$r$ in this approximation, one finds

$$\mathcal{E}=-2.2\times10^{-5}(G_W{}^4\mathfrak{N}^2/R^8)1/r. \qquad (20)$$

Thus, the weak interaction of two electrons with distant matter could produce an effective, attractive, long-range, $1/r$ potential. The effective coupling depends not only on the amount of the distant matter but also, in view of the results of Sec. III, on the fact that this matter is distributed in a highly inhomogeneous way. Comparing this force with gravity, we have

$$\text{(induced weak potential)/(gravitational potential)} \approx 10^{-91}\mathfrak{N}^2/R^8 \quad (R \text{ in cm}). \qquad (21)$$

If we take $\mathfrak{N}\approx10^{78}$ as roughly the number of electrons in the universe and, therefore, certainly the largest conceivable fluctuation, we find a force comparable with gravity only for $R\lesssim10^8$ cm. At such a radius the strong forces are already much more important than either gravity or the weak force. Actually perturbation theory for this problem breaks down when $G_W\mathfrak{N}/R^2\lesssim1$ or for the size fluctuation used above $R\approx10^{23}$ cm (density$\approx10^{-15}$ g/cm$^3$). We have not completely investigated the region below where the perturbation theory fails. The above arguments, however, do show that for fluctuations with

$$\mathfrak{N}/R^2\lesssim10^{32}\ \text{cm}^{-2},\quad \mathfrak{N}/R^4\lesssim10^{46}\ \text{cm}^{-4}, \qquad (22)$$

the effects of the long-range weak forces are negligible compared with gravity. There are no celestial objects from neutron stars to the universe itself which do not satisfy these conditions.

## V. CONCLUSION

The answer to the question of why the weak interactions are unimportant for the structure of the universe on the large scales lies not so much in the intrinsic nature or strength of the weak interactions as it does in the absence of large inhomogeneities or anisotropies.

## ACKNOWLEDGMENTS

Appreciation is expressed to R. F. Sawyer for several conversations and to the Aspen Center for Physics where this work was finished.

PHYSICAL REVIEW D VOLUME 1, NUMBER 2 15 JANUARY 1970

# Nonlinear Limit on Primeval Adiabatic Perturbations*

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(Received 12 June 1969)

The amplitude of adiabatic perturbations that might have existed in the early universe is limited by the nonlinear transfer of energy to high frequencies. For flat cosmological models, this limit, together with the effect of linear dissipation, yields a residual perturbation in the matter distribution comparable with the semiempirical estimate of what is needed to produce massive galaxies.

## INTRODUCTION

In the primeval fireball cosmology[1] it may be possible to come to some understanding of the origin of the galaxies from a detailed study of the time variation of density perturbations in the evolving universe. Following Zel'dovich,[2] one can separate the perturbation into two parts. In the initially isothermal part, the radiation is uniformly distributed. In the initially adiabatic part, the fractional perturbation to the radiation energy density is $\frac{4}{3}$ the fractional perturbation to the matter density, so the entropy per nucleon is constant. Until the primeval fireball temperature falls to 3000°K, the matter and radiation are locked together, and the initially adiabatic perturbation can act like an acoustic wave.[3]

There are two useful points of attack on initially adiabatic perturbations. The first is the consideration of the linear dissipation of such irregularities.[4] The second is the question of nonlinear effects. If the fraction perturbation $\delta\rho/\rho$ to the total mass density reached unity before the characteristic length $\lambda$ of the irregularity exceeded $ct$, that part of space would collapse back in on itself, perhaps forming a black hole.[5] If $\delta\rho/\rho$ were

* Supported in part by the National Science Foundation, under Grants Nos. GP-9433 (formerly GP-7976) and GP-9114, and by the U. S. Office of Naval Research under Contract No. Nonr-220(47).

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[1] G. Gamow, Phys. Rev. **74**, 505 (1948); R. A. Alpher, *ibid.* **74**, 1577 (1948); R. H. Dicke, P. J. E. Peebles, P. G. Roll, and D. T. Wilkinson, Astrophys. J. **142**, 414 (1965).

[2] Ya. B. Zel'dovich, Usp. Fiz. Nauk **89**, 647 (1966) [English transl.: Soviet Phys.—Usp. **9**, 602 (1967)].

[3] E. M. Lifshitz, J. Phys. Radium **10**, 116 (1946).

[4] The problem was first considered independently by R. W. Mitchie (unpublished), by P. J. E. Peebles [in Proceedings of the Texas Conference on Relativistic Astrophysics, New York, 1967 (to be published)], and by J. Silk [Astrophys. J. **151**, 459 (1968)].

[5] Ya. B. Zell'dovich and I. D. Novikov, Astron. Zh. **43**, 758 (1966) [English transl.: Soviet Astron.—AJ **10**, 602 (1967)].

somewhat smaller, the perturbation would reach the regime $\lambda \ll ct$, and would act there like a strong acoustic wave, tending to dissipate itself in a shock. My purpose here is to discuss the consequence of this latter nonlinear effect on the possible residual matter perturbation following decoupling of matter and radiation. It seems unlikely that a third question, the possible development of turbulence, would be important here, for although the Reynolds number may be enormous, the contemplated displacement from equilibrium still is a small fraction of a wavelength.

I consider a "weakly nonlinear" irregularity, which can be treated in a second-order perturbation calculation. In this approximation, each acoustic mode acts like a simple harmonic oscillator that can exchange energy with other modes, presumably tending thereby to establish equipartition. The higher frequencies are attenuated by the finite photon mean free path, and, since the nonlinear relaxation rate increases with increasing wave number, this can make the power spectrum fall off toward larger wave numbers.

The purpose of the following calculation is to find the characteristic time scale for the transfer of energy among the modes as a function of the amplitude of the perturbation. Evidently this effect will be important when the time scale becomes comparable to the characteristic expansion time from the cosmological model. Since the acoustic oscillation period for the wavelengths considered is very much shorter than these characteristic times, or the linear dissipation time, it is a reasonable approximation in the calculation to ignore the expansion of the universe and the linear dissipation. Finally, the fluid is taken to be a mixture of matter and radiation, and it is assumed that the entropy per nucleon is constant.

## CALCULATION

Since the fluid is assumed to be ideal, it is described by the equations[6]

$$(\rho c^2+P)\gamma^2\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]+c^2\nabla P+\mathbf{v}\frac{\partial P}{\partial t}=0,$$
$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho c^2+\frac{Pv^2}{c^2}\right)\right]+\nabla\cdot[\gamma^2\mathbf{v}(\rho c^2+P)]=0, \tag{1}$$

where $\gamma$ is the usual time dilation factor. These two equations may be combined in a wave equation of the sort

$$\frac{\partial^2}{\partial t^2}\left[\left(\rho c^2+\frac{Pv^2}{c^2}\right)\gamma^2\right]=c^2\nabla^2P+\nabla\cdot\left\{(\rho c^2+P)\gamma^2(\mathbf{v}\cdot\nabla)\mathbf{v}-\mathbf{v}\frac{\partial}{\partial t}[\gamma^2(\rho c^2+P)]+\mathbf{v}\frac{\partial P}{\partial t}\right\}. \tag{2}$$

[6] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1959), p. 499.

In a second-order calculation, one can replace the velocity wherever it appears in Eq. (2) with the velocity derived from the first-order theory. With space periodic in some large volume $V$, we write

$$\rho=\rho_0(1+\textstyle\sum \delta_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}). \tag{3}$$

From the first-order theory, for a pressure wave, we derive

$$\mathbf{v}=S\textstyle\sum i\mathbf{k}\dot{\delta}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}/k^2, \tag{4}$$

where

$$\dot{\delta}_{\mathbf{k}}=\partial\delta_{\mathbf{k}}/\partial t,\quad S=(\tfrac{3}{4}+R)/(1+R),\quad R=3\rho_mc^2/4\mathcal{E}_r, \tag{5}$$

$\rho_m$ and $\mathcal{E}_r/c^2$ being the contributions to the mass density by matter and radiation. The equation of state expanded to second order in $\delta$ is

$$P=P_0+\rho_0c_1^2\delta+\rho_0B\delta^2,\quad c_1^2=\frac{c^2}{3(1+R)},$$
$$B=\frac{c^2R(1+4R/3)}{24(1+R)^3}. \tag{6}$$

On substituting Eqs. (3), (4), and (6) into Eq. (2), discarding all terms beyond the second order, and then extracting the $\mathbf{k}$ component, one finds

$$\frac{\partial^2\delta_{\mathbf{k}}}{\partial t^2}+k^2c_1^2\delta_{\mathbf{k}}=-Bk^2\sum_{\mathbf{k}'}\delta_{\mathbf{k}'}\delta_{\mathbf{k}''}+S\sum_{\mathbf{k}'}\left[\frac{\mathbf{k}'\cdot\mathbf{k}''}{k'^2k''^2}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{(\mathbf{k}\cdot\mathbf{k}')(\mathbf{k}'\cdot\mathbf{k}'')}{k'^2k''^2}+\frac{\mathbf{k}\cdot\mathbf{k}'}{k'^2}\right]\dot{\delta}_{\mathbf{k}'}\dot{\delta}_{\mathbf{k}''}, \tag{7}$$

where

$$\mathbf{k}''=\mathbf{k}-\mathbf{k}'. \tag{8}$$

To obtain an approximate solution, suppose that in the right-hand side $\delta_{\mathbf{k}}$ is the given function

$$\delta_{\mathbf{k}}=f(k)[e^{-ikc_1t+i\varphi_{\mathbf{k}}}+e^{ikc_1t-i\varphi_{-\mathbf{k}}}]/\sqrt{2}, \tag{9}$$

where the $\varphi_{\mathbf{k}}$ are randomly chosen phases. We now approximate the expected shape of the power spectrum $f(k)^2$ by the assumption that $f^2$ is flat when $k$ is less than a cutoff $k_m$, and $f^2=0$, $k>k_m$. Then the right-hand side of Eq. (7) becomes a sum over given sinusoidally oscillating terms, and it is straightforward to write down the solution $\delta_{\mathbf{k}}(t)$ for $k>k_m$. It is convenient to normalize this solution by introducing the variance $v$ of the initial perturbation:

$$v\equiv\langle(\delta\rho/\rho)^2\rangle=\sum_{k<k_m}|\delta_{\mathbf{k}}|^2=\frac{Vf^2k_m^3}{6\pi^2}. \tag{10}$$

The contribution to the variance by the higher frequencies generated through the nonlinearity is given by the equation

$$\frac{\delta v}{v^2} \equiv \frac{1}{v_0} = \sum_{k_m \leq k \leq 2k_m} \frac{\langle|\delta_k(t)|^2\rangle}{v^2}$$

$$= \frac{213\pi}{1120} k_m c_1 t \frac{(1+4R/3)^2(1+7R/4)^2}{(1+R)^4}. \qquad (11)$$

This solution is valid when $k_m c_1 t \gg 1$ and $\delta v \ll v$.

## APPLICATION TO COSMOLOGICAL MODEL

Equation (11) may be applied to the cosmological problem in the following simple-minded way. The age $t$ of a cosmological model is also a characteristic expansion time. Thus when this value for $t$ is substituted into Eq. (11), the value $v_0$ thus defined is an approximate upper limit to the variance at wave number $k_m$ that can exist without appreciable nonlinear transfer of energy to higher frequencies. Explicitly, let $M(k)$ be the mass with a sphere with diameter equal to $2\pi/k$, and write the variance of the matter perturbation as

$$v = \int \mathcal{P} d \ln M , \quad \mathcal{P} = \frac{V}{6\pi^2} k^3 |\delta_k|^2. \qquad (12)$$

The function $\mathcal{P}$ is the contribution to the variance for unit increment of the logarithm of the mass. Then the limit is

$$\mathcal{P}(k) \lesssim v_0(k). \qquad (13)$$

If the initial values led to a violation of this limit at any epoch, the nonlinear interaction would reduce the perturbation, leaving a residual which ought to be comparable to $v_0$.

In the cosmologically flat general-relativity model, the ratio of $v_0^{1/2}$ to the amplitude $\delta(t)$ computed in the linear approximation[4,7] reaches a minimum at $T \sim 20\,000°$K, nearly independent of wave number. This means that if the amplitude were equal to the limit $v_0(t)^{1/2}$ at this epoch, the amplitude would be below the corresponding limits at earlier and later epochs. The resulting limit on the residual perturbation to the distribution of matter evaluated at 2000°K is shown in Fig. 1 for the flat general-relativity model.

If it is assumed (a) that massive protogalaxies formed through gravitational forces alone and (b) that the protogalaxies formed at $10^8$–$10^9$ yr, then at recombination we must have had $\mathcal{P}_f^{1/2} \sim 10^{-2}$–$10^{-3}$.[8] In the flat general-relativity model, the computed limit on $\mathcal{P}_f^{1/2}$ is $2\times10^{-3}$ for a mass of $2\times10^{11} M_\odot$. In the scalar-tensor model[9] this limit would be larger by a factor of $\sim 20$. Thus, in either theory it is possible to assume that massive systems like our own galaxy formed from an initially adiabatic perturbation (if the universe is cosmologically flat). This picture is to a degree independent of initial conditions, for if the initial power were higher, it would be pulled down to the computed value by the nonlinear interaction.

[7] We use the results obtained from the numerical integration of the radiation collision equation by P. J. E. Peebles and J. T. Yu (to be published).

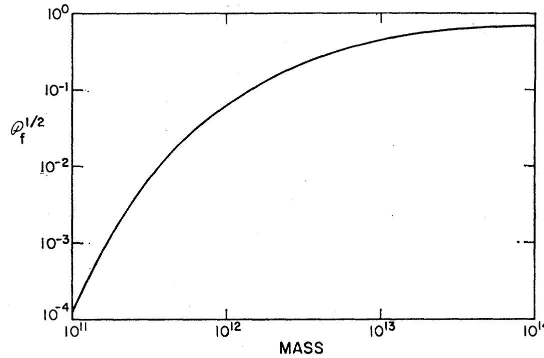


FIG. 1. Upper limit on the residual matter perturbation $\mathcal{P}_f^{1/2}$ [Eq. (12)] from the nonlinear limit [Eq. (13)] and the linear-dissipation result. This curve is evaluated at 2000°K, for the flat general-relativity model. The mass unit is solar masses.

The initial conditions do play a key role—when the mass $\gg 10^{11} M_\odot$, the value of $\mathcal{P}_f^{1/2}$ must fall below the limiting curve, for otherwise very massive systems would have formed too soon.[8] One might speculate that the initial value of $\mathcal{P}^{1/2}$ increased rapidly with increasing wave number. The combination of this (at this point *ad hoc*) initial condition with the dissipation at large wave number could yield a curve $\mathcal{P}_f^{1/2}(M)$ which peaks up to $\sim 0.01$ at $\sim 3\times10^{11} M_\odot$. The result would be the formation of massive protogalaxies, followed at a later epoch by the formation of clusters of galaxies.

This calculation shows that massive galaxies could have formed at a reasonable time from initially adiabatic perturbations. This result is, of course, at best only a small beginning toward a theory of the origin of the galaxies. Apart from the vexing question of the initial conditions, there remains the problem of the numerous smaller galaxies for which one apparently must invoke some different process, like fragmentation of massive protogalaxies, or initially isothermal perturbations.

[8] P. J. E. Peebles, J. Roy. Astron. Soc. Can. **63**, 4 (1969).
[9] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).