Comments and Addenda

The Comments and Addenda section is for short communications which are not of such urgency as to justify publication in Physical Review Letters *and are not appropriate for regular articles. It includes only the following types of communications: (1) comments on papers previously published in* The Physical Review *or* Physical Review Letters; *(2) addenda to papers previously published in* The Physical Review *or* Physical Review Letters, *in which the additional information can be presented without the need for writing a complete article. Manuscripts intended for this section may be accompanied by a brief abstract for information retrieval purposes. Accepted manuscripts will follow the same publication schedule as articles in this journal, and galleys will be sent to authors.*

Degeneracy of the Mass Spectrum for Infinite-Component Fields

A. I. OKSAK
Institute for High-Energy Physics, Serpukhov, USSR

AND

I. T. TODOROV*
Institute for Advanced Study, Princeton, New Jersey 08540

(Received 10 December 1969)

The "no-go" theorem of Grodsky and Streater is extended to nonrenomalizable local theories in which the two-point function may increase faster than any polynomial in momentum space. As a result the nonexistence of a local irreducible infinite-component field with a nontrivial mass spectrum is demonstrated without "technical" assumptions.

THE difficulties in describing a nondegenerate mass spectrum (as a function of spin) by using local infinite-component fields were first encountered in the study of simple infinite-component wave equations[1,2] and led finally to the discovery of the "no-go" theorem by Grodsky and Streater.[3] However, this negative result was not conclusive, since heavy use was made in Ref. 3 (as well as in the more heuristic approach in Ref. 2) of the "technical" assumption that the two-point function is polynomially bounded in momentum space. This assumption is not at all natural in a theory involving infinitely many different spin values. In fact, the assumption of polynomial boundedness being discarded, models have been constructed which provide a nontrivial mass spectrum.[4,5] These models are, however, trivial in the sense that they deal with fields which are infinite sums of conventional finite-component fields. It is clear that any desired mass spectrum can be fitted this way. The question arises whether a Lorentz-irreducible local infinite-component field exists in the wider class of nonrenormalizable theories which has a nontrivial mass spectrum. We will show that the answer to this question is "no." Our main tool will be a generalization of the Källén-Lehmann representation of the two-point function, applicable to infinite-component fields.

The representations of the spinor Lorentz group $SL(2,C)$ will be realized as argument transformations in the set of functions $\phi(z)$ of two complex variables $z=(z_1,z_2)$[6–8]: $[T(A)\phi](z)=\phi(zA)$. Any irreducible representation $\chi=[k,c]$ of $SL(2,C)$ is realized in the space of homogeneous functions ϕ of index of homogeneity χ:

$$\phi(\rho^{1/2}e^{i\alpha/2}z)=\rho^{c-1}e^{ik\alpha}\phi(z) \quad \text{for } \rho>0 \text{ and } \alpha \text{ real}, \quad (1)$$

where $k=0, \pm\frac{1}{2}, \pm1, \ldots$, and c is an arbitrary complex number. A field ψ transforming according to an arbitrary representation of $SL(2,C)$ will be written as a (generalized) function of two variables $\psi=\psi(x,z)$. Its components will be labeled by

$$\psi(x,f)=\int\int\psi(x,z)f(z)d^2zd^2\bar{z},$$

* On leave from the Joint Institute for Nuclear Research, Dubna, USSR, and from the Physical Institute of the Bulgarian Academy of Sciences, Sofia, Bulgaria.

1 E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. **159**, 1222 (1967); W. Rühl, Commun. Math. Phys. **6**, 312 (1967); D. Tz. Stoyanov and I. T. Todorov, J. Math. Phys. **9**, 2146 (1968).

2 H. D. I. Abarbanel and Y. Frishman, Phys. Rev. **171**, 1442 (1968).

3 I. T. Grodsky and R. F. Streater, Phys. Rev. Letters **20**, 695 (1968); R. F. Streater, in *Nobel Symposium 8. Elementary Particle Theory. Relativistic Groups and Analyticity* (Almqvist and Wiksell, Stockholm, 1968), pp. 149–156.

4 H. D. I. Abarbanel, Joseph Henry Laboratories, Princeton University Report, 1969 (unpublished).

5 A. Chodos, Phys. Rev. D **1**, 2937 (1970); N. Wu, University of Maryland Technical Report No. 903, 1968 (unpublished).

6 I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions, Vol. 5: Integral Geometry and Representation Theory* (Academic, New York, 1966); see also Appendix B to Vol. 1: I. M. Gel'fand and G. E. Shilov, *Properties and Operations* (Academic, New York, 1964).

7 I. T. Todorov and R. P. Zaikov, J. Math. Phys. **10**, 2014 (1969); I. T. Todorov, in *Nobel Symposium 8. Elementary Particle Theory. Relativistic Groups and Analyticity* (Almqvist and Wiksell, Stockholm, 1968), pp. 133–148.

8 A. I. Oksak and I. T. Todorov, Commun. Math. Phys. **14**, 271 (1969).

where $f(z)$ is any sufficiently smooth function which vanishes for $z=0$ and for $|z|\to\infty$.

We assume that the Fourier transform $\tilde{F}$ of the two-point function $F_{\psi\psi^*}(x-y)$ [i.e., of the vacuum expectation value of the product $\psi(x,f)\psi^*(y,g)$] is defined as a distribution in momentum space with the following properties.

(i) $\tilde{F}(p;f,g)\in D'(R_4)$ (i.e., it is a Schwartz distribution) and supp $\tilde{F}(p;f,g)\subset\bar{V}_+=\{p\in R_4: p_0\geq|\mathbf{p}|\}$ (in other words, $\tilde{F}$ vanishes outside the forward light cone, which is an implication of the *spectrum condition*).

(ii) Define the correspondence $A\to\Lambda=\Lambda(A)$ between the 2×2 matrices A of $SL(2,C)$ and the restricted Lorentz transformation Λ by $A\tilde{p}A^*=\widetilde{\Lambda p}$, where $\tilde{p}=p^\mu\sigma_\mu$ (σ_j are the Pauli matrices; σ_0 is the 2×2 unit matrix). Then the condition of *Lorentz covariance* implies that $\tilde{F}(\Lambda(A)p;zA^{-1},wA^{-1})=\tilde{F}(p;z,w)$.

(iii) For each $y\in V_+$ (i.e., $y_0>0$, $y^2>0$) the product $e^{-yp}\tilde{F}(p;f,g)$ is a tempered distribution [i.e., it belongs to $S'(R_4)$]. As a consequence, $\tilde{F}$ admits a Laplace transform

$$F(\zeta;f,g)=\frac{1}{(2\pi)^2}\int e^{-ip\zeta}\tilde{F}(p;f,g)d^4p\,,$$

which is analytic in ζ in the tube domain $T_-=R_4-iV_+$. This requirement defines a *localizable nonrenormalizable theory* (or a *nonrenormalizable theory of the first kind*[9,10]). This class of theories includes as a special case the Jaffe strictly localizable fields.[11]

(iv) The functions $F_{\psi\psi^*}(\zeta;f,g)$ and $\sigma F_{\psi^*\psi}(-\zeta;g,f)$ (where $\sigma=\pm1$ is independent of ζ, f, and g) allow an analytic continuation to real spacelike vectors $\zeta=x$ and coincide for such vectors (*locality*). According to Streater's theorem[12] this implies that $F(\zeta;f,g)$ $(=F_{\psi\psi^*})$ can be continued analytically in the extended tube $\mathcal{T}=\{\zeta\in C_4:\zeta^2\in\mathfrak{M}=C_1\diagdown[0,\infty)\}$.

(v) The field $\psi(x;z)$ transforms according to an *irreducible representation* $\chi=[k,c]$ of $SL(2,C)$ (acting on the second argument); consequently, $\psi^*(x;w)$ transforms under the complex conjugate representation $\chi=[-k,\bar{c}]$. According to (1), this leads to the homogeneity condition

$$F(\zeta;\rho_1^{1/2}e^{i\alpha_1/2}z,\rho_2^{1/2}e^{i\alpha_2/2}w)=\rho_1^{c-1}\rho_2^{\bar{c}-1}e^{ik(\alpha_1-\alpha_2)}F(\zeta;z,w)\,. \quad (2)$$

[In order to incorporate the irreducible finite-component fields for which $c-|k|$ is a positive integer we have to assume in addition to (2) that F is a polynomial in z, $\bar{z}$ and w, $\bar{w}$ in that case.]

[9] B. Schroer, J. Math. Phys. **5**, 1361 (1964).

[10] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **47**, 1966 (1964) [Soviet Phys. JETP **20**, 1320 (1965)]; S. S. Khoruzhii, Dokl. Akad. Nauk SSSR **177**, 206 (1967) [Soviet Phys. Doklady **12**, 1019 (1968)].

[11] A. Jaffe, Phys. Rev. Letters **17**, 661 (1966); Phys. Rev. **158**, 1454 (1967); Society for Industrial and Applied Mathematics, J. Appl. Math. **15**, 1046 (1967).

[12] R. F. Streater, J. Math. Phys. **3**, 256 (1962).

The positivity of the metric in the Hilbert space of physical states implies the positive definiteness of the kernel $\tilde{F}(p;z,w)$ (cf. Ref. 8). However, we will not need this property in what follows.

The local covariant two-point function can be written as a superposition of scalar functions of ζ^2, satisfying the Källén-Lehmann representation with covariant coefficients which are polynomials in ζ.

Theorem 1. Let the two-point function $F(\zeta;z,w)$ satisfy conditions (i)–(v) stated above. Then F can be written as a finite sum:

$$F(\zeta;z,w)=\sum_{n=0,1,2,\ldots}\sum_i\zeta_{\mu_1}\cdots\zeta_{\mu_n}\Gamma_{(i)}{}^{\mu_1\cdots\mu_n}(z,w)A_{ni}(\zeta^2)\,, \quad (3)$$

where $\Gamma_{(i)}{}^{\mu_1\cdots\mu_n}(z,w)$ are traceless symmetric invariant tensors of rank n satisfying the homogeneity condition (2), and $A_{ni}(t)$ are analytic in the cut plane $\mathfrak{M}$ $(=C_1\diagdown[0,\infty))$. All coefficients A_{ni} vanish unless c is real or $c+\bar{c}$ is an integer.

Outline of the proof.[13] (A) Let $\tilde{\zeta}=(\zeta^{\alpha\dot{\beta}})=\zeta^\mu\sigma_\mu$ (α, $\dot{\beta}=1,2$). The analyticity of $F(\zeta;f,g)$ in the extended tube and the invariance condition (ii) may be written in the form of a system of linear partial differential equations:

$$\frac{\partial F}{\partial\bar{\zeta}_\mu}=\frac{\partial F}{\partial\bar{\zeta}^{\alpha\dot{\beta}}}=0,\quad \mu=0,1,2,3\quad(\alpha,\beta=1,2);\quad (4)$$

$$\left(\zeta^{\alpha\dot{\kappa}}\frac{\partial}{\partial\zeta^{\beta\dot{\kappa}}}-z_\beta\frac{\partial}{\partial z_\alpha}-w_\beta\frac{\partial}{\partial w_\alpha}\right)F=0\,,\quad \alpha=1,2,\quad\beta=3-\alpha;\quad (5a)$$

$$\left(\zeta^{\kappa\dot{\alpha}}\frac{\partial}{\partial\zeta^{\kappa\dot{\beta}}}-\bar{z}_{\dot{\beta}}\frac{\partial}{\partial\bar{z}_{\dot{\alpha}}}-\bar{w}_{\dot{\beta}}\frac{\partial}{\partial\bar{w}_{\dot{\alpha}}}\right)F=0\,,\quad \dot{\alpha}=1,2,\quad\dot{\beta}=4-\dot{\alpha}\quad (5b)$$

[summation is to be carried out over the repeated index $\dot{c}$ (or c) from 1 to 2].

(B) The general solution of Eqs. (4) and (5) in the domain where

$$s=z\epsilon w=z_1w_2-z_2w_1\neq0\quad\left[\epsilon=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right]\quad (6)$$

can be expressed in terms of the holomorphic in ζ algebraic invariants s, $\bar{s}$ and

$$t=\zeta^2=\det\tilde{\zeta}\,,\quad u_{11}=z\tilde{\zeta}\bar{z}\,,\quad u_{12}=z\tilde{\zeta}\bar{w}\,,\quad u_{21}=w\tilde{\zeta}\bar{z}\,,\quad u_{22}=w\tilde{\zeta}\bar{w}\,. \quad (7)$$

There exists exactly one relation among these invariants in the domain (6), namely,

$$\det u=u_{11}u_{22}-u_{12}u_{21}=ts\bar{s}\,. \quad (8)$$

[13] A detailed proof of this theorem is given in A. I. Oksak and I. T. Todorov, Joint Institute for Nuclear Research, Dubna, USSR, Report No. P2-4812, 1969 (unpublished).

For $s \neq 0$ it allows us to determine t as a function of u and s. Thus we have

$$F(\zeta; z,w) = \mathfrak{F}(u; s), \quad (9)$$

where $\mathfrak{F}$ is a distribution in s in the domain (6) and is analytic in u in the extended tube, i.e., for $\det u \in \mathfrak{M}$.

On the other hand, the general solution of Eqs. (4) and (5) with support on the surface $z\epsilon w=0$ but with $u_{11}=z\tilde{\zeta}\bar{z}\neq 0$ is a finite sum of the form

$$F(\zeta; z,w) = \sum_{m,n} \left(\frac{\partial}{\partial w}\epsilon^{-1}\tilde{\zeta}\bar{z}\right)^m \left(z\tilde{\zeta}\epsilon\frac{\partial}{\partial \bar{w}}\right)^n \times [g_{mn}(t,u_{11}; q)\delta(s)], \quad (10)$$

where $q=w_1/z_1$ $(=w_2/z_2)$ and the distributions g_{mn} are analytic in the first two arguments [for $t(=\zeta^2)\in\mathfrak{M}$ and $u_{11}\neq 0$].

(C) The sum of (9) and (10) gives the general form of the invariant analytic two-point function. Next, we impose the irreducibility condition (2). It cannot be satisfied in the domain (6) (with a nonvanishing $\mathfrak{F}$) unless c is real and then

$$F(\zeta; z,w) = \mathfrak{F}(u; s) = u_{12}^{2k}|s|^{2(c-k-1)}f(t,\kappa), \quad (11)$$

where

$$\kappa = \frac{u_{11}u_{22}+u_{12}u_{21}}{u_{11}u_{22}-u_{12}u_{21}} = 1+2\frac{u_{12}u_{21}}{|s|^2 t}, \quad (12)$$

and f is analytic in $\mathfrak{M}\times C$. (We have assumed, without loss of generality, that $k\geq 0$.) The distribution F given by (11) for $s\neq 0$ allows a continuation preserving (i)–(v) in the domain containing $s=0$ if and only if $f(t,\kappa)$ is a polynomial in κ of the form

$$f(t,\kappa) = \sum_{l=0}^{N} a_l(t)\kappa^l/\Gamma(c-l-k), \quad (13)$$

where $a_l(t)$ are analytic in $\mathfrak{M}$.

(D) An invariant distribution of the type (10) (with support on the surface $s=0$) can satisfy the homogeneity condition (2) only if $c+\bar{c}=n$ is an integer. We will assume that $n\geq 0$ [which can be achieved by using the equivalence between the representations χ and $-\chi$ for $c\neq\pm(|k|+N)$, $N=1, 2, \ldots$]. In this case the combination of (10) and (2) gives

$$F(\zeta; z,w) = (z\tilde{\zeta}\bar{z})^n \sum_{m=0,1,2,\ldots} b_m(\zeta^2) \times\left[\left(\frac{\partial}{\partial w}\epsilon^{-1}\tilde{\zeta}\bar{z}\right)\left(z\tilde{\zeta}\epsilon\frac{\partial}{\partial w}\right)\right]^m [q^{\bar{c}-k+m}\bar{q}^{\bar{c}+k+m}\delta(z\epsilon w)], \quad (14)$$

where $q=w_1/z_1$; the functions $b_m(t)$ are analytic for $t\in\mathfrak{M}$ and only a finite number of them do not vanish.

(E) Equations (11)–(14) determine the general form of the two-point function satisfying (i)–(v). The last step in the argument consists of showing that the representation (3) unifies all the different cases. The linearly independent irreducible tensors $\Gamma_i^{\mu_1\cdots\mu_n}(z,w)$ can be determined explicitly, by making use of (11)–(14) and observing that $\Box_\zeta\{\zeta_{\mu_1}\cdots\zeta_{\mu_n}\Gamma_i^{\mu_1\cdots\mu_n}\}=0$. (For fixed χ and n, the index i may take at most two values.)

According to our previous results,[7,8] the spin decomposition of the two-point function in momentum space (for $k\geq 0$) reads

$$\tilde{F}(p; z,w) = \theta(p_0)(z\tilde{p}\bar{z})^{c-k-1}(w\tilde{p}\bar{w})^{\bar{c}-k-1}(z\tilde{p}\bar{w})^{2k} \times \sum_{j=k,k+1,\ldots} \rho_j(p^2)P_{j-k}^{(0,2k)}(\nu), \quad (15)$$

where

$$\nu = \frac{|z\tilde{p}\bar{w}|^2 - p^2|z\epsilon w|^2}{z\tilde{p}\bar{z}w\tilde{p}\bar{w}}, \quad (16)$$

$P_n^{(\alpha,\beta)}(\nu)$ are the Jacobi polynomials, and $\operatorname{supp}\rho_j(\tau) \subset (0,\infty)$.

The combination of Eqs. (3) and (15) implies the degeneracy of the mass spectrum for irreducible infinite-component fields.

Theorem 2. Let the two-point function satisfy conditions (i)–(v) (with $k\geq 0$) and let there be only a finite number of nonvanishing terms in the decomposition (15) for $(0<)a<p^2<b$. Then $c-k=N+1$ is a positive integer and $\tilde{F}$ is a polynomial in z, $\bar{z}$, w, and $\bar{w}$ in the above interval:

$$\tilde{F}(p; z,w) = \theta(p_0)(z\tilde{p}\bar{z}w\tilde{p}\bar{w})^N(z\tilde{p}\bar{w})^{2k} \times \sum_{j=k}^{k+N} \rho_j(p^2)P_{j-k}^{(0,2k)}(\nu). \quad (17)$$

The statement results from a straightforward comparison of Eq. (15) with the Fourier transform of (3). Thus, we have proved that only the finite-component part in the Lorentz decomposition of ψ (which exists only for $c-|k|=1, 2, \ldots$) can contribute to a given mass level which is not infinitely degenerate with respect to spin.

The implications of this negative result to the single-particle states saturation of current algebra and superconvergence sum rules and to the infinite-component wave equations were made explicit in Ref. 3.

One of the authors (I.T.T.) would like to thank Professor C. Kaysen for his hospitality at the Institute for Advanced Study in Princeton, where this work was partly done.