towards the right, this now being the manifestation of a reduced strength for the effective $(V+\delta V)$ potential. However, unlike the case of B.E., where the correction is always represented by a small decrease over the NR value, the corresponding quantity for $a_{1/2}$ can now show a more fluctuating variation depending on the particular value being considered for λ. In particular, for $\lambda=25.8\alpha^3$, which gives B.E. = 7.2 MeV, the value of $a_{1/2}$ changes from $+0.36$ F to -0.33 F as a result of the relativistic correction.

To summarize, we have considered relativistic corrections to certain three-body parameters in a rather simple model which is realistic as far as the results depend on the totally symmetric part of the three-body wave function. There is only one kind of "deuteron," and hence only one adjustable strength parameter for the two-body potential V. Only a limited aspect of the relativistic correction to the three-body problem has been considered, viz., one which bears on the Shirokov correction δV to the potential, and which vanishes in the two-body c.m. frame. This has the advantage of yielding no correction to the two-body parameters, and hence does not require any concomitant readjustments in the parameters of V. The three-body effect of this correction is unique, predicting about 5% decrease in the "triton's" B.E., and a lateral shift of the $a_{1/2}$ curves versus λ to the right. This simple investigation leaves open the broader and more involved aspects of relativistic corrections which bear on the actual dynamics of two- and three-body systems.

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Sign of the Absorptive Elastic Amplitude below the Physical Threshold

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We show that for elastic processes which have absorptive channels below the physical threshold, the absorptive partial-wave amplitude $a_l(s)$ has the sign of $(-1)^l$ for s below the physical threshold, where l is the orbital angular momentum. The result is valid for any stable particles of arbitrary spins. More generally, for elastic partial wave transitions between states of definite parity $P=\epsilon\eta_1\eta_2$ (where η_1 and η_2 are the intrinsic parities and $\epsilon=\pm1$), the sign of the absorptive amplitude below threshold is that of ϵ.

INTRODUCTION

The positivity of the absorptive part of the partial-wave elastic amplitudes has been extensively used, together with analyticity of the scattering amplitude as a function of the energy and momentum transfer, to derive a variety of bounds and inequalities for the scattering amplitude and cross sections. This positivity follows immediately from unitarity at physical values of the energy. However, there are systems such as nucleon-antinucleon or $\bar{K}$-nucleon which can go into channels such as π-π in the first case and π-Λ in the second (or even many-particle channels) with a total mass smaller than the total mass of the initial system. In those cases there will be a range of energies below the physical threshold in which the absorptive amplitude for elastic scattering does not vanish. In this region the absorptive amplitude has to be defined by analytic continuation in the masses of the external particles and therefore its sign is not immediately known. In this note, we establish that the sign of the absorptive part $a_l(s)$ of an elastic partial wave amplitude $f_l(s)$ is given by $(-1)^l$, where l is the orbital angular momentum. We first consider the case of spinless particles.

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I. SPINLESS PARTICLES

Let $F(s,t)$ be the amplitude for elastic scattering of particles m_1 and m_2 with initial and final momenta (p_1,p_2) and (p_1',p_2'), respectively, and let us assume that there are channels with energy threshold below the physical threshold $(m_1+m_2)^2=s_0$.

The absorptive amplitude is given by

$$A(s,t)=(4p_{10}p_{10}')^{1/2}\sum_n \langle p_1'|j_2(0)|n\rangle \times\langle n|j_2(0)^\dagger|p_1\rangle(2\pi)^4\delta^4(p_n-p_1-p_2)\,. \quad (1)$$

Below the physical threshold one cannot have $p_1+p_2=p_n$ with physical momenta. The absorptive amplitude in this region is defined by taking particle m_2 off the mass shell and then analytically continuing in p_2^2 $(p_2'^2)$ to the physical value $p_2^2=p_2'^2=m_2^2$. To do that, we use Dyson's representation[1] for the matrix element:

$$\int\langle n|[j_2(0)^\dagger,j_1(x)^\dagger]_R|0\rangle e^{-ip_1\cdot x}d^4x =F(n;p_1,p_2)=\int\frac{d^4ud\lambda^2\psi(n;u,\lambda^2)}{(k+u)^2-\lambda^2}\,, \quad (2)$$

where $k=\frac{1}{2}(p_1-p_2)$ and $p_n=p_1+p_2$.

[1] F. Dyson, Phys. Rev. 110, 1460 (1958).

The matrix element $(2p_{10})^{1/2}\langle n|j_2(0)^\dagger|p_1\rangle$ differs from $F(n;p_1,p_2)$ in terms that arise from equal-time commutators. These terms are polynomials in $\mathbf{k}$ and we defer to later a discussion of their contributions.

In the center-of-mass system, we have $\mathbf{p}_1+\mathbf{p}_2=0$ and $p_{10}+p_{20}=\sqrt{s}$. Then (2) can be written as

$$F(n;p_1,p_2)=\int\frac{d^4ud\lambda^2\psi(n;u,\lambda^2)}{\Lambda-2\mathbf{k}\cdot\mathbf{u}}=\phi(n;p_2^2,\mathbf{k}),\quad(3)$$

where

$$\Lambda=[u_0+(m_1^2-p_2^2)/2\sqrt{s}]^2-\mathbf{u}^2-\mathbf{k}^2-\lambda^2.\quad(4)$$

For $\mathbf{k}^2>0$ one can show[2] that, over the whole region of integration where $\psi(n;u,\lambda^2)$ has its support, $\Lambda^2-4\mathbf{k}^2\mathbf{u}^2$ is positive so that the denominator in the integrand in (4) never vanishes.

The absorptive part of the partial-wave amplitude is given by

$$a_l(s)=\frac{1}{2}\int A(s,t)P_l(\hat{k}\cdot\hat{k}')d(\hat{k}\cdot\hat{k}')$$
$$=\frac{2l+1}{16\pi}\int A(s,t)\sum_m Y_l^m(\hat{k}')Y_l^m(\hat{k})^*d\hat{k}'d\hat{k},\quad(5)$$

where Y_l^m are normalized spherical harmonics, $\hat{k}$ and $\hat{k}'$ are unit vectors in the directions of $\mathbf{k}$, $\mathbf{k}'$, respectively, and we have used the addition theorem for Legendre polynomials. Inserting (3) in the expression (1) for $A(s,t)$ and taking it into (5), one obtains, for values of p_2^2 such that $\mathbf{k}^2>0$,

$$a_l(s,p_2^2)=\frac{2l+1}{16\pi}\sum_{n,m}\bar{\phi}_l^m(n;p_2^2,k)\phi_l^m(n;p_2^2,k),\quad(6)$$

where

$$\phi_l^m(n;p_2^2,k)=\int d\hat{k}\,Y_l^m(\hat{k})^*\phi(n;p_2^2,\mathbf{k}),\quad(7)$$

with

$$k=[s^2-2s(m_1^2+p_2^2)+(m_1^2-p_2^2)^2]^{1/2}/2\sqrt{s}.\quad(8)$$

Using the property $Y_l^m(-\hat{k})=(-1)^lY_l^m(\mathbf{k})$, one can write

$$\phi_l^m(n;p_2^2,k)=\frac{1}{2}\int dk\,Y_l^m(\hat{k})^*$$
$$\times[\phi(n;p_2^2,\mathbf{k})+(-1)^l\phi(n;p_2^2,-\mathbf{k})].\quad(9)$$

For $k^2>0$, $\bar{\phi}_l^m$ is just ϕ_l^{m*}. Now as one continues $\phi(n;p_2^2,k)$ in p_2^2 to its physical value, the only singularity that one encounters is a branch point at $p_2^2=(\sqrt{s}-m_1)^2$, where $k=0$. As p_2^2 increases, k becomes imaginary. Then it follows from (3) that at $p_2^2=m_2^2$ the analytic continuation of

$$\bar{\phi}(n;p_2^2,\mathbf{k})=\int\frac{d^4ud\lambda^2\psi(n;u,\lambda^2)^*}{\Lambda-2\mathbf{k}\cdot\mathbf{u}}\quad(10)$$

is $\phi(n;m_2^2,-\mathbf{k})^*$ and

$$\bar{\phi}(n;m_2^2,\mathbf{k})+(-1)^l\bar{\phi}(n;m_2^2,-\mathbf{k})$$
$$=(-1)^l[\phi(n;m_2^2,\mathbf{k})+(-1)^l\phi(n;m_2^2,-\mathbf{k})]^*.\quad(11)$$

Therefore, the analytic continuation of $\bar{\phi}_l^m(n;p_2^2,k)$ is $(-1)^l\phi_l^m(n;m_2^2,k)^*$ and it follows from (6) that $a_l(s)$ has the sign of $(-1)^l$.[3] Now it is clear that the addition to $\phi(n;p_2^2,\mathbf{k})$ of polynomials in $\mathbf{k}$ does not affect this result. We also remark that should there be a need for subtractions in the Dyson representation the effect of such subtractions would be to add to and multiply ϕ by polynomials in $(k_0,\mathbf{k})$. Since k_0 is linear in p_2^2, the result of our analysis would remain the same. As a consequence of this result one can find a domain of s and t, with s below the physical threshold and positive t, in which $A(s,t)>0$. Indeed if t_0 is the lowest threshold in the crossed t channel, then for $0<t<t_0$ one can write

$$A(s,t)=\sum_l(2l+1)a_l(s)P_l(1+t/2k^2).\quad(12)$$

In the region $(1+t/2k^2)<-1$ each term in this expansion will be positive. Therefore $A(s,t)>0$ in the region:

$$t<t_0,\quad s<(m_1+m_2)^2,\quad su+(m_1^2-m_2^2)^2>0.\quad(13)$$

In the equal-mass case the last condition reduces to $u>0$. Unfortunately in all elastic processes $t_0=4m_\pi^2$ and owing to the smallness of the pion mass the region in s for which $A(s,t)>0$ is rather small and in most cases does not include the whole unphysical region below threshold (one exception for instance is Σ-π scattering with Λ-π as an unphysical region).

We next proceed on to discuss the case of elastic scattering of particles with spin.

II. PARTICLES WITH SPINS

We consider the elastic scattering of particles with spins S_1, S_2 as described by a set of helicity amplitudes[4]:

$$F_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\mathbf{k},\mathbf{k}')=\langle k'\theta'\phi';\lambda_1'\lambda_2'|T|k\theta\phi;\lambda_1\lambda_2\rangle$$
$$\times(4p_{10}p_{20})^{1/2}(4p_{10}'p_{20}')^{1/2},\quad(14)$$

where (k,θ,ϕ) and (k',θ',ϕ') are the polar coordinates of $\mathbf{k}$ and $\mathbf{k}'$, respectively, in an arbitrary center-of-mass frame. The partial-wave helicity amplitudes $f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s)$ for transitions in states of total angular momentum J are given by[4]

$$f_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s)=\frac{1}{16\pi^2}\int F_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\mathbf{k},\mathbf{k}')$$
$$\times\sum_m D_{m\lambda'}{}^J(\phi',\theta',-\phi')D_{m\lambda'}(\phi,\theta,-\phi)^*d\hat{k}d\hat{k}',\quad(15)$$

where the D^J's are the matrix elements of an irreducible

[2] Silvan S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, New York, 1961), p. 810.

[3] I was informed that Dr. G. Mahoux arrived independently at this result.

[4] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

representation of the rotation group and $\lambda=\lambda_1-\lambda_2$, $\lambda'=\lambda_1'-\lambda_2'$.

The absorptive part of $F_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\mathbf{k},\mathbf{k}')$ is given by

$$A_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(\mathbf{k}',\mathbf{k})=(4p_{10}p_{10}')^{1/2}\sum_m \langle p_1'\lambda_1'|j_{\lambda_2'}(0)|n\rangle \times\langle n|j_{\lambda_2}(0)^\dagger|p_1\lambda_1\rangle(2\pi)^4\delta(p_n-p_1-p_2). \quad (16)$$

Now one cannot write down a Dyson representation for helicity amplitudes because the helicity labels are the projections of the spins on the directions of the momenta and therefore momentum dependent. So in order to be able to use Dyson's representation, we have to express the two-particle helicity states $|k,\theta,\phi;\lambda_1\lambda_2\rangle$ in terms of the covariant spinor states $|p_1S_1\mu_1;p_2S_2\mu_2\rangle$. The relation is[5]

$$|k,\theta,\phi;\lambda_1\lambda_2\rangle =\sum_{\mu_1\mu_2} D^{S_1}(L(1))_{\mu_1\lambda_1}D^{S_2}(L(2))_{\mu_2\lambda_2}(-1)^{S_2-\lambda_2} \times|p_1S_1\mu_1;p_2S_2\mu_2\rangle, \quad (17)$$

where

$$L(1)=R_n(\theta)B_3(p_1),\quad L(2)=R_n(\theta)R_2(\pi)B_3(p_2), \quad (18)$$

with B_3 a boost along the z axis and $R_n(\theta)$ a rotation of an angle θ about the axis $\mathbf{n}=\boldsymbol{\varepsilon}_3\times\mathbf{k}$. The matrix elements for the boost $B_3(p_i)$ are given by

$$D^S(B_3(p_i))_{\mu\lambda}=\left(\frac{p_{i0}-k}{p_{i0}+k}\right)^{\lambda/2}\delta_{\mu\lambda}=\alpha_\lambda(p_{i0},k)\delta_{\mu\lambda}. \quad (19)$$

The rotations are given by

$$D^S(R_n(\theta))_{\mu\lambda}=D_{\mu\lambda}{}^S(\phi,\theta,-\phi)=D_{\mu\lambda}{}^S(\hat{k}). \quad (20)$$

Then we have

$$\begin{aligned} D^{S_1}(L(1))_{\mu_1\lambda_1}&=D_{\mu_1\lambda_1}{}^{S_1}(\hat{k})\alpha_{\lambda_1}(p_{10},k),\\ (-1)^{S_2-\lambda_2}D^{S_2}(L(2))_{\mu_2\lambda_2}&=D_{\mu_2-\lambda_2}{}^{S_2}(\hat{k})\alpha_{\lambda_2}(p_{20},k). \end{aligned} \quad (21)$$

Therefore the absorptive helicity amplitudes in the unphysical region will be obtained by analytic continuation in p_2^2 of (16) with the matrix elements in the sum given by

$$\langle n|j_{\lambda_2}{}^\dagger(0)|p_1\lambda_1\rangle(2p_{10})^{1/2}=\sum_{\mu_1\mu_2} D_{\mu_1\lambda_1}{}^{S_1}(\hat{k})\alpha_{\lambda_1}(p_{10},k) \times D_{\mu_2-\lambda_2}{}^{S_2}(\hat{k})\alpha_{\lambda_2}(p_{20},k)\phi_{\mu_1\mu_2}(n;p_2{}^2,\mathbf{k}), \quad (22)$$

where $\phi_{\mu_1\mu_2}(n;p_2{}^2,\mathbf{k})$ has a representation of the form of (3) with $\psi(n;u,\lambda^2)$ replaced by $\psi_{\mu_1\mu_2}(n;u,\lambda^2;p_i)$ which is a polynomial in the p_i's. We remark that, for fixed s, the boosts $\alpha_\lambda(p_{i0},k)$ are analytic functions of $p_2{}^2$ with branch points at $p_2{}^2=0$ and $p_2{}^2=(\sqrt{s}\pm m_1)^2$. We therefore take the $p_2{}^2$ plane cut along the intervals $(-\infty,0)$ and $[(\sqrt{s}-m_1)^2,(\sqrt{s}+m_1)^2]$ and take the boosts to be real and positive in the interval $(0,(\sqrt{s}-m_1)^2)$. In going to the physical value of $p_2{}^2$, we come across the branch point of k at $p_2{}^2=(\sqrt{s}-m_1{}^2)$ and take the path along the side of the cut on which k is positive imaginary, which is the value it takes on in the physical sheet.

The absorptive part of the partial-wave helicity amplitudes will be given by

$$a_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s,p_2{}^2)=\frac{1}{16\pi}\sum(2J+1) \times\bar{\phi}_{m\lambda_1'\lambda_2'}{}^J(n;p_2{}^2,k)\phi_{m\lambda_1\lambda_2}{}^J(n;p_2{}^2,k), \quad (23)$$

where

$$\phi_{m\lambda_1\lambda_2}{}^J(n;p_2{}^2,k)=\sum_{\mu_1\mu_2}\frac{1}{\sqrt{(4\pi)}}\left(\frac{2}{2J+1}\right)^{1/2}\int D_{m\lambda}{}^J(\hat{k})^* \times D_{\mu_1\lambda_1}{}^{S_1}(\hat{k})\alpha_{\lambda_1}(p_{10},k)D_{\mu_2-\lambda_2}{}^{S_2}(\hat{k})\alpha_{\lambda_2}(p_{20},k) \times\phi_{\mu_1\mu_2}(n;p_2{}^2,\mathbf{k})d\hat{k}. \quad (24)$$

Under the transformation $\hat{k}\to-\hat{k}$, we have $\phi\to\phi+\pi$, $\theta\to\pi-\theta$, and

$$\begin{aligned} D_{\mu_i\lambda_i}{}^{S_i}(\hat{k})=e^{-i\mu_i\phi}d_{\mu_i\lambda_i}{}^{S_i}(\theta)e^{i\lambda_i\phi}&\to D_{\mu_i\lambda_i}{}^{S_i}(-\hat{k})\\ &=e^{-i\mu_i\phi}d_{\mu_i-\lambda_i}{}^{S_i}(\theta)e^{i\lambda_i\phi}(-1)^{S_i+\lambda_i}. \end{aligned} \quad (25)$$

Similarly,

$$\begin{aligned} D_{m\lambda}{}^J(\hat{k})^*&\to D_{m\lambda}{}^J(-\hat{k})^*\\ &=e^{im\phi}d_{m-\lambda}{}^J(\theta)e^{-i\lambda\phi}(-1)^{J+\lambda}. \end{aligned} \quad (26)$$

Therefore taking into account that

$$\alpha_{\lambda_i}(p_{i0},k)=\alpha_{-\lambda_i}(p_{i0},-k),$$

one can write

$$\begin{aligned} \phi_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k) &=\sum_{\mu_1\mu_2}\frac{1}{\sqrt{(4\pi)}}\left(\frac{2}{2J+1}\right)^{1/2}\int D_{m-\lambda}{}^J(\hat{k})\\ &\quad\times D_{\mu_1-\lambda_1}{}^{S_1}(\hat{k})\alpha_{-\lambda_1}(p_{10},-k)\\ &\quad\times D_{\mu_2-\lambda_2}{}^{S_2}(\hat{k})\alpha_{-\lambda_2}(p_{20},-k)(-1)^{J-S_1-S_2}\\ &\quad\times\phi_{\mu_1\mu_2}(n;p_2{}^2,-\mathbf{k})d\hat{k}\\ &=(-1)^{J-S_1-S_2}\phi_{m,-\lambda_1,-\lambda_2}{}^J(n;p_2{}^2,-k). \end{aligned} \quad (27)$$

For real k, one has

$$\bar{\phi}_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k)=\phi_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k)^*. \quad (28)$$

On the other hand, as $p_2{}^2$ takes on its physical value, k becomes purely imaginary for s below the physical threshold, and the analytic continuation of $\bar{\phi}_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k)$ will be given by

$$\begin{aligned} \bar{\phi}_{m,\lambda_1,\lambda_2}{}^J(n;m_2{}^2,k) &=\phi_{m,\lambda_1,\lambda_2}{}^J(n;m_2{}^2,-k)^*\\ &=(-1)^{J-S_1-S_2}\phi_{m,-\lambda_1,-\lambda_2}{}^J(n;p_2{}^2,k)^*. \end{aligned} \quad (29)$$

[5] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) 46, 239 (1968). For the phase convention of helicity states we follow Jacob and Wick (Ref. 4).

We are thus led to consider the combinations

$$\phi_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k)=(1/\sqrt{2})[\phi_{m,\lambda_1,\lambda_2}{}^J(n;p_2{}^2,k)+\epsilon(-1)^{J-S_1-S_2}\phi_{m,-\lambda_1,-\lambda_2}{}^J(n;p_2{}^2,k)],\quad(30)$$

with $\epsilon=\pm1$, corresponding to transitions from states of definite parity $P=\eta_1\eta_2\epsilon$, where the η's are the intrinsic parities of the particles. Then we have

$$\bar{\phi}_{m,\lambda_1,\lambda_2}{}^{J\epsilon}(n;m_2{}^2,k)=(1/\sqrt{2})[(-1)^{J-S_1-S_2}\phi_{m,-\lambda_1,-\lambda_2}{}^J(n;m_2{}^2,k)+\epsilon\phi_{m,\lambda_1,\lambda_2}{}^J(n;m_2{}^2,k)]^*=\epsilon\phi_{m,\lambda_1,\lambda_2}{}^{J\epsilon}(n;m_2{}^2,k)^*.\quad(31)$$

It follows that below the physical threshold the matrices

$$\epsilon a_{[\lambda'],[\lambda]}{}^{J\epsilon}(s)=\tfrac{1}{2}\epsilon\{a_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s)+a_{-\lambda_1'-\lambda_2'-\lambda_1-\lambda_2}{}^J(s)+\epsilon(-1)^{J-S_1-S_2}\times[a_{\lambda_1'\lambda_2'-\lambda_1-\lambda_2}{}^J(s)+a_{-\lambda_1'-\lambda_2'\lambda_1\lambda_2}{}^J(s)]\}\quad(32)$$

are positive Hermitian. We shall, from now on assume parity conservation. Then the above expression reduces to

$$\epsilon a_{[\lambda'],[\lambda]}{}^{J\epsilon}(s)=\epsilon[a_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s)+\epsilon(-1)^{J-S_1-S_2}a_{-\lambda_1'-\lambda_2'\lambda_1\lambda_2}{}^J(s)].\quad(33)$$

In particular, the diagonal elements of these two matrices are positive; hence it follows that $(-1)^{J-S_1-S_2}\times a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J(s)$ is positive.

In the physical region it is the matrices $a_{[\lambda'],[\lambda]}{}^{J\epsilon}(s)$ which are positive Hermitian. Since the parity of a two-particle state with relative orbital angular momentum l is $P=\eta_1\eta_2(-1)^l$, it follows that for transitions between states of initial and final angular momentum l and l' and total spin S and S', the absorptive partial waves $a_{l'S'lS}{}^J(s)$ with $(l'-l)$ even are the matrix elements of a Hermitian matrix of definite sign $(-1)^l$. This can also be obtained directly from the relation between the set of amplitudes $a_{l'S'lS}{}^J(s)$ and $a_{\lambda_1'\lambda_2'\lambda_1\lambda_2}{}^J(s)$. This result is independent of the assumption of parity conservation.

We can also generalize the result obtained for the transition amplitude for the scattering of spinless particles as a function of the invariants s and t.

It has been proved that the amplitudes[5,6]

$$\frac{s^{|\lambda'-\lambda|/2}(sk^2)^{S_1+S_2-M}}{(\sin\frac{1}{2}\theta)^{|\lambda'-\lambda|}(\cos\frac{1}{2}\theta)^{|\lambda'+\lambda|}}F_{\lambda_1'\lambda_2'\lambda_1\lambda_2}(s,t),\quad(34)$$

with $M=\max\{|\lambda'|,|\lambda|\}$, are free of kinematical singularities.

We consider the combinations

$$F_{\lambda_1\lambda_2}{}^\epsilon(s,t)=(sk^2)^{S_1+S_2+|\lambda|}\left\{[\cos^2(\tfrac{1}{2}\theta)]^{|\lambda|}F_{\lambda_1\lambda_2\lambda_1\lambda_2}(s,t)+\epsilon(-1)^{2|\lambda|}\left(\frac{s_0}{s}\right)^{|\lambda|}[\sin^2(\tfrac{1}{2}\theta)]^{|\lambda|}F_{-\lambda_1-\lambda_2\lambda_1\lambda_2}(s,t)\right\}.\quad(35)$$

These amplitudes are also free of kinematical singularities.

The absorptive part of $F_{\lambda_1\lambda_2}{}^\epsilon(s,t)$ in the s channel, for positive values of t below the lowest threshold t_0 of the crossed t channel, can be written as a partial-wave expansion:

$$\begin{aligned}A_{\lambda_1\lambda_2}{}^\epsilon(s,t)&=(sk^2)^{S_1+S_2+|\lambda|}\sum_J(2J+1)\times a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J(s)d_{\lambda\lambda}{}^J(\theta)[\cos^2(\tfrac{1}{2}\theta)]^{|\lambda|}+\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J(s)d_{\lambda-\lambda}{}^J(\theta)[\sin^2(\tfrac{1}{2}\theta)]^{|\lambda|}\\&=(sk^2)^{S_1+S_2+|\lambda|}\sum_J(2J+1)\times\left\{a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J(s)P_{J-|\lambda|}{}^{0,2|\lambda|}(x)[\tfrac{1}{2}(x+1)]^{2|\lambda|}+\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J(s)P_{J-|\lambda|}{}^{2|\lambda|,0}(x)\times[\tfrac{1}{2}(x-1)]^{2|\lambda|}\right\},\end{aligned}\quad(36)$$

where $x=\cos\theta$ and $P_n{}^{\alpha,\beta}(x)$ are Jacobi polynomials.

For $x>1$, $P_n{}^{\alpha,\beta}(x)$ is positive and using Rodrigues's formula for the Jacobi polynomials,[7] one can show that

$$(x+1)^\alpha P_n{}^{0,\alpha}(x)>(x-1)^\alpha P_n{}^{\alpha,0}(x).\quad(37)$$

Then, since for physical values of s we have from unitarity

$$a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J(s)+\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J>0,\quad(38)$$

it follows that $A_{\lambda_1\lambda_2}{}^\epsilon(s,t)$ is positive for s above the physical threshold and t in the interval $0<t<t_0$.

Now let us take s below the physical threshold so that $k^2<0$. If

$$t+4k^2>0,\quad(39)$$

we have $x<-1$ and we use the following property of Jacobi polynomials:

$$P_n{}^{\alpha,\beta}(x)=(-1)^nP_n{}^{\beta,\alpha}(-x).\quad(40)$$

Then we write

$$A_{\lambda_1\lambda_2}{}^\epsilon(s,t)=(sk^2)^{S_1+S_2+|\lambda|}\sum_J\tfrac{1}{2}(2J+1)\times\left\{\left[a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J+\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}(-1)^{J-|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J\right]\right.$$

[6] Y. Hara, Phys. Rev. **136**, B507 (1964); L. L. C. Wang, *ibid.* **142**, 1187 (1966).

[7] *Handbook of Mathematical Functions*, edited by M. Abramovitz and I. A. Stegun [Natl. Bur. Std. (U. S.), Washington, D. C., 1964], p. 785.

$$\times\left[\left(\frac{x+1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{0,2|\lambda|}(x)+(-1)^{J-|\lambda|}\left(\frac{x-1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{2|\lambda|,0}(x)\right]$$
$$+\left[a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J-\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}(-1)^{J-|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J\right]$$
$$\times\left[\left(\frac{x+1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{0,2|\lambda|}(x)-(-1)^{J-|\lambda|}\left(\frac{x-1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{2|\lambda|,0}(x)\right]\Bigg\}. \quad (41)$$

Now it follows from (33) that the sign of

$$\left[a_{\lambda_1\lambda_2\lambda_1\lambda_2}{}^J\pm\epsilon\left(\frac{s_0}{s}\right)^{|\lambda|}(-1)^{J-|\lambda|}a_{-\lambda_1-\lambda_2\lambda_1\lambda_2}{}^J\right] \quad (42)$$

is that of $\pm\epsilon(-1)^{S_1+S_2-|\lambda|}$, and from (37) and (40) it follows that for $x<-1$ the sign of

$$\left(\frac{x+1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{0,2|\lambda|}(x)\pm(-1)^{J-|\lambda|}\left(\frac{x-1}{2}\right)^{2|\lambda|}P_{J-|\lambda|}{}^{2|\lambda|,0}(x) \quad (43)$$

is that of $\pm(-1)^{2|\lambda|}$ so that the sign of the summation in (41) is that of $\epsilon(-1)^{S_1+S_2+|\lambda|}$. On the other hand, since $k^2<0$, the factor in front of the summation in (41) has the sign of $(-1)^{S_1+S_2+|\lambda|}$, then if follows that $\epsilon A_{\lambda_1\lambda_2}{}^{\epsilon}(s,t)$ is positive. We have thus constructed a set of amplitudes $F_{\lambda_1\lambda_2}{}^+(s,t)$ and[8] $(s-s_0)^{-1}F_{\lambda_1\lambda_2}{}^-(s,t)$ whose absorptive parts are positive in the s-channel in the domain of s and t given by

$$0<t<t_0,\quad t+4k^2>0. \quad (44)$$

These amplitudes do not coincide with those considered by Mahoux and Martin[9] out of which they constructed an amplitude satisfying the positivity condition in both s and u channels. Their amplitudes are essentially $F_{\lambda_1\lambda_2\lambda_1\lambda_2}(s,t)$ (apart from factors which remove kinematical singularities), whereas below the physical region positivity holds for $F_{-\lambda_1-\lambda_2\lambda_1\lambda_2}(s,t)$ (with the appropriate factors) rather than for $F_{\lambda_1\lambda_2\lambda_1\lambda_2}(s,t)$.

It might be possible to construct an amplitude satisfying the positivity condition in both channels out of a linear combination of the amplitudes $F_{\lambda_1\lambda_2}{}^+(s,t)$ and $(s-s_0)^{-1}F_{\lambda_1\lambda_2}{}^-(s,t)$. If this were to be the case, this result could be used to generalize the analysis of Martin[10] for π-π scattering, based on analyticity and the positivity properties of the absorptive amplitudes, even to the case of processes with unphysical regions.

However, as pointed out before, the applicability of these results is severely restricted by the fact that $t_0=4m_\pi{}^2$ is very small as compared to other physical masses.

CONCLUSIONS

We have considered elastic processes which have an unphysical region corresponding to the existence of channels below the physical threshold. We have shown that, for s below the physical threshold, the absorptive partial wave amplitudes for elastic transitions between states of given parity $P=\eta_1\eta_2\epsilon$ (where η_1 and η_2 are the intrinsic parities of the particles and $\epsilon=\pm1$) has the sign of ϵ. The result applies for elastic scattering of particles of arbitrary spins. In particular, for transitions between states of orbital angular momentum l the sign of the absorptive partial-wave amplitudes is $(-1)^l$. More generally, one obtains that below the physical threshold, the matrix for partial-wave transitions between states of given J and $P=\eta_1\eta_2\epsilon$ has an absorptive part which is Hermitian and of definite sign ϵ.

We have then constructed analytic amplitudes as functions of s and t which, for positive t below the threshold t_0 for the crossed t channel, have absorptive parts in the s channel which are positive both above and below the physical threshold, provided that $4k^2+t>0$. Since $t_0=4m_\pi{}^2$ this condition is rather restrictive owing to the smallness of the pion mass and severely limits the applicability of these results.

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[8] From an inspection of the threshold behavior, one can show that $F_{\lambda_1\lambda_2}{}^-(s,t)$ vanishes at $s=s_0$.

[9] G. Mahoux and A. Martin, Phys. Rev. **174**, 2140 (1968).

[10] A. Martin, Nuovo Cimento **58A**, 303 (1968); **63A**, 167 (1969); G. Auberson *et al.*, CERN Report No. TH 1032, 1969 (unpublished).