r^2|\alpha|^2+(t^2|\kappa|^2r^2|\alpha|^2)^2/(2!)^2+\cdots]\\&=\exp(-|\alpha|^2)\exp(-t^2|\kappa|^2r^2)I_0(2t|\kappa|r|\alpha|).\end{aligned}\qquad (A1)$$

In the above formula, the approximate ($t\ll1/|\kappa|r^{1/2}$, $r\gg1$) relation

$$\exp(i\kappa t\alpha^*R_-)|r,0\rangle\simeq|r,0\rangle+i\kappa t\alpha^*r|r,-1\rangle+[(i\kappa t\alpha^*r)^2/2!]|r,-2\rangle+\cdots \qquad (A2)$$

is used. The Fourier transform $\tilde{T}$ of T with respect to the real and imaginary part of α, $\alpha=(q+ip)/\sqrt{2}$, is

$$\tilde{T}(x,k)=\frac{1}{2\pi}\int dp\,dg\,e^{i(xp-kq)}e^{-\frac{1}{2}(p^2+q^2)}I_0(\sqrt{2}t|\kappa|r(p^2+q^2)^{1/2})\exp(-t^2|\kappa|^2r^2). \qquad (A3)$$

Going over to radial integration variables ρ and φ [$\rho=(p^2+q^2)^{1/2}$, $xp-kq=\rho(x^2+k^2)^{1/2}\sin\varphi$] and using Parceval's formula,[10]

$$\int_0^{2\pi}d\varphi\,\exp[i\rho(x^2+k^2)^{1/2}\sin\varphi]=2\pi J_0(\rho(x^2+k^2)^{1/2}),$$

we obtain[11]

$$\begin{aligned}\tilde{T}(x,k)&=\exp(-t^2|\kappa|^2r^2)\int_0^\infty d\rho\,\rho e^{-\frac{1}{2}\rho^2}I_0(\sqrt{2}t|\kappa|r\rho)J_0(\rho(x^2+k^2)^{1/2})\\&=\exp[-\tfrac{1}{2}(x^2+k^2)]J_0(\sqrt{2}t|\kappa|r(x^2+k^2)^{1/2}).\end{aligned}\qquad (A4)$$

The weight function $P(\alpha,t)$ is given therefore by the following Fourier integral:

$$P(\alpha,t)=\frac{1}{2\pi^2}\int dx\,dk\,e^{-i(xp-kq)}J_0(\sqrt{2}t|\kappa|r(x^2+k^2)^{1/2}). \qquad (A5)$$

In radial variables it reduces to

$$P(\alpha,t)=\frac{1}{2\pi}\int_0^\infty d\rho\,\rho J_0(t|\kappa|r\rho)J_0(|\alpha|\rho). \qquad (A6)$$

This is the representation of the Dirac δ function on the plane[12]:

$$P(\alpha,t)=(1/2\pi|\alpha|)\delta(|\alpha|-t|\kappa|r). \qquad (A7)$$

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[10] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (University Press, Cambridge, 1922), Sec. 2.2.
[11] Reference 10, Sec. 13.31.
[12] Reference 10, Sec. 14.4.

(2) Evaluation of Weight Fuuntion for Stimulated Radiation

The function $T(\alpha,t)$ is now given by the formula

$$\begin{aligned} T(\alpha,t) &= \exp(-t^2|\kappa|^2r^2)\sum_{\psi_A}|\langle\alpha|\langle\psi_A|\exp(i\kappa t\alpha^*R_-)\exp(i\kappa^*t\beta R_+)|r,0\rangle|\beta\rangle|^2 \\ &= \exp(-t^2|\kappa|^2r^2)|\langle\alpha|\beta\rangle|^2\langle r,0|\exp(-i\kappa t\beta^*R_-)\exp(-i\kappa^*t\alpha R_+)\exp(i\kappa t\alpha^*R_-)\exp(i\kappa^*t\beta R_+)|r,0\rangle \\ &= \exp(-t^2|\kappa|^2r^2)\exp(-|\alpha-\beta|^2)\|\exp[-i\kappa^*t(\alpha-\beta)R_+]|r,0\rangle\|^2 \\ &\simeq \exp(-t^2|\kappa|^2r^2)\exp(-|\alpha-\beta|^2)I_0(2t|\kappa|r|\alpha-\beta|)\,, \end{aligned} \tag{A8}$$

where we used the approximate commutativity of R_+ and R_- and the smallness of $t|\kappa|r^{1/2}$. Shifting the integration variables by $p\to p'=p+p_0$, $q\to q'=q+q_0$, where $\beta=(q_0+ip_0)/\sqrt{2}$, we can reduce the Fourier integral for $\tilde{T}(x,k)$ to the previous form (A4):

$$\tilde{T}(x,k)=e^{i(xp_0-kq_0)}\exp[-\tfrac{1}{2}(x^2+k^2)]J_0(\sqrt{2}t|\kappa|r(x^2+k^2)^{1/2})\,. \tag{A9}$$

The Fourier integral for the weight function $P(\alpha,t)$ now has the form

$$P(\alpha,t)=\frac{1}{2\pi^2}\int dx\,dk\,e^{-i[x(p-p_0)-k(q-q_0)]}J_0(\sqrt{2}t|\kappa|r(x^2+k^2)^{1/2})\,, \tag{A10}$$

which gives after the integration the formula

$$P(\alpha,t)=\delta(|\alpha-\beta|-t|x|r)1/(2\pi|\alpha-\beta|)\,. \tag{A11}$$

The density operator $\rho_I(t)$ can be therefore represented as

$$\rho_I(t)=\int d^2\alpha|\alpha+\beta\rangle\langle\alpha+\beta|\delta(|\alpha|-t|\kappa|r)/|\alpha|\,. \tag{A12}$$

The final expression for the complete density operator $\rho(\beta,t)$ can be expressed in terms of the displacement operator $D(\gamma)$ in the form

$$\begin{aligned} \rho(\beta,t) &= \exp(-it\omega a^\dagger a)\rho_I(t)\exp(it\omega a^\dagger a) \\ &= D(\beta e^{-i\omega t})\rho(0,t)D^{-1}(\beta e^{-i\omega t})\,, \end{aligned} \tag{A13}$$

where

$$\rho(0,t)=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\;|t|\kappa|re^{i\varphi}\rangle\langle t|\kappa|re^{i\varphi}|\,.$$

In order to evaluate the correlation functions $G^{(n,n)}(t)$, we use the relation

$$D^{-1}(\gamma)(a^\dagger)^m a^n D(\gamma)=(a^\dagger+\gamma^*)^m(a+\gamma)^n\,.$$

For even-order correlation functions we obtain

$$\begin{aligned} G^{(n,n)}(t) &= \frac{1}{2\pi}\int_0^{2\pi}d\varphi(t|\kappa|re^{-i\varphi}+\beta^*)^n(t|\kappa|re^{i\varphi}+\beta)^n \\ &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k,l=0}^{n}\binom{n}{k}\binom{n}{l}(t|\kappa|r)^{k+l}\exp[i\varphi(k-l)](\beta^*)^{n-k}\beta^{n-l} \\ &= \sum_{k=0}^{n}\binom{n}{k}^2(t|\kappa|r)^{2k}|\beta|^{2n-2k}\,. \end{aligned} \tag{A14}$$