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# Relativistic Corrections to the Three-Body Problem

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Relativistic corrections are evaluated for a model three-body problem consisting of three identical bosons in $s$-wave pairs through separable potentials. For the potentials, Shirokov's second-order (in $v/c$) correction, which vanishes in the c.m. frame of a two-body system, is considered. No corrections are considered for the kinetic energy, as it is argued that any such correction is intimately linked with the precise structure assumed for a three-particle dynamics and cannot merely be incorporated through the modification $p^2/2m \rightarrow (m^2+p^2)^{1/2}-m$ for each particle, which affects even the two-particle dynamics on the energy shell (and hence also the parameters of the two-body potential). The reduction in the quantity analogous to the binding energy of the triton is found to be of the order of 5%, while the curve for the counterpart of the doublet $n$-$d$ scattering length ($a_{1/2}$) versus the two-body strength parameter ($\lambda$) is slightly shifted to the right as a result of the relativistic correction.

## I. INTRODUCTION

INVESTIGATIONS of the three-body problem seem now to have reached a level of sophistication where it has become physically meaningful to speak of finer effects in more quantitative terms.[1–6] Already we seem to be getting fairly clear ideas about the relative effects of the tensor force and the repulsive core[7–9] on important three-body parameters which are sensitive to such details of the two-body interaction, after the limited success achieved with the long-range $s$-wave attraction several years ago.[10–14] Relativistic corrections to the two-body interaction represent an additional effect which is generally estimated to be comparable to that of the repulsive core. It is the purpose of this paper to make a semiquantitative estimate of this effect on the three-body problem by considering a model three-boson system of identical particles, interacting in pairs through $s$-wave forces. In particular, we wish to calculate the effect of relativistic corrections to these forces on two important low-energy parameters, which are the analogs of the triton binding energy (B.E.) and the neutron-deuteron doublet scattering length ($a_{1/2}$).

It need hardly be recalled that this model is a fairly good facsimile of the actual triton state, to the extent that its spatially symmetric ($s$) configuration accounts for about 95% of the total wave function. In any case, one would expect the model to provide a fairly realistic reproduction of such gross attributes as the binding energy and the $n$-$d$ scattering length, provided the latter is identified with the "doublet" ($a_{1/2}$) variety (the quartet scattering length $a_{3/2}$ has no counterpart in this simple model). To the same extent, a calculation of relativistic corrections to B.E. and $a_{1/2}$ in the model should provide a faithful index of the nature and extent of such corrections to the more realistic three-body problem which takes account of greater detail in the two-body interaction. On the other hand, such a simple model would be too inadequate for other three-body parameters which are expected to be more sensitive to the noncentral parts of the interaction, e.g., polarization in $n$-$d$ scattering, charge and magnetic form factors of $H^3$ and $He^3$, and so on. In this paper, we shall consider merely the relativistic corrections to the simple three-boson problem, and hence we must limit ourselves to only the analogs of B.E. and $a_{1/2}$. For simplicity, we shall continue to use the notation B.E. and $a_{1/2}$ to imply the analogous quantities in the three-boson problem.

Now the question of relativistic corrections to a two- or three-body problem can itself be viewed at different levels of sophistication. Thus from the two-body point of view, a relativistic correction might be taken to imply not merely an interaction which takes account of "retardation effects," but an enlargement of the dynamical framework itself from the simple three-dimensional Schrödinger equation to something like the four-dimensional Bethe-Salpeter equation.[15] A less sophisticated dynamical modification would be to use relativistic kinematics within a three-dimensional framework such as the (now old-fashioned) Tamm-Dancoff equation[16] or its more modern versions, the Blankenbecler-Sugar[17] and Logunov-Tavkhelidze

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[15] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
[16] See, e.g., H. A. Bethe and F. de Hoffman, *Mesons and Fields* (Row, Peterson, New York, 1955), Vol. II, p. 199.
[17] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

equations.[18] In any one of these modifications, it is of course implied that both the dynamical framework and the (corrected) interaction should reduce to the usual Schrödinger equation and the (uncorrected) static interaction, respectively, when one considers the nonrelativistic (NR) limit. Therefore, insofar as the two-body potential is parametrized to fit certain two-body data, the parameters of the corrected potential must in general be *changed*, so that the same sort of data might be fitted within the relativistically corrected dynamical framework. As a result, the significance of any comparison of the relativistic corrections with the NR framework is likely to be considerably clouded by the freedom of parametrization of the two-body interaction, especially since the correction is not expected to be more than $\sim 10\%$. To add further to this confusion, the formulation of a proper relativistic framework to the three-body problem is itself a formidable problem which has so far not crossed even the formal mathematical barriers, not to speak of any uniqueness in approach.

It would thus appear that any serious attempt at relativistic corrections to the three-body problem is fraught with ambiguities, not only in the parametrization of the two-body potential but also in the definition of what constitutes a relativistic dynamical framework for the two-body problem, not to speak of a three-body system. Fortunately, however, there is one limited aspect of relativistic corrections to the two-body potential which is not only free from such ambiguities at the two-body level, but promises to give a clear (though limited) answer to the corresponding three-body problem even without having to enlarge the dynamical framework beyond the simple Schrödinger equation. Such a correction to the two-body potential, which was first suggested by Shirokov,[19] is based on the consideration that the potential can be so modified as to make it invariant under the group of Lorentz transformations up to and including terms of $O(v^2/c^2)$. The correction is, moreover, proportional to the total c.m. momentum of the two-body system, so that it vanishes exactly in the c.m. frame of that system. This fact makes it possible to view the relativistic correction to the three-body problem as an *off*-c.m. effect, an aspect which may be studied independently of any relativistic modification to the dynamical framework at the two-body and three-body levels. In particular, at the two-body level, this correction to a given parametric potential does not produce any change in its parameters, as long as the dynamics is assumed to be governed by the same Schrödinger equation. However, at the three-body level, the above correction to the two-body potential, which can now make off-c.m. contributions to the two-body energy, is now capable of making definite predictions on the relativistic corrections to suitable three-body parameters. In this paper, we shall calculate only the Shirokov corrections to the three-body parameters B.E. and $a_{1/2}$, and not try to concern ourselves with the bigger, but much more ambiguous question of what constitutes a proper relativistic modification to the dynamical framework, whether at the two-body or the three-body levels.

In Sec. II we obtain the necessary equations for the bound and scattering states of a three-boson system when the relativistic (Shirokov) corrections to the two-body potential are taken into account. (The uncorrected potential is, of course, taken to be a separable, $s$-wave structure.) In Sec. III we describe the numerical results for B.E. and $a_{1/2}$, with and without the relativistic corrections, as a function of the strength parameter of the two-body interaction. It is found that while B.E. registers an expected decrease ($\sim 5\%$) as a result of relativistic corrections, $a_{1/2}$, as well as its relativistic correction, exhibit a more fluctuating behavior as a function of the two-body strength parameter.

## II. NECESSARY FORMALISM

For any two-body potential $V$ involving two equal masses ($M$), the second-order relativistic correction $\delta V$ can be expressed, in momentum space, as

$$\langle \mathbf{pP}|\delta V|\mathbf{p'P'}\rangle = -(4M^2)^{-1}[P^2+\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{P})(\mathbf{P}\cdot\nabla_p) + \tfrac{1}{2}(\mathbf{p'}\cdot\mathbf{P})(\mathbf{P}\cdot\nabla_{p'})]\langle \mathbf{p}|V|\mathbf{p'}\rangle, \quad (1)$$

where $\mathbf{p}$ and $\mathbf{p'}$ are the relative momenta in the initial and final states, and $\mathbf{P}$ is the total c.m. momentum of the two-body system. Note that while the nonrelativistic potential $V$ is independent of $\mathbf{P}$, the correction $\delta V$ is proportional to $P^2$ and hence vanishes in the c.m. frame. For the NR potential $V$ in the c.m. frame, we take the $s$-wave separable form

$$\langle \mathbf{p}|V|\mathbf{p'}\rangle = -\lambda M^{-1}g(p)g(p'), \quad g(p)=(\beta^2+p^2)^{-1}, \quad (2)$$

while its representation in any arbitrary frame characterized by the momentum pairs $(\mathbf{P}_1\mathbf{P}_2)$ and $(\mathbf{P}_1'\mathbf{P}_2')$ is

$$\langle \mathbf{P}_1\mathbf{P}_2|V|\mathbf{P}_1'\mathbf{P}_2'\rangle = \delta(\mathbf{P}-\mathbf{P'})\langle \mathbf{p}|V|\mathbf{p'}\rangle, \quad (3)$$

where

$$2\mathbf{p}=\mathbf{P}_1-\mathbf{P}_2, \quad \mathbf{P}=\mathbf{P}_1+\mathbf{P}_2, \quad (4)$$

with similar definitions for the primed quantities. Since the potential $V$ is separable, it is clear from (1) that the correction $\delta V$ is also a sum of separable terms, so that it should admit of the standard manipulations[1] when it is inserted into the three-body Schrödinger equation. As mentioned already in Sec. I, we refrain from taking any relativistic correction to the kinetic energy, since any such correction is linked with the larger question of what constitutes the correct relativistic dynamics of

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[19] Yu. M. Shirokov, Zh. Eksperim. i Teor. Fiz. **36**, 474 (1959) [Soviet Phys. JETP **9**, 330 (1959)].

a two-body system.[20] Our modified three-body equation may now be written as

$$M(T-E)\Psi=-M(V_{12}+V_{23}+V_{31}+\delta V_{12}+\delta V_{23}+\delta V_{31})\Psi\,, \quad (5)$$

$$M(T-E)=\tfrac{1}{2}(P_1{}^2+P_2{}^2+P_3{}^2)-ME\,, \quad (6)$$

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0\,, \quad (7)$$

where, according to (1)–(4),

$$M\langle \mathbf{P}_1\mathbf{P}_2|V_{12}|\mathbf{P}_1'\mathbf{P}_2'\rangle = -\lambda g(p_{12})\{[1-\tfrac{1}{4}P_3{}^2M^{-2}+(1/4M^2)(\mathbf{p}_{12}\cdot\mathbf{P}_3)^2g(p_{12})] + (\mathbf{p}_{12}'\cdot\mathbf{P}_3)^2(1/4M^2)g(p_{12}')\}g(p_{12}')\delta(\mathbf{P}_3-\mathbf{P}_3')\,. \quad (8)$$

This gives rise, in the standard manner,[1] to the following structure for $\Psi$:

$$M(T-E)\Psi(\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3)=\sum_{123} g(p_{ij})\{[(1-(1/4M^2)P_k{}^2) + (1/4M^2)(\mathbf{p}_{ij}\cdot\mathbf{P}_k)^2g(p_{ij})]F_1(\mathbf{P}_k)+F_2(\mathbf{P}_k)\}\,, \quad (9)$$

where the spectator functions $F_1(\mathbf{P}_k)$ and $F_2(\mathbf{P}_k)$ satisfy the equations

$$[\lambda^{-1}-h_1(P)(1-P^2/4M^2)-h_2(P)]F_1(\mathbf{P})-h_1(P)F_2(\mathbf{P}) =2\int d\mathbf{q}\{[K_1(\mathbf{P},\mathbf{q})(1-q^2/4M^2)+K_2{}^T(\mathbf{P},\mathbf{q})]F_1(\mathbf{q}) +K_1(\mathbf{P},\mathbf{q})F_2(\mathbf{q})\}\,, \quad (10)$$

$$\lambda^{-1}F_2(\mathbf{P})-h_2(P)F_1(\mathbf{P})=2\int d\mathbf{q}\, K_2(\mathbf{P},\mathbf{q})F_1(\mathbf{q})\,, \quad (11)$$

and the various functions are defined by

$$\begin{Bmatrix} h_1(P) \\ h_2(P)\end{Bmatrix}=\int d\mathbf{q}\, g^2(q)(q^2+\tfrac{3}{4}P^2-ME)^{-1} \times\begin{Bmatrix} 1 \\ g(q)(\mathbf{q}\cdot\mathbf{P})^2/4M^2\end{Bmatrix}\,, \quad (12)$$

$$K_1(\mathbf{P},\mathbf{q})=g(\mathbf{q}+\tfrac{1}{2}\mathbf{P})g(\mathbf{P}+\tfrac{1}{2}\mathbf{q}) \times(q^2+P^2+\mathbf{q}\cdot\mathbf{P}-ME)^{-1}\,, \quad (13)$$

$$K_2(\mathbf{P},\mathbf{q})=K_2{}^T(\mathbf{q},\mathbf{P}) =\{[\mathbf{P}\cdot(\mathbf{q}+\tfrac{1}{2}\mathbf{P})]^2/4M^2\}g^2(\mathbf{q}+\tfrac{1}{2}\mathbf{P}) \times g(\mathbf{P}+\tfrac{1}{2}\mathbf{q})(q^2+P^2+\mathbf{q}\cdot\mathbf{P}-ME)^{-1}\,. \quad (14)$$

In deriving Eqs. (10) and (11), we have neglected terms of order higher than $O(P^2/4M^2)$, in keeping with the spirit of retaining only the relativistic corrections of $O(v^2/c^2)$ to the potentials. Substitution of (11) in (10) leads to the following integral equation in the single function $F_1(\mathbf{P})$:

$$[\lambda^{-1}-h_1(P)(1-P^2/4M^2)-2h_2(P)]F_1(\mathbf{P}) =2\int d\mathbf{q}\Big[F_1(\mathbf{q})K_1(\mathbf{P},\mathbf{q})\Big(1-\frac{q^2}{4M^2}+\lambda h_2(q)\Big) +[K_2{}^T(\mathbf{P},\mathbf{q})+K_2(\mathbf{P},\mathbf{q})]F_1(\mathbf{q}) +2\lambda K_1(\mathbf{p},\mathbf{q})\int d\mathbf{q}'\, K_2(\mathbf{q},\mathbf{q}')F_1(q')\Big]\,, \quad (15)$$

where the approximation

$$h_1(P)\approx\lambda^{-1}\,, \quad (16)$$

which may be compared with the definition

$$\lambda^{-1}=\int d\mathbf{q}\, g^2(q)(q^2+\alpha^2)^{-1}\,, \quad (17)$$

has been made in one of the terms proportional to $h_2(P)$ [since the latter is already of $O(P^2/4M^2)$]. Equation (15) represents the central equation of our theory. For the bound-state problem, it must be solved as a straightforward eigenvalue problem after setting

$$E=-\text{B.E.}=-\alpha_T{}^2/M\,. \quad (18)$$

For the problem of $n$-$d$ scattering, on the other hand, when $EM=\frac{3}{4}k^2-\alpha^2$, one must first insert the appropriate boundary condition on $F_1(\mathbf{P})$ so as to express it in terms of the corresponding scattering amplitude. For this purpose, we exhibit the pole structure for $F_1(\mathbf{P})$ by rewriting its coefficient on the left-hand side of Eq. (15) as

$$\lambda^{-1}-h_1(P)[1-(1/4M^2)P^2]-2h_2(P) =\tfrac{3}{4}(P^2-k^2)\int d\mathbf{q}\frac{g^2(q)\{1-(P^2/4M^2)[1-2(\mathbf{q}\cdot\hat{P})^2g(q)]\}}{(q^2+\alpha^2)(q^2+\alpha^2+\frac{3}{4}P^2-\frac{3}{4}k^2)} +\frac{P^2}{4M^2}\int d\mathbf{q}\frac{g^2(q)}{(q^2+\alpha^2)}[1-2(\mathbf{q}\cdot\hat{P})^2g(q)]\,, \quad (19)$$

which shows how the pole at $P^2=k^2$ gets shifted to a neighboring point $P^2=k'^2$ given by the zero of (19). Since the second term is small, an approximate solution for the pole position works out as

$$k'^2=k^2\Big(1-\frac{1}{3M^2}\frac{C}{f(k^2)}\Big)\,, \quad (20)$$

[20] One possibility is to write the total kinetic energy as $2[(p^2+M^2)^{1/2}-M]$, which yields the correction $-\frac{1}{4}M^{-3}p^4$ to the NR value $p^2/M$, but then the two-body amplitude derived from the corresponding Schrödinger equation would be an analytic function only of $E$ (total energy) and not of $E^2$. As a result, such an amplitude would, e.g., lack the MacDowell symmetry (Ref. 21) [S. W. MacDowell, Phys. Rev. **116**, 774 (1959)], $f_l(E)=f_l(-E)$, which is a very desirable property for a relativistic amplitude. On the other hand, a Blankenbecler-Sugar-type equation (Ref. 17) yields a structure of the form $4E_p{}^2-E^2$, which more or less resembles the structure of the NR equation. In this paper we take the view that the kinetic-energy term should be left untouched, so that the two-body system remains unaffected by the relativistic correction, and defend this prescription on the ground that a more profound relativistic modification which preserves the MacDowell symmetry may well have a closer resemblance to the NR structure of the Schrödinger equation than the naive modification suggested by the replacement $p^2/M\rightarrow 2(p^2+M^2)^{1/2}-2M$.

where

$$C=\int d\mathbf{q}\, g^2(q)(q^2+\alpha^2)^{-1}[1-\tfrac{4}{3}q^2g(q)],$$
$$f(P^2)=\int d\mathbf{q}\frac{g^2(q)\{1-(P^2/4M^2)[1-\tfrac{4}{3}q^2g(q)]\}}{(q^2+\alpha^2)(q^2+\alpha^2+\tfrac{3}{4}P^2-\tfrac{3}{4}k^2)}. \quad (21)$$

Note that $k'^2$ vanishes with $k^2$, so that the pole position remains unaffected for zero-energy $n$-$d$ scattering.

As a result of these manipulations, Eq. (19) may be approximated as

$$\text{left-hand side}=\tfrac{3}{4}(P^2-k'^2)f(P^2). \quad (22)$$

We now set the boundary condition

$$F_1(\mathbf{P})=(2\pi)^3\delta(\mathbf{P}-\mathbf{k}')+4\pi a(\mathbf{P})/(P^2-k'^2-i\epsilon), \quad (23)$$

where $a_{1/2}$ is defined as

$$a_{1/2}=-a(0) \quad (24)$$

in the limit $k'^2=0$.

## III. RESULTS AND DISCUSSION

As already stated in the Introduction, a comparison of this simple model with experiment can be meaningful only in respect to the triton binding energy and the $n$-$d$ doublet scattering length $a_{1/2}$ and not, e.g., the quartet scattering length $a_{3/2}$, which depends on the mixed-symmetric component of the orbital function (a quantity which is not simulated in this model). Within the above limitation, however, a comparison with the experimental B.E. and $a_{1/2}$ is almost as good as that of an (more realistic) $s$-wave model which distinguishes between the triplet ($V_t$) and singlet ($V_s$) forces, provided one identifies the strength of the $N$-$N$ force in this model as the average $\frac{1}{2}(V_s+V_t)$ between $V_s$ and $V_t$. The only disadvantage is that one has only a single strength parameter ($\lambda$), which simultaneously affects the "deuteron" ($N^2$) and "triton" ($N^3$) parameters without separate handles on the singlet and triplet strengths. We do not consider the variations with the inverse-range parameter $\beta$, since these have been found to be somewhat insensitive to this input quantity so long as the intrinsic strength parameter[21] is held fixed. For purposes of numerical evaluation we have therefore arbitrarily considered a central value

$$\beta=6.255\alpha \quad (\alpha^2/M=2.225\text{ MeV})$$

roughly similar to Yamaguchi's original parameter.[22] Table I gives the results for B.E. as a function of $\lambda$, together with the corresponding values of the deuteron binding energy $\alpha^2/M$. A comparison of the nonrelativistic values with those obtained from relativistic corrections shows a decrease which is typically of the order of 4–5%.[23] The decrease in B.E. is a direct reflection of the fact that the relativistic correction $\delta V$ is of opposite sign (repulsive) to that of the main potential $V$ (attractive). $V$ does not, of course, affect the deuteron binding energy, since it vanishes exactly in the two-body c.m. frame. We recall in this connection that the present result for the triton's binding energy is not inconsistent with an earlier calculation[24] which had indicated an increase in B.E. due to relativistic effects. The apparent contradiction is resolved through the observation that while the Shirokov correction to the potential decreases the B.E., this decrease was more than offset by an increase brought about by the correction $(-P_i^4/8M^3)$ to the NR kinetic energy $(P_i^2/2M)$ for each particle. For reasons explained in the Introduction, this last correction has not been considered in this paper. This is not to claim that the kinetic-energy correction does not exist. Rather the latter is connected with the bigger question of what should be regarded as the appropriate relativistic dynamics for two- and three-body systems. We have not committed ourselves at this stage to this important question, which would in general imply a reappraisal of the two-body potential parameters; we have merely considered a limited aspect of the relativistic correction, viz., one bearing on the potential $V$, off the center of mass, and this yields a decrease in the triton's B.E. without affecting the two-body parametrization of $V$ in the least.

TABLE I. Binding energies of $N^2$ and $N^3$ as functions of $\lambda$.

| Binding energy of ideal triton Nonrelativistic (MeV) | Relativistic (MeV) | Strength parameter $\lambda$ ($\alpha^{-3}$) | Binding energy of deuteron (MeV) |
|---|---|---|---|
| 7.20 | 6.91 | 25.81 | 0.036 |
| 8.48 | 8.20 | 26.48 | 0.097 |
| 12.91 | 12.40 | 28.55 | 0.464 |
| 14.50 | 13.92 | 29.22 | 0.637 |
| 18.50 | 17.76 | 30.82 | 1.149 |
| 25.50 | 24.48 | 33.36 | 2.225 |

The behavior of $a_{1/2}$ as a function of $\lambda$ is somewhat more involved and does not seem to have been investigated in any great detail in the literature, in contrast with the two-body system, for which the variation of the scattering length with the potential strength has, of course, been thoroughly studied and found to show the characteristic (infinite) discontinuities at the successive bound-state limits for $\lambda$.[3] The behavior of $a_{1/2}$ as a function of $\lambda$, on the other hand, is in general beset with nontrivial three-body effects manifested in both

[21] A parameter akin to $a^2V_0$, where $V_0$ and $a$ are, respectively, the depth and range of a square-well potential.

[22] Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).

[23] We note from Table I that the "best" value of B.E. (∼ 11–12 MeV) which may be expected with purely $s$-wave forces seems to be associated with $\alpha^2/M\approx 0.5$ MeV, which is about one-fourth the experimental value. Such a reduction is expected in this simple model where a single effective force plays a role intermediate between the stronger triplet force $V_t$ and the weaker singlet force $V_s$, so that the effect of a larger number of deuteron pairs is, so to say, compensated by assigning a smaller value to the deuteron's binding energy.

[24] V. K. Gupta, B. S. Bhakar, and A. N. Mitra, Phys. Rev. Letters 15, 974 (1965).

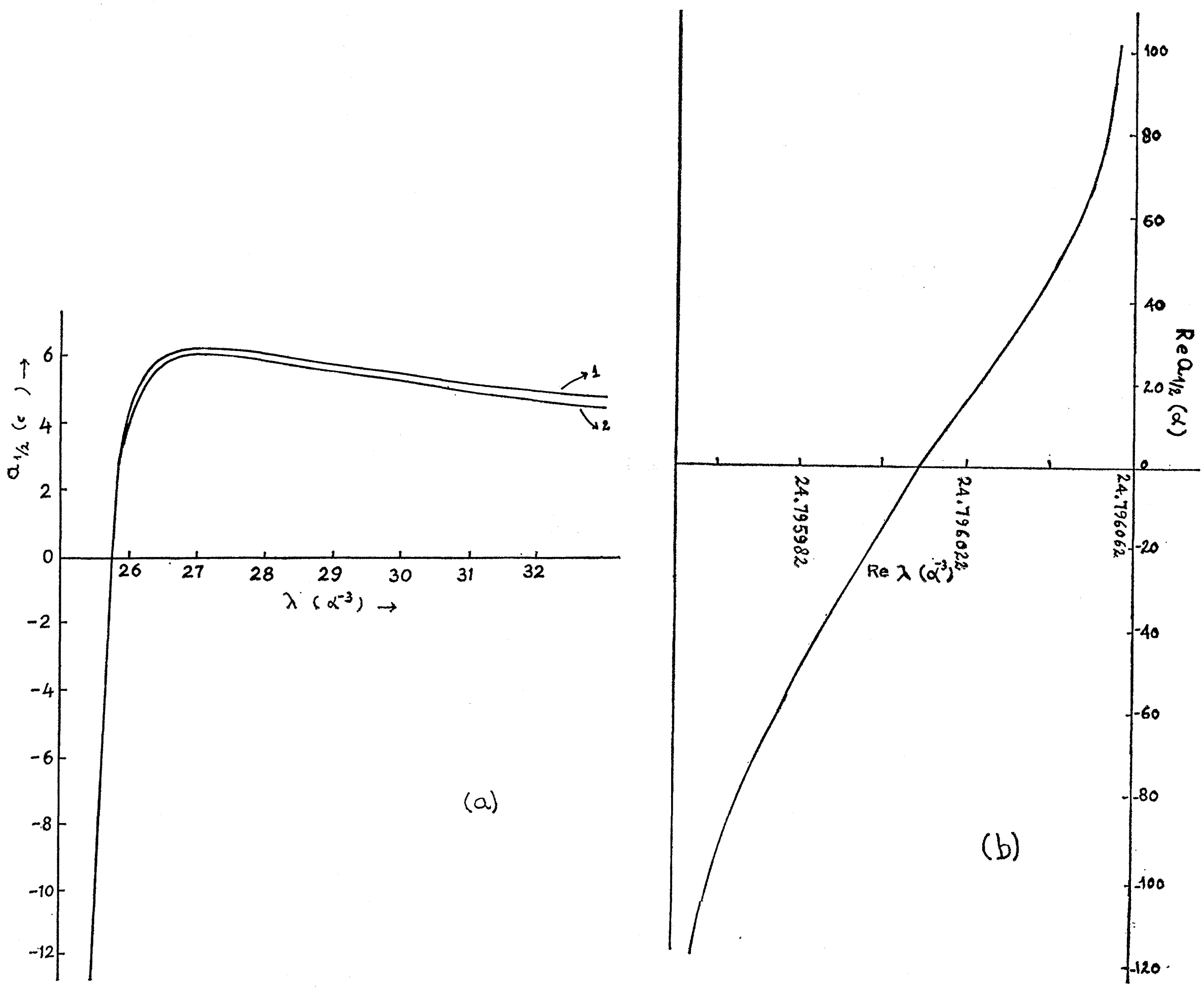


FIG. 1. (a) Variation of the doublet scattering length $a_{1/2}$ as a function of the two-body strength parameter ($\lambda \gtrsim \lambda_c$), both $a_{1/2}$ and $\lambda$ being expressed in units of the deuteron binding-energy parameter ($\alpha$). While curve (1) represents the NR behavior, curve (2) gives the corresponding relativistic correction. (b) Variation of the real part of the doublet scattering length $\mathrm{Re}(a_{1/2})$ as a function of the real part of the two-body strength parameter ($\mathrm{Re}\lambda < \lambda_c$).

pole and cut structures. In connection with this investigation of relativistic corrections to $a_{1/2}$, it is therefore useful to keep on record the results for $a_{1/2}$ versus $\lambda$ in this simple model. Figure 1(a), which shows this variation for a fairly wide range of $\lambda$ values, does indicate a rather rapid fluctuation, though not quite as violent as in the two-body case.[25] For values of $\lambda$ down to $\lambda_c = 24.80\alpha^3$, which corresponds exactly to zero binding energy for the deuteron, one has, of course, a precise mathematical definition for the $n$-$d$ scattering length, based on a 2+1 breakup of the $N^3$ system. Below this critical value for $\lambda$ one cannot, strictly speaking, have an asymptotic 2+1 breakup, and must therefore be content with a nontrivial three-body system. Formally, however, it is possible to define the behavior of $a_{1/2}$ with $\lambda$ in the sense of an analytic continuation, the price now being a complex value for this quantity (since the kernel of its integral equation will now be complex). Since the imaginary part of the kernel builds up rather slowly from a zero value at threshold ($\lambda = \lambda_c$), we have, as a first approximation, neglected the imaginary part and evaluated the real part of $a_{1/2}$ on the basis of the real (but approximate) integral equation, for values of $\lambda$ down to about $24.7\alpha^3$ [see Fig. 1(b)]. As expected, the real part of $a_{1/2}$ for $\lambda \sim \lambda_c$ does seem to show a fairly smooth transition from its value at threshold ($\lambda = \lambda_c$).

For ease of comparison, the relativistic correction to $a_{1/2}$ as a function of $\lambda$ is also shown in Fig. 1(a) alongside the nonrelativistic curve. As expected, the shapes of the two curves are very similar, the corrected one merely being laterally displaced from the uncorrected one

[25] Note that for a single separable potential, like the one considered here, it is not possible to have more than one bound state for a two-body system, so that the characteristic discontinuities in the scattering length at successive bound-state limits obtainable with local potentials are absent in this model.

towards the right, this now being the manifestation of a reduced strength for the effective $(V+\delta V)$ potential. However, unlike the case of B.E., where the correction is always represented by a small decrease over the NR value, the corresponding quantity for $a_{1/2}$ can now show a more fluctuating variation depending on the particular value being considered for $\lambda$. In particular, for $\lambda=25.8\alpha^3$, which gives B.E.$=7.2$ MeV, the value of $a_{1/2}$ changes from $+0.36$ F to $-0.33$ F as a result of the relativistic correction.

To summarize, we have considered relativistic corrections to certain three-body parameters in a rather simple model which is realistic as far as the results depend on the totally symmetric part of the three-body wave function. There is only one kind of "deuteron," and hence only one adjustable strength parameter for the two-body potential $V$. Only a limited aspect of the relativistic correction to the three-body problem has been considered, viz., one which bears on the Shirokov correction $\delta V$ to the potential, and which vanishes in the two-body c.m. frame. This has the advantage of yielding no correction to the two-body parameters, and hence does not require any concomitant readjustments in the parameters of $V$. The three-body effect of this correction is unique, predicting about 5% decrease in the "triton's" B.E., and a lateral shift of the $a_{1/2}$ curves versus $\lambda$ to the right. This simple investigation leaves open the broader and more involved aspects of relativistic corrections which bear on the actual dynamics of two- and three-body systems.

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# Sign of the Absorptive Elastic Amplitude below the Physical Threshold

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We show that for elastic processes which have absorptive channels below the physical threshold, the absorptive partial-wave amplitude $a_l(s)$ has the sign of $(-1)^l$ for $s$ below the physical threshold, where $l$ is the orbital angular momentum. The result is valid for any stable particles of arbitrary spins. More generally, for elastic partial wave transitions between states of definite parity $P=\epsilon\eta_1\eta_2$ (where $\eta_1$ and $\eta_2$ are the intrinsic parities and $\epsilon=\pm1$), the sign of the absorptive amplitude below threshold is that of $\epsilon$.

## INTRODUCTION

The positivity of the absorptive part of the partial-wave elastic amplitudes has been extensively used, together with analyticity of the scattering amplitude as a function of the energy and momentum transfer, to derive a variety of bounds and inequalities for the scattering amplitude and cross sections. This positivity follows immediately from unitarity at physical values of the energy. However, there are systems such as nucleon-antinucleon or $\bar{K}$-nucleon which can go into channels such as $\pi$-$\pi$ in the first case and $\pi$-$\Lambda$ in the second (or even many-particle channels) with a total mass smaller than the total mass of the initial system. In those cases there will be a range of energies below the physical threshold in which the absorptive amplitude for elastic scattering does not vanish. In this region the absorptive amplitude has to be defined by analytic continuation in the masses of the external particles and therefore its sign is not immediately known. In this note, we establish that the sign of the absorptive part $a_l(s)$ of an elastic partial wave amplitude $f_l(s)$ is given by $(-1)^l$, where $l$ is the orbital angular momentum. We first consider the case of spinless particles.

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## I. SPINLESS PARTICLES

Let $F(s,t)$ be the amplitude for elastic scattering of particles $m_1$ and $m_2$ with initial and final momenta $(p_1,p_2)$ and $(p_1',p_2')$, respectively, and let us assume that there are channels with energy threshold below the physical threshold $(m_1+m_2)^2=s_0$.

The absorptive amplitude is given by

$$A(s,t)=(4p_{10}p_{10}')^{1/2}\sum_n \langle p_1'|j_2(0)|n\rangle \times\langle n|j_2(0)^\dagger|p_1\rangle(2\pi)^4\delta^4(p_n-p_1-p_2). \quad (1)$$

Below the physical threshold one cannot have $p_1+p_2=p_n$ with physical momenta. The absorptive amplitude in this region is defined by taking particle $m_2$ off the mass shell and then analytically continuing in $p_2^2$ $(p_2'^2)$ to the physical value $p_2^2=p_2'^2=m_2^2$. To do that, we use Dyson's representation[1] for the matrix element:

$$\int\langle n|[j_2(0)^\dagger,j_1(x)^\dagger]_R|0\rangle e^{-ip_1\cdot x}d^4x =F(n;p_1,p_2)=\int\frac{d^4u d\lambda^2\psi(n;u,\lambda^2)}{(k+u)^2-\lambda^2}, \quad (2)$$

where $k=\frac{1}{2}(p_1-p_2)$ and $p_n=p_1+p_2$.

[1] F. Dyson, Phys. Rev. 110, 1460 (1958).