Differential Equations for Perturbations on the Schwarzschild Metric

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Differential equations for perturbations on the Schwarzschild metric have appeared incorrectly in the literature. This paper presents the correct set of equations.

I. INTRODUCTION

THE purpose of this paper is to present the correct equations for the Regge-Wheeler perturbations on the Schwarzschild metric. These equations were first derived by Regge and Wheeler.[1] Manasse[2] pointed out that the equations appearing in the literature contained errors and were inconsistent with the Einstein field equations. Brill and Hartle[3] rederived the odd-parity equations, which once again contained some errors as published.

We have obtained the equations for both odd- and even-parity perturbations and have verified their internal consistency which provides a check on their correctness. To avoid typographical errors, we have independently checked the proofs until we were able to return a copy free of errors. As a number of physicists working or planning to work with these perturbations have asked for copies of the equations, we have written this paper and hope it will clear up any confusion that may exist.

II. ODD-PARITY EQUATIONS

The notation for the remainder of the paper is the same as that of Ref. 1. The spherically symmetric line element is

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (1)$$

For such a spherically symmetric background, $\delta R_{\mu\nu}$ has the same angular dependence and same number of independent radial factors as the perturbations in Eqs. (12) and (13) of Regge and Wheeler. Hence the components of $\delta R_{\mu\nu}$ that we list in both the odd- and even-parity cases are sufficient to determine all the components.

In the Regge-Wheeler canonical gauge, the first-order perturbations of the Ricci tensor are[4]

$$\delta R_{23} = -\frac{1}{2}\left\{ike^{-\nu}h_0 + e^{-\lambda}\left[\frac{1}{2}(\nu_r - \lambda_r)h_1 + \frac{dh_1}{dr}\right]\right\} \times\left(\cos\theta\frac{d}{d\theta} - \sin\theta\frac{d^2}{d\theta^2}\right)P_L e^{-ikt}, \qquad (2a)$$

$$\delta R_{13} = \frac{1}{2}\left\{ike^{-\nu}\left(\frac{dh_0}{dr} - 2\frac{h_0}{r}\right) + \left[\frac{L(L+1)}{r^2} - e^{-\nu}k^2 + \frac{e^{-\lambda}}{r}\left(\lambda_r - \nu_r - \frac{2}{r}\right)\right]h_1\right\}\sin\theta\frac{dP_L}{d\theta}e^{-ikt}, \qquad (2b)$$

$$\delta R_{03} = -\left\{\frac{1}{2}e^{-\lambda}\frac{d^2h_0}{dr^2} + ik\left[\frac{1}{2}e^{-\lambda}\frac{dh_1}{dr} + \frac{e^{-\lambda}}{r}h_1 - \frac{e^{-\lambda}}{4}(\nu_r + \lambda_r)h_1\right] - \frac{e^{-\lambda}}{4}(\nu_r + \lambda_r)\frac{dh_0}{dr} + \left(\frac{e^{-\lambda}}{r}\nu_r - \frac{L(L+1)}{2r^2}\right)h_0\right\}\sin\theta\frac{dP_L}{d\theta}e^{-ikt}, \qquad (2c)$$

where

$$\lambda_r = \frac{d\lambda}{dr}, \quad \lambda_{rr} = \frac{d^2\lambda}{dr^2}, \quad \text{etc.}$$

The angular factors in Eqs. (2) are identically zero for $L=0$, and similarly, δR_{23} is identically zero for $L=1$. For higher values of L, the empty-space field equations $\delta R_{\mu\nu}=0$ imply that the bracketed factors vanish, giving three radial equations. The second-order radial equation is a consequence of the other two, provided

$$(\nu-\lambda)_{rr} + \tfrac{1}{2}(\nu_r - \lambda_r)^2 - \tfrac{1}{2}[\nu_r^2 - \lambda_r^2] + (1/r)(\nu_r - 3\lambda_r) = 0. \qquad (3)$$

This condition is satisfied by the Schwarzchild metric

$$e^{\nu} = e^{-\lambda} = 1 - 2m/r. \qquad (4)$$

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[1] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[2] F. Manasse, J. Math. Phys. **4**, 746 (1963).

[3] D. R. Brill and J. B. Hartle, Phys. Rev. **135**, B271 (1964). The results derived in this paper were correct, however, since the authors considered the equations only in the limit of large values of L.

[4] K. Thorne and A. Compolottaro (private communication) have checked Eqs. (2) on an IBM-7094 computer using a program of Thorne and Zimmerman and agree with our results.

Eliminate h_0 from the first-order equations to obtain

$$\frac{d^2h_1}{dr^2}+\left[\tfrac{3}{2}(\nu_r-\lambda_r)-\frac{2}{r}\right]\frac{dh_1}{dr}+\left[\tfrac{1}{2}(\nu-\lambda)_{rr}+\tfrac{1}{2}(\nu_r-\lambda_r)^2 -e^{\lambda}\frac{L(L+1)}{r^2}+k^2e^{\lambda-\nu}+\frac{2}{r^2}\right]h_1=0. \quad (5)$$

Define

$$Q=e^{(\nu-\lambda)/2}h_1/r=(1-2m/r)h_1/r$$

and

$$dr^*=e^{(\lambda-\nu)/2}dr=(1-2m/r)^{-1}dr, \quad (6)$$

and find

$$\frac{d^2Q}{dr^{*2}}+(k^2-V_{\mathrm{eff}})Q=0, \quad (7)$$

where

$$V_{\mathrm{eff}}=\frac{e^{\nu}L(L+1)}{r^2}-\frac{3}{r}\frac{d}{dr^*}\left[e^{(\nu-\lambda)/2}\right],$$

or, for the Schwarzschild background metric,

$$V_{\mathrm{eff}}=\left[\frac{L(L+1)}{r^2}-\frac{6m}{r^3}\right]\left(1-\frac{2m}{r}\right).$$

III. EVEN-PARITY EQUATIONS

In the Regge-Wheeler canonical gauge,[5] the independent first-order perturbations of the Ricci tensor are

$$\delta R_{01}=\left\{ik\left[\frac{dK}{dr}+\left(\frac{1}{r}-\frac{\nu_r}{2}\right)K-\frac{1}{r}H_2\right]+e^{-\lambda}\left[\frac{e^{\lambda}L(L+1)}{2r^2}-\frac{\nu_{rr}}{2}-\frac{\nu_r^2}{4}+\frac{\nu_r\lambda_r}{4}-\frac{\nu_r}{r}\right]H_1\right\}P_Le^{-ikt},$$

$$\delta R_{12}=\left[\tfrac{1}{2}ike^{-\nu}H_1+\frac{1}{2}\frac{dH_0}{dr}-\frac{1}{2}\frac{dK}{dr}+\left(\frac{\nu_r}{4}-\frac{1}{2r}\right)H_0+\left(\frac{\nu_r}{4}+\frac{1}{2r}\right)H_2\right]\frac{dP_L}{d\theta}e^{-ikt},$$

$$\delta R_{02}=\left[\tfrac{1}{2}ik(H_2+K)+\tfrac{1}{2}e^{-\lambda}\frac{dH_1}{dr}+\tfrac{1}{4}(\nu_r-\lambda_r)e^{-\lambda}H_1\right]\frac{dP_L}{d\theta}e^{-ikt},$$

$$\delta R_{00}=\left\{\tfrac{1}{2}k^2(H_2+2K)+ike^{-\lambda}\left[\left(\frac{\lambda_r}{2}-\frac{2}{r}\right)H_1-\frac{dH_1}{dr}\right]-\tfrac{1}{2}e^{\nu-\lambda}\frac{d^2H_0}{dr^2}+e^{\nu-\lambda}\left(\frac{\lambda_r}{4}-\frac{\nu_r}{2}-\frac{1}{r}\right)\frac{dH_0}{dr}-\tfrac{1}{4}e^{\nu-\lambda}\nu_r\frac{dH_2}{dr}+\tfrac{1}{2}e^{\nu-\lambda}\nu_r\frac{dK}{dr}\right.$$
$$\left.+e^{\nu-\lambda}\left[\frac{\nu_r\lambda_r}{4}-\frac{\nu_{rr}}{2}-\frac{\nu_r^2}{4}-\frac{\nu_r}{r}+\frac{1}{2r^2}e^{\lambda}L(L+1)\right]H_0+e^{\nu-\lambda}\left[\frac{\nu_r\lambda_r}{4}-\frac{\nu_{rr}}{2}-\frac{\nu_r^2}{4}-\frac{\nu_r}{r}\right]H_2\right\}P_Le^{-ikt}, \quad (8)$$

$$\delta R_{11}=\left[ike^{-\nu}\left(\frac{dH_1}{dr}-\frac{\lambda_r}{2}H_1\right)-\tfrac{1}{2}k^2e^{\lambda-\nu}H_2+\frac{1}{2}\frac{d^2H_0}{dr^2}-\frac{d^2K}{dr^2}+\left(\frac{\nu_r}{2}-\frac{\lambda_r}{4}\right)\frac{dH_0}{dr}+\left(\frac{\nu_r}{4}+\frac{1}{r}\right)\frac{dH_2}{dr}\right.$$
$$\left.+\left(\frac{\lambda_r}{2}-\frac{2}{r}\right)\frac{dK}{dr}+\frac{L(L+1)}{2r^2}e^{\lambda}H_2\right]P_Le^{-ikt},$$

$$\delta R_{22}=\left\{-\tfrac{1}{2}k^2e^{-\nu}r^2K+ikre^{-(\lambda+\nu)}H_1-\tfrac{1}{2}e^{-\lambda}r^2\frac{d^2K}{dr^2}+\tfrac{1}{2}re^{-\lambda}\frac{dH_0}{dr}+\tfrac{1}{2}re^{-\lambda}\frac{dH_2}{dr}+(\tfrac{1}{4}r^2e^{-\lambda}\lambda_r-\tfrac{1}{4}r^2e^{-\lambda}\nu_r-2re^{-\lambda})\frac{dK}{dr}\right.$$
$$\left.+[\tfrac{1}{2}re^{-\lambda}(\nu_r-\lambda_r)+e^{-\lambda}]H_2+[\tfrac{1}{2}re^{-\lambda}(\lambda_r-\nu_r)-e^{-\lambda}+\tfrac{1}{2}L(L+1)]K\right\}P_Le^{-ikt}+\tfrac{1}{2}(H_0-H_2)\frac{d^2P_L}{d\theta^2}e^{-ikt}.$$

The angular factors containing derivatives vanish for $L=0$; also, the two angular factors in the expression for δR_{22} are not independent when $L=1$. For all higher values of L, the empty-space field equations $\delta R_{\mu\nu}=0$ imply that the bracketed factors vanish, and specializing to the Schwarzschild background metric yields the following system of radial equations:

$$H_0=H_2\equiv H, \quad (9a)$$

$$\frac{dK}{dr}+\frac{1}{r}(K-H)-\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}K-i\frac{L(L+1)}{2kr^2}H_1=0, \quad (9b)$$

[5] For the background metric of Eq. (1), the Regge-Wheeler perturbations h_{00} and h_{11} defined by Regge and Wheeler for the special case of the Schwarzschild exterior metric would be generalized to $h_{00}=e^{\nu}H_0(r)P_L(\cos\theta)e^{-ikt}$ and $h_{11}=e^{\lambda}H_2(r)P_L(\cos\theta)e^{-ikt}$.

$$\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)H_1\right]+ik(H+K)=0, \tag{9c}$$

$$ikH_1+\left(1-\frac{2m}{r}\right)\left(\frac{dH}{dr}-\frac{dK}{dr}\right)+\frac{2m}{r^2}H=0, \tag{9d}$$

$$\left(1-\frac{2m}{r}\right)^2\frac{d^2H}{dr^2}+\frac{2}{r}\left(1-\frac{2m}{r}\right)\frac{dH}{dr}-k^2H-L(L+1)\left(1-\frac{2m}{r}\right)\frac{H}{r^2}+2ikm\frac{H_1}{r^2}+2ik\left(1-\frac{2m}{r}\right)\frac{1}{r^2}\frac{d}{dr}(r^2H_1)$$
$$-2k^2K-2\left(1-\frac{2m}{r}\right)\frac{m}{r^2}\frac{dK}{dr}=0, \tag{9e}$$

$$2ik\left(1-\frac{2m}{r}\right)\frac{dH_1}{dr}+2ikm\frac{H_1}{r^2}-k^2H+\left(1-\frac{2m}{r}\right)^2\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dH}{dr}\right)-\frac{2}{r^2}\frac{d}{dr}\left(r^2\frac{dK}{dr}\right)\right]$$
$$+\left(1-\frac{2m}{r}\right)\left[\frac{4m}{r^2}\frac{dH}{dr}-\frac{2m}{r^2}\frac{dK}{dr}+L(L+1)\frac{H}{r^2}\right]=0, \tag{9f}$$

$$\frac{d}{dr}\left\{\left(1-\frac{2m}{r}\right)\left[\frac{d}{dr}(r^2K)-2rH\right]\right\}-L(L+1)K+k^2r^2\left(1-\frac{2m}{r}\right)^{-1}K-2ikrH_1=0. \tag{9g}$$

The three first-order equations may be used to derive any of the second-order equations, provided the following algebraic relationship holds:

$$F(r)\equiv-\left[\frac{6m}{r}+(L-1)(L+2)\right]H+\left[(L-1)(L+2)+\frac{2m}{r}\right.$$
$$\left.-2\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{m^2}{r^2}+r^2k^2\right)\right]K+\left[2ikr-\frac{iL(L+1)}{k}\frac{m}{r^2}\right]H_1=0. \tag{10}$$

If F is constructed out of the solutions of the first-order equations (9a)–(9d), it can be shown that

$$\frac{dF}{dr}+\frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}F=0. \tag{11}$$

Hence, if the boundary conditions are chosen so that F vanishes anywhere, it vanishes everywhere.

Finally, the three first-order equations together with Eq. (10) give a second-order equation in a single unknown. Define

$$\bar{S}=\left(1-\frac{2m}{r}\right)\frac{H_1}{r},$$

$$x=\frac{r}{2m},\quad x^*=\frac{r^*}{2m},\quad \bar{k}=2mk.$$

Then

$$\frac{d^2\bar{S}}{dx^{*2}}+\left[(L-1)(L+2)+\frac{3}{x}\right]\frac{4}{x}\left(1-\frac{3}{2x}\right)\frac{1}{D(x)}\frac{d\bar{S}}{dx^*}+\left\{\bar{k}^2-\frac{1}{D(x)}\left(1-\frac{3}{2x}\right)\left[8\bar{k}^2-2L(L+1)\frac{1}{x^3}\right.\right.$$
$$\left.\left.-\frac{4(L-1)(L+2)}{x^2}\left(1-\frac{1}{x}\right)-\frac{12}{x^3}\left(1-\frac{1}{x}\right)\right]-\frac{2}{x^2}\left(1-\frac{3}{2x}\right)\left(1-\frac{1}{x}\right)-\frac{L(L+1)}{x^2}\left(1-\frac{1}{x}\right)\right\}\bar{S}=0, \tag{12}$$

where

$$D(x)=\frac{4}{x}+2(L-1)(L+2)-2\left(\frac{1}{4x^2}+\bar{k}^2x^2\right)\left(\frac{x}{x-1}\right).$$

Note that $D(x)$ approaches infinity as r tends to $2m$ or ∞. Also, any zeros of $D(x)$ are only apparent singularities, as the numerator can be shown to vanish at these points.

Similarly, with the quantities x, x^*, $\bar{k}$, and $D(x)$ defined as before, the differential equation for the function $S=H_1/r$ is

$$\frac{d^2S}{dx^{*2}}+\left\{\frac{2}{x}+\frac{1}{D(x)}\left[(L-1)(L+2)+\frac{3}{x}\right]\frac{4}{x}\left(1-\frac{3}{2x}\right)\right\}\frac{dS}{dx^*}+\left\{\bar{k}^2-\frac{1}{D(x)}\left(1-\frac{3}{2x}\right)\right.$$
$$\left.\times\left[8\bar{k}^2-\frac{2L(L+1)}{x^3}-\frac{4(L-1)(L+2)}{x^2}-\frac{12}{x^3}\right]-\frac{2}{x^2}\left(1-\frac{3}{2x}\right)-\frac{L(L+1)}{x^2}\left(1-\frac{1}{x}\right)\right\}S=0. \quad (13)$$

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Remarks on Electromagnetic Mass Differences*

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An $O(4,2)$ model for the electromagnetic transitions of the $I=\frac{1}{2}$ nucleon resonances is inserted into a finite-energy sum rule for forward off-shell Compton scattering in order to estimate the neutron-proton mass splitting. The result is encouraging. Comments are made on the importance of the nonpole terms and of the high-momentum-transfer behavior of the form factor in the case of the pion mass difference.

I. NEUTRON-PROTON MASS DIFFERENCE

IN a recent paper,[1] Buccella *et al.* apply finite-energy sum rules to the computation of the kaon mass splitting. Provided we accept their arguments that the result obtained by setting the lower limit of the Regge region, $N(q^2)$, equal to the photon laboratory energy ν for the highest resonance included, the K^*, the calculation is a complete success. This value,

$$N(q^2)=q^2/2m_K+0.6\text{ GeV}, \quad (1)$$

is certainly too low, but the hope is that the error made in the asymptotic behavior is at least roughly canceled by the omission of higher resonant states.

In order to calculate the nucleon mass difference, one requires the ratio of the A_2 coupling to nucleons to the coupling to kaons. Since high-energy scattering data[2] permit this ratio of Regge residues to lie anywhere between 0 and 1.2, Buccella *et al.* take it from the work of Cabibbo, Horwitz, and Ne'eman.[3] In this theory, however, the A_2 coupling to the baryon octet is pure F and $\gamma_{NNA_2}/\gamma_{KKA_2}=1$. The prescription for incorporating some D coupling to conform to the experimental[4] $F/D\approx-2$ is therefore unclear. The method selected in Ref. 1, setting the K coupling equal to the nucleon F coupling and simply adding the appropriate amount of D, yields $F+D=\frac{1}{2}$ and, as is crucial for good agreement with the experimental $m_p-m_n=-1.3$ MeV, $\gamma_{NNA_2}/\gamma_{KKA_2}=\frac{1}{2}$. But, *a priori*, one might have insisted equally well on the usual normalization $F+D=1$, in which case one still has $\gamma_{NNA_2}/\gamma_{KKA_2}=1$. So, one must consider this ratio as a free parameter in the computation and an independent determination of the neutron-proton mass splitting acquires considerable interest.

The analysis of Ref. 1 gives for the mass shift of any hadron,

$$\Delta m=\sum_r \Delta m^{(r)}+\Delta m^{\text{sub}}+\Delta m^{\text{as}}, \quad (2)$$

where $\Delta m^{(r)}$ is obtained by substituting the contribution of the rth resonance into the Cottingham formula[5]

$$\Delta m=-\frac{1}{4\pi}\int_0^\infty \frac{dq^2}{q^2}\int_{-q}^{+q} d\nu(q^2-\nu^2)^{1/2}T(q^2,\nu), \quad (3)$$

$$T(q^2,\nu)=3q^2t_1(q^2,i\nu)-(2\nu^2+q^2)t_2(q^2,i\nu), \quad (4)$$

$$\Delta m^{\text{sub}}=-\frac{3}{4\pi}\int_0^\infty q^2dq^2\beta(q^2)\frac{[N(q^2)]^{\alpha(0)}}{\alpha(0)}, \quad (5)$$

and Δm^{as} is numerically negligible. In order to eliminate possible wrong-signature fixed poles, we shall calculate

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