Broken Chiral Symmetry. III. $SW(3)$ Theory and Soft-Pion Corrections*

V. S. MATHUR AND S. OKUBO
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
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Starting with the $SW(3)$ group as the fundamental symmetry group, we develop a theory for the continuous breaking of this symmetry, in analogy with the theory based on the $W(3)$ group discussed earlier. Despite what appears at first sight, there are considerable differences in the two theories, and in the present case one can avoid many difficulties associated with the $W(3)$ theory. Several refinements of the arguments given earlier are also presented, which lead to mass formulas for π and K as well as the η meson. A method to calculate finite-mass corrections to soft-pion theorems is also proposed and applied to a few problems.

I. INTRODUCTION

IN previous papers,[1] we have proposed a possible approach to studying the consequences of breaking the chiral $W(3)\equiv U^{(+)}(3)\otimes U^{(-)}(3)$ symmetry realized through Goldstone bosons, and have obtained several interesting results using only some rather general principles. Our arguments are based on the observation that by varying one of the symmetry-breaking parameters in the theory, one can realize various subgroups of the $W(3)$ group at suitable discrete points. We have presented arguments that this parameter can lie only in a finite "physical" domain, whose end points correspond to special subgroups involving degenerate vacuum states and the attendant zero-mass pseudoscalar bosons. This restriction was shown in I to arise from the following considerations. In order to realize the special subgroups with degenerate vacua, one can show from general considerations of the two-point functions and positive definiteness of Hilbert space, that the end points must be essentially singular. Now regarding the physical quantities or matrix elements as *continuous* functions of the symmetry-breaking parameter under consideration in the physical domain, and using general principles such as current algebra, soft-meson methods, and variational techniques, we can impose considerable restrictions on the functional dependence, which leads to several interesting results, some of which have been derived in I. In this way, one can, for instance, fix the "physical" value of the symmetry-breaking parameters, and one finds, in agreement with the work of Gell-Mann, Oakes, and Renner,[2,3] that whereas the Hamiltonian is approximately invariant under the $W(2)$ symmetry, the vacuum state is predominantly an $SU(3)$ singlet.

Another important aspect of our work is that our approach provides a clear separation between the points where one may use soft pions, or soft kaons, or $SU(3)$ symmetry, so that all of these results may then be used as constraints for continuation from one point to another, in contrast to using them simultaneously [which would imply working in the $SW(3)$ limit], as is often done in the literature. In principle, then, one can also compute corrections to soft-meson theorems, which were discussed briefly in I.

The purpose of this paper is twofold. Firstly, we reinvestigate the problem using the $SW(3)\equiv SU^{(+)}(3)\otimes SU^{(-)}(3)$ group rather than the $W(3)$ group as the underlying fundamental symmetry, and, as usual, break the $SW(3)$ symmetry by terms that transform as the $(3,3^*)\oplus(3^*,3)$ representation of the group.[2] Secondly, we discuss in greater detail the application of our technique in evaluating corrections to the soft-meson theorems.

At first glance, the change of the fundamental symmetry group from $W(3)$ to $SW(3)$ may appear to be rather trivial. However, there are some aspects of the problem which are considerably modified and it is these features which we would like to study in the first part of this paper. In particular, we find that we can no longer present heuristic arguments for the existence of scalar mesons as discussed in I. Also in contrast to II, we do not have to introduce a large and complicated structure for the η-X mixing, which, of course, is a great advantage. Another attractive feature of the present analysis, which is also related to the η-X mixing problem, is that in this case we can discuss much more precisely the question of the zero-mass Goldstone η meson. In fact, we have proposed here an expression for the η mass aside from the formulas for π and K, which were derived in I and remain unaltered in the present case.

Thus, in general, the $SW(3)$ theory eliminates the troublesome results of the $W(3)$ theory, while retaining its good features. From the point of view of the quark model, however, this situation is somewhat puzzling since in practice if one constructs a quark-model Lagrangian which satisfies $SW(3)$ symmetry, one seems to find that it is also invariant under the $W(3)$ group, if we insist on the conservation of the total quark number.

In Sec. II, we present the basic structure of the $SW(3)$ theory, starting with the model of Gell-Mann,

* Work supported in part by the U. S. Atomic Energy Commission.
[1] S. Okubo and V. S. Mathur, Phys. Rev. D 1, 2046 (1970); also Phys. Rev. Letters 23, 1412 (1969). The former paper will be referred to as I; V. S. Mathur, J. Subba Rao, and S. Okubo, Phys. Rev. D 1, 2058 (1970), referred to as II.
[2] M. Gell-Mann, Physics 1, 63 (1964).
[3] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); see also S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).

Oakes, and Renner, and emphasizing those features which are different from the $W(3)$ theory. We also present some refinements of the arguments presented in I, which are equally applicable to both types of theories. In particular, we have given a detailed discussion of the way $SW(3)$- or $W(3)$-symmetry limit may be achieved, which plays an important role in our continuation arguments. Some of these points were somewhat obscure in I.

We also investigate a symmetry-breaking model in Sec. III, where we add a term proportional to the $(8,1)\oplus(1,8)$ representation in addition to the usual $(3,3^*)\oplus(3^*,3)$ term in the Hamiltonian for both the cases corresponding to the fundamental symmetry group being $SW(3)$ and $W(3)$. As expected, since the number of parameters has increased, the situation is far more complicated.

In the final part of the paper, we have discussed in some detail ways to compute corrections to soft-pion and soft-kaon theorems. This technique has been applied to a discussion of the K_{l3} decays and the Goldberger-Treiman relations.

II. $SW(3)$ THEORY

We shall use the same notation as in Paper I, and write the strong-interaction Hamiltonian density $H(x)$ as[2,3]

$$H(x)=H_0(x)+\epsilon_0 S^{(0)}(x)+\epsilon_8 S^{(8)}(x)\,, \tag{1}$$

where $H_0(x)$ is now assumed to be invariant only under the chiral group $SW(3)\equiv SU^{(+)}(3)\otimes SU^{(-)}(3)$ rather than the larger symmetry group $W(3)\equiv U^{(+)}(3)\otimes U^{(-)}(3)$. We also assume that the scalar densities $S^{(\alpha)}(x)$ together with the pseudoscalar densities $P^{(\alpha)}(x)$ ($\alpha=0,1,\ldots,8$) transform according to the $(3,3^*)\oplus(3^*,3)$ representation of the semidirect product group $G=SW(3)\circledS Z_2$. As before, we shall assume that in a certain domain which will be specified shortly, the $SW(3)$ symmetry is realized through the emergence of an octet of massless pseudoscalar mesons, with the vacuum invariant under $SU(3)$ rather than the full symmetry group $SW(3)$. Also we shall use the current-algebra postulate so that, although the $SW(3)$ symmetry may be broken, the $SW(3)$ algebra will be assumed to be valid as equal-time commutation relations.

Defining the $SW(3)$ generators[2] $F^{(\alpha)}$ and $F_5{}^{(\alpha)}$ ($\alpha=1,\ldots,8$) by

$$\begin{aligned} F^{(\alpha)}(t)&=-i\int_{x0=t} d^3x\, V_4{}^{(\alpha)}(x)\,,\\ F_5{}^{(\alpha)}(t)&=-i\int_{x0=t} d^3x\, A_4{}^{(\alpha)}(x)\,, \end{aligned} \tag{2}$$

where $V_\mu{}^{(\alpha)}(x)$ and $A_\mu{}^{(\alpha)}(x)$ denote the octets of vector and axial-vector current densities, one can derive the partial-conservation laws

$$\begin{aligned} \partial_\mu V_\mu{}^{(\alpha)}(x)&=\epsilon_8 f_{\alpha8\beta}S^{(\beta)}(x)\,,\\ \partial_\mu A_\mu{}^{(\alpha)}(x)&=(\epsilon_0 d_{\alpha0\beta}+\epsilon_8 d_{\alpha8\beta})P^{(\beta)}(x)\,, \end{aligned} \tag{3}$$

where $\alpha=1,\ldots,8$, while the summation over β runs from $\beta=0$ to $\beta=8$. The fact that $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$ ($\alpha=0,1,\ldots,8$) form a $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ group, leads[2,3] to the following algebra at equal times:

$$\begin{aligned} [F^{(\alpha)}(t),S^{(\beta)}(x)]_{x_0=t}&=if_{\alpha\beta\gamma}S^{(\gamma)}(x)\,,\\ [F^{(\alpha)}(t),P^{(\beta)}(x)]_{x_0=t}&=if_{\alpha\beta\gamma}P^{(\gamma)}(x)\,,\\ [F_5{}^{(\alpha)}(t),S^{(\beta)}(x)]_{x_0=t}&=id_{\alpha\beta\gamma}P^{(\gamma)}(x)\,,\\ [F_5{}^{(\alpha)}(t),P^{(\beta)}(x)]_{x_0=t}&=-id_{\alpha\beta\gamma}S^{(\gamma)}(x)\,, \end{aligned} \tag{4}$$

where again $\alpha=1,\ldots,8$, but β and γ run from 0 to 8.

If we write the usual spectral representation for the commutator ($\alpha,\beta=1,\ldots,8$)

$$\begin{aligned} \langle 0|[A_\mu{}^{(\alpha)}(x),A_\nu{}^{(\beta)}(y)]|0\rangle &=\int_0^\infty dm^2\bigg[\bigg(\delta_{\mu\nu}-\frac{1}{m^2}\partial_\mu\partial_\nu\bigg)\rho_{\alpha\beta}{}^{(1)}(m,A)\\ &\quad-\frac{1}{m^2}\rho_{\alpha\beta}{}^{(0)}(m,A)\partial_\mu\partial_\nu\bigg]\Delta(x-y,m)\,, \end{aligned} \tag{5}$$

and a similar one for the vector currents, we obtain, on taking divergences of both sides with respect to x and setting $x_0=y_0$, the relations

$$\begin{aligned} I_{\alpha\beta}&\equiv\int_0^\infty dm^2\,\rho_{\alpha\beta}{}^{(0)}(m,A)\\ &=-(\epsilon_0 d_{0\alpha\gamma}+\epsilon_8 d_{8\alpha\gamma})(\xi_0 d_{0\beta\gamma}+\xi_8 d_{8\beta\gamma})\,,\\ K_{\alpha\beta}&\equiv\int_0^\infty dm^2\,\rho_{\alpha\beta}{}^{(0)}(m,V)=-\epsilon_8\xi_8 f_{8\alpha\gamma}f_{8\beta\gamma}\,, \end{aligned} \tag{6}$$

where the summation over γ runs from 0 to 8 again, and where

$$\xi_0=\langle 0|S^{(0)}(0)|0\rangle\,,\quad \xi_8=\langle 0|S^{(8)}(0)|0\rangle \tag{7}$$

are the only nonzero vacuum expectation values of the scalar density operators. As in I, defining the real parameters a, b, and γ by

$$a=\frac{1}{\sqrt2}\frac{\epsilon_8}{\epsilon_0}\,,\quad b=\frac{1}{\sqrt2}\frac{\xi_8}{\xi_0}\,,\quad \gamma=-\tfrac{2}{3}\epsilon_0\xi_0\,, \tag{8}$$

Eq. (6) leads to

$$\begin{aligned} I_{33}&=\gamma(1+a)(1+b)\,,\\ I_{44}&=\gamma(1-\tfrac12 a)(1-\tfrac12 b)\,,\\ I_{88}&=\gamma(1-a-b+3ab)\,,\\ K_{44}&=(9/4)\gamma ab\,. \end{aligned} \tag{9}$$

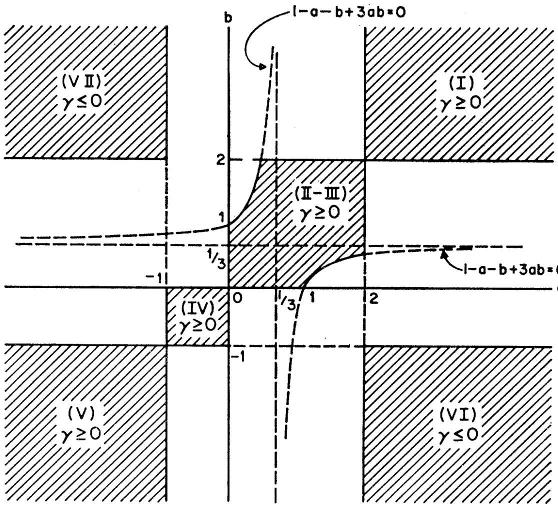

FIG. 1. Allowed domains for the case of the $SW(3)$ group.

These are exactly the same relations as in I. However, the important difference is that now we do not have any equations for I_{00} and I_{08}, since α and β in Eq. (6) cannot take the value 0 for the $SW(3)$ theory. It is interesting to note that Eq. (9) still leads to the sum rule

$$\tfrac{1}{4}I_{33}+\tfrac{3}{4}I_{88}-I_{44}=K_{44}. \qquad (10)$$

Now, let us utilize the positivity condition[4] of the spectral weights, which implies

$$I_{33}\geq 0,\quad I_{44}\geq 0,\quad I_{88}\geq 0,\quad K_{44}\geq 0. \qquad (11)$$

It is easy to see that Eqs. (9) and (11) give rise to the following restrictions:

$$\begin{array}{ll} \text{(I)} & a\geq 2,\ b\geq 2,\ \gamma\geq 0, \\ \text{(II–III)} & 2\geq a\geq 0,\ 2\geq b\geq 0,\ 1-a-b+3ab\geq 0,\ \gamma\geq 0, \\ \text{(IV)} & 0\geq a\geq -1,\ 0\geq b\geq -1,\ \gamma\geq 0, \\ \text{(V)} & a\leq -1,\ b\leq -1,\ \gamma\geq 0, \\ \text{(VI)} & a\geq 2,\ b\leq -1,\ \gamma\leq 0, \\ \text{(VII)} & a\leq -1,\ b\geq 2,\ \gamma\leq 0, \end{array} \qquad (12)$$

where we have used the special notation (II–III) for the second region for reasons which will be explained below. These six distinct allowed domains are most conveniently projected on the a-b plane as in Fig. 1. Comparing the present domains with those[1] of the $W(3)$ symmetry depicted in Fig. 2, we find that the restriction in the $SW(3)$ case fuses the two domains II and III of the $W(3)$ group into a single region (II–III) of Eq. (12). Therefore, the point $a=\frac{1}{2}$, which played an important role in Paper I, is no longer a special point in the present case. This difference is simply due to the fact that we do not have any relations for I_{00} and I_{08} here. Note also that the positivity requirement on I_{88} slices off the opposite corners of the otherwise square domain (II–III), which then assumes two hyperbolic boundaries.

One can now repeat the same arguments as in I, and observe that the boundaries for the domains in Fig. 1 are related to various subgroups of the $SW(3)$ group as follows:

(1) $a=0$ implies $\partial_\mu V_\mu^{(\alpha)}(x)=0$ ($\alpha=1,\ldots,8$), (i.e., it corresponds to the validity of the usual $SU(3)$ group.

(2) $a=-1$ leads to $\partial_\mu A_\mu^{(\alpha)}(x)=0$ ($\alpha=1, 2, 3$). Together with the ordinary isospin group $SU(2)$, the point $a=-1$ then corresponds to the validity of the $SW(2)=SU^{(+)}(2)\otimes SU^{(-)}(2)$ subgroup. Note that we no longer have an additional conservation law $\partial_\mu A_\mu^{(-1)}(x)=0$ as in the $W(3)$ case, so that our subgroup at $a=-1$ is $SW(2)$ rather than $W(2)$. In particular, we do not have to worry about the question whether η and X mesons would become soft at $a=-1$.

(3) $a=2$ leads to $\partial_\mu A_\mu^{(\alpha)}(x)=0$ ($\alpha=4, 5, 6, 7$). Setting $X^{(\alpha)}=F^{(\alpha)}$ for $\alpha=1, 2, 3, 8$ and $X^\alpha=F_5^{(\alpha)}$ for $\alpha=4, 5, 6, 7$, we find that $X^{(\alpha)}$ ($\alpha=1,\ldots,8$) generates the chimeral $SU(3)$ group discussed in I.

As before, one can prove the following theorem. At $a=2$, 0, or -1, we have $b=a$, if and only if the vacuum state is nondegenerate under the subgroups in question. Conversely, at $b=2$, 0, or -1, we would get $a=b$ under the same condition.

An important difference from the $W(3)$ theory discussed in I is that we no longer have a conservation law at the point $a=\frac{1}{2}$. Hence $a=\frac{1}{2}$ or $b=\frac{1}{2}$ gives no additional constraints, and all our previous arguments for the existence of scalar mesons do not hold in the present case.

We may now assume that b is a continuous function of a and ϵ_0, $b=f(a,\epsilon_0)$. We shall first study how b

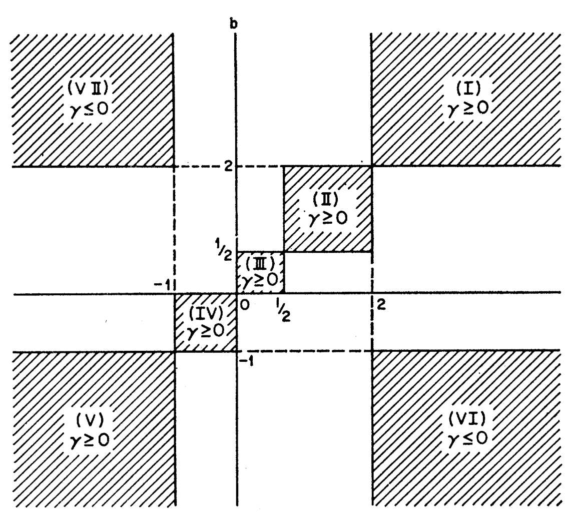

FIG. 2. Allowed domains for the $W(3)$ group.

[4] S. Okubo, Nuovo Cimento **44A**, 1015 (1966); G. Pocsik, *ibid.* **43A**, 541 (1966).

behaves in the $SW(3)$ limit. As assumed, the $SW(3)$ symmetry is realized[5] in the limit in which the masses of the mesons in the pseudoscalar octet tend to zero, and the vacuum state is invariant under $SU(3)$. Now the Hamiltonian (1) will become invariant under $SW(3)$ symmetry as ϵ_0, $\epsilon_8 \to 0$ in any particular manner, so that the parameter $a=\epsilon_8/\sqrt{2}\epsilon_0$ can take all possible values. Also, since the vacuum state is invariant under $SU(3)$, we must have $\xi_8=0$. From the definition (8), this in turn implies $b=0$, if $\xi_0\neq 0$. Thus if $\xi_0\neq 0$, one would get in the $SW(3)$ limit $b=0$ for all a. However, a glance at Fig. 1 shows that b cannot assume the value zero for $1<a\leq 2$, since this value falls in the forbidden zone. Thus the following possibilities arise: (1) If we accept $b=0$ for all a in the $SW(3)$ limit, we are forced to conclude that for all $a>1$, b as a function of ϵ_0 must possess discontinuities at $\epsilon_0=0$, and there is no way of reaching the $SW(3)$ limit uniformly in this region. (2) In the region $a>1$, on the other hand, we may not have $b=0$. In this case, one can show that the hyperbolic bounds to the region II–III (see Fig. 1) can describe the realization of the $SW(3)$ limit. For this purpose, note that $1-a-b+3ab=0$ implies $I_{88}=0$ from Eq. (9). Then the positivity[4] of the spectral function demands the stronger result

$$\partial_\mu A_\mu^{(8)}(x)|0\rangle=0. \qquad (13)$$

Now if we assume that the vacuum state is unique over the hyperbola, Eq. (13) implies[6] $\partial_\mu A_\mu^{(8)}(x)=0$, so that from Eq. (3) one then obtains $\epsilon_0=\epsilon_8=0$, i.e., exact $SW(3)$ symmetry. However, even if the vacuum state is degenerate, but is invariant under the $SU(3)$ group, one can realize the $SW(3)$ limit, again over the hyperbola. This follows from Eqs. (3) and (13), since we would then have

$$[F^{(\alpha)},\partial_\mu A_\mu^{(8)}(x)]|0\rangle = (i/\sqrt{3})(\sqrt{2}\epsilon_0-\epsilon_8)f_{\alpha 8\beta}P^{(\beta)}(x)|0\rangle=0, \qquad (14)$$

so that for $P^{(\beta)}(x)|0\rangle\neq 0$, we must have $\epsilon_0=\epsilon_8=0$ for $a\neq 1$. Also, since the $SU(3)$ invariance[7] of the vacuum implies $\xi_8=0$, and since $b\neq 0$ $(a>1)$, we must also have in this case $\xi_0=0$ for $a>1$. Note that the $SW(3)$ limit can now be reached continuously and uniformly. Writing

$$b=f(a,\epsilon_0), \qquad (15)$$

we would then have in the $SW(3)$ limit

$$\begin{aligned} b=f(a,0)&=0, & -1\leq a\leq 1 \\ &=(a-1)/(3a-1), & 1\leq a\leq 2. \end{aligned} \qquad (16)$$

Notice that $g(a)\equiv f(a,0)$ is a continuous function of a. We have in Eq. (16) restricted ourselves to the domain $-1\leq a\leq 2$, since, as we shall see presently, it is impossible to continue b outside this domain.

In Paper I, we showed that the points $a=-1$ and $a=2$ correspond to essential singularities of the theory. This argument applies here too. However, we can prove this result in a more convincing and instructive fashion if we make the ansatz[5] that the $SW(3)$ limit $(\epsilon_0, \epsilon_8\to 0)$ should in fact be achieved continuously and uniformly for all a, so that we accept the possibility (2) mentioned above. Then b as a function of a and ϵ_0 will satisfy the constraint (16). We can now prove that at $a=-1$, b cannot assume the value -1 for all ϵ_0. For if this is so, we find from Eq. (15) that for arbitrary ϵ_0, we must have

$$-1=f(-1, \epsilon_0). \qquad (17)$$

Because of our ansatz, Eq. (17) is also valid in particular when $\epsilon_0=0$, but the resulting equation contradicts Eq. (16) at $a=-1$. Clearly Eq. (16) requires that at $a=-1$ or, for that matter, at any particular value of a less than unity, b must be a function of ϵ_0, say, $b(\epsilon_0)\equiv f(-1, \epsilon_0)$, such that $b(0)=0$. Now, since at $a=-1$, we see that b cannot take the value -1, the transition from the region IV to V (see Fig. 1) as a moves across the point -1, must necessarily be discontinuous. If we assume that b as a function of a is also an *analytic* function of a except at some isolated points, we have shown that the point $a=-1$ corresponds to an essential singularity of the theory. Similarly, we can prove that b cannot attain the value $b=2$ when we let a increase to $a=2$. Thus $a=2$ is also an essentially singular point.

We would like to remark that in these arguments we did not assume that the pion (kaon) becomes a zero-mass Goldstone boson at $a=-1$ $(a=2)$ for nonzero values of ϵ_0, as was done in I. Indeed, at $a=-1$ $(a=2)$, since b cannot assume the values $b=-1$ $(b=2)$, our theorem stated before and proved in I shows that the vacuum state must be degenerate at $a=-1$ $(a=2)$. We must then have zero-mass particles appearing at $a=-1$ and $a=2$, for all nonzero values of ϵ_0, which, from considerations of the divergence conditions (3), can easily be identified as the pion and the kaon, respectively. It is interesting to note that we have shown that the points $a=-1$ and $a=2$, which are essential singularities in the theory, are also the points where vacuum states are degenerate. The connection between these properties has been known for some other models in the literature, e.g., the Nambu–Jona-Lasinio model.[8]

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[5] This ansatz has been proposed and applied by Dashen and Weinstein in a series of papers: R. F. Dashen, Phys. Rev. **183**, 1245 (1969); R. F. Dashen and M. Weinstein, *ibid.* **183**, 1261 (1969); Phys. Rev. Letters **22**, 1337 (1969).

[6] P. Federbush and K. Johnson, Phys. Rev. **120**, 1926 (1960); R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

[7] Another possibility is to relax our requirement so that we assume only the $SU(2)$-invariant vacuum rather than the full $SU(3)$-invariant one in the $SW(3)$ limit for the range $1\leq a\leq 2$, while maintaining the $SU(3)$-invariant vacuum in the same limit for $|a|\leq 1$. Then we may avoid setting $b=0$, $\xi_0=0$, and $f=0$ in this range. This possibility will be investigated elsewhere.

[8] Y. Nambu and Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

The fact that we have been able to show that the $SW(2)$ and chimeral symmetries are realized through the appearance of zero-mass pions and kaons, respectively, may appear quite surprising in view of the fact that these symmetries as well as the larger symmetry $SW(3)$ can, in principle, also be realized as ordinary symmetry groups with unique vacuum states. One may thus well ask what would have happened if the $SW(3)$ symmetry were realized as an ordinary symmetry with a nondegenerate vacuum. In this case it is most likely that b as a function of a would be simply given by $b=a$ in the $SW(3)$ limit, instead of the structure exhibited in Eq. (16), so that the subgroup symmetries would also be realized without Goldstone bosons.

Our argument based on the continuity ansatz is, of course, also applicable at $a=0$. Here since $SU(3)$ symmetry is exact, one expects $b=0$, for arbitrary ϵ_0, if we accept the usual assumption that the vacuum state is not degenerate in the $SU(3)$ limit. The result $b=0$ is, however, consistent with Eq. (16), according to which, if b is independent of ϵ_0 for $a<1$, it must be zero. Thus a smooth transition is possible between the domains (II–III) and (IV) as one passes across $a=0$, so that $a=0$ is not a singular point.

Before going into further details, we may also prove that ξ_0 may have an essential singularity at $\epsilon_0=0$ for fixed values of a, provided that ξ_0 can assume a nonzero value in the limit $\epsilon_0\to 0$. This is due to the fact that since $\gamma\geq 0$ for the physical domain of a, we have $\epsilon_0\xi_0\leq 0$, so that $\xi_0\leq 0$ for $\epsilon_0\geq 0$ and $\xi_0\geq 0$ for $\epsilon_0\leq 0$. Thus, if ξ_0 does not go to zero as $\epsilon_0\to 0$, it follows that ξ_0 must have a discontinuity at $\epsilon_0=0$. Assuming, moreover, that ξ_0 is an analytic function of ϵ_0 except at a few isolated points, this implies also that ξ_0 has an essential singularity at $\epsilon_0=0$, since the value of ξ_0 must discontinuously jump at $\epsilon_0=0$. In such a case unless ξ_8 also has an essential singularity at $\epsilon_0=0$ which cancels the one in ξ_0, the parameter $b=\xi_8/\sqrt{2}\xi_0$ may be essentially singular at $\epsilon_0=0$.

Returning to the original discussion, we note that the arguments presented here are evidently applicable to both the cases when $SW(3)$ or $W(3)$ is the fundamental symmetry group. In the $W(3)$ case, however, the ansatz leading to Eq. (16) is now replaced by

$$f(a,0)=0\,,\quad -1\leq a<\tfrac{1}{2}$$
$$f(a,0)=\tfrac{1}{2}\,,\quad \tfrac{1}{2}\leq a\leq 2\,. \tag{18}$$

We see that now b as a function of a need not have a singularity at $a=\frac{1}{2}$, if at this point we also have $b=\frac{1}{2}$, i.e., the vacuum state is nondegenerate. This provides a justification for the assumption made in I that $a=\frac{1}{2}$ is not a singular point for nonzero ϵ_0. However, at $\epsilon_0=0$, $g(a)=f(a,0)$ is evidently discontinuous at $a=\frac{1}{2}$ in contrast to the $SW(3)$ case [see Eq. (16)]. This, in our opinion, is an unpleasant feature of the $W(3)$ theory.

At this stage, one might well ask if there is any point in Fig. 1 where one might realize a zero-mass η, just as at $a=-1$ and $a=2$, one obtains a zero-mass pion and kaon, respectively. In fact a moment's reflection shows that the vanishing of the η mass has to do with the hyperbolic boundaries of the domain (II–III). Let us first imagine, for the sake of argument, that if $b=f(a,\epsilon_0)$ could be solved as a function of a and ϵ_0 for the "realistic" case $\epsilon_8\neq 0$, $\epsilon_0\neq 0$, the resulting curve for b for some fixed $\epsilon_0\neq 0$ touches the hyperbola in Fig. 1 at some suitable point (or points). Then Eq. (13) should be valid at such a point. If we have no zero-mass Goldstone particles emerging at this point, Eq. (13) implies[6] $\partial_\mu A_\mu^{(8)}(x)=0$. In the event that the vacuum state is degenerate, this conclusion does not necessarily follow. However, as mentioned before, an inspection of Eq. (3) shows that $\partial_\mu A_\mu^{(8)}(x)=0$ implies $\epsilon_0=\epsilon_8=0$, i.e., the exact $SW(3)$ limit. Since, in fact, we started with $\epsilon_0\neq 0$, $\epsilon_8\neq 0$, we obviously have to discard this solution. Similarly, we discard the case when the vacuum state is an $SU(3)$ scalar, since as shown before this case also corresponds to the $SW(3)$ symmetry limit. Thus, if we want $\epsilon_0\neq 0$, $\epsilon_8\neq 0$, the vacuum state is presumably symmetric only under the smaller $SU(2)$ group. Also note that Eq. (13) in particular demands that

$$\langle\eta(k)|\,\partial_\mu A_\mu^{(8)}(x)\,|0\rangle=0\,. \tag{19}$$

But the left-hand side of Eq. (19) is proportional to m_η^2, so that η can in fact become massless at the point under discussion. Actually, however, the same argument is also applicable to the X meson. Thus it is not possible to decide whether one obtains a zero-mass η or X or both, if indeed a zero-mass particle emerges. It may, however, be more appropriate to assume that a zero-mass η in fact emerges, whether or not one obtains a massless X. Such a possibility seems to be consistent with the assumption of an octet of massless pseudoscalar mesons in the $SW(3)$ limit $\epsilon_0\to 0$, which, as mentioned before, is also realized on the hyperbola. This argument is particularly relevant if the η-X mixing is not large, as we will see shortly.

Some questions naturally arise at this stage. At the point under consideration on the hyperbola, what symmetry group does one realize, and is this larger than the symmetry [presumably $SU(2)$] of the vacuum? Also, would this point be an essentially singular point of the theory? With respect to the first question, one could argue that there is no reason to believe that if a zero-mass particle exists, the symmetry of the Hamiltonian should be necessarily higher than that of the vacuum state. Thus, unlike π and K, the η may not be a Goldstone boson in the ordinary sense, and the vanishing of the η mass for nonzero ϵ_0 would then have to be a dynamical accident of the theory. Also in this case, unlike the situation of $a=-1$ and $a=2$, we have no compelling reason one way or another with regard to the existence of an essential singularity, except for the comment that should it exist, a smooth transition to the $SW(3)$ limit would be hard to understand.

There exists, however, another possibility which bypasses these problems. This arises if b as a function of a for a fixed $\epsilon_0\neq 0$ does not make contact at all with the hyperbola. In this case one cannot realize a zero-mass η, except as a Goldstone boson in the $SW(3)$ limit ($\epsilon_0\to 0$) when b as a function of a is given by the hyperbola itself. From the point of view of our present discussion, this possibility is much more attractive. However, if this is the case, it is clear that the soft-η technique would make sense only in the $SW(3)$ limit, unlike soft-π and -K methods which are valid for subgroups of $SW(3)$ at $a=-1$ and $a=2$, respectively.

Before proceeding further, we would like to point out that one can analyze the functional dependence (15) in somewhat greater detail, which suggests that the numerical value of b is quite small in comparison with unity for the entire physical range of a, $-1\leq a\leq 2$. First of all, notice that both a and b are dimensionless, whereas ϵ_0 and ϵ_8 are not. Hence, if we have no other constant with the dimension of mass in the theory, b must only be a function of a. But this contradicts Eq. (16), unless b is identically zero for $a<1$. We know, however, that an extra constant M with the dimension of mass must be present in the Hamiltonian H_0 in Eq. (1), so that in the $SW(3)$ limit one can have baryons, vector mesons, etc., with nonzero mass. Thus, assuming that ϵ_0 has dimensions of mass as in the quark model (the other cases may be similarly worked out), we must have

$$b=f(a,\ \epsilon_0/M)\,. \tag{20}$$

Since we expect[9] $|\epsilon_0/M|\ll 1$, assuming that $\epsilon_0=0$ is not a singular point, we may expand b in a power series in ϵ_0/M to obtain, for the range $-1\leq a\leq 1$,

$$b=(\epsilon_0/M)^2 g(a)+O((\epsilon_0/M)^3)\,,\quad -1\leq a\leq 1\,. \tag{21}$$

Note that in deriving Eq. (21) we have used Eq. (16) and the fact that b cannot depend linearly on ϵ_0/M. The reason for this is that we must have $ab\geq 0$, for our solution of Fig. 1 irrespective of the sign of ϵ_0. Now, since $(\epsilon_0/M)^2\ll 1$, Eq. (21) suggests a rather small value of b in the range $-1\leq a\leq 1$. This is also consistent with the numerical solution at the physical point computed in I and II. For the range $1\leq a\leq 2$, one obtains, using Eq. (16), the result

$$b=\frac{a-1}{3a-1}+\left(\frac{\epsilon_0}{M}\right)^2 h(a)+O((\epsilon_0/M)^3)\,,\quad 1\leq a\leq 2\,. \tag{22}$$

Once again the linear term ϵ_0/M makes no contribution since we know that for $\epsilon_0\neq 0$, one must have $b>(a-1)/(3a-1)$ for all a lying in the range $1\leq a\leq 2$, irrespective of the sign of ϵ_0. Since the first term on the right-hand side of Eq. (22) can only be as large as $\frac{1}{5}$ for the range of a under consideration, the value of b is still expected to be small compared with unity. This explains why for some matrix elements the use of $SU(3)$ for any a in the range $-1\leq a\leq 2$ may lead to sensible results. Such matrix elements would not depend sensitively on a, but rather only through b. Also the smallness of b may be interpreted to suggest that the vacuum is roughly $SU(3)$ invariant. This then provides an understanding of the result of Gell-Mann, Oakes, and Renner, according to which at the physical point where the Hamiltonian is approximately $SU(2)\otimes SU(2)$ invariant, the vacuum state behaves roughly as an $SU(3)$ singlet.

However, as we noted earlier, b may have an essential singularity at $\epsilon_0=0$. In that case, Eqs. (21) and (22) and the discussion based upon them will not hold. It may be possible, however, that numerically the effect of the essential singularity at $\epsilon_0=0$ is small so that the conclusions may possibly be still valid approximately.

III. MASS FORMULAS

As long as the $SW(3)$ symmetry limit can be reached uniformly, our arguments leading to the mass formulas for m_π^2 and m_K^2 and the other relations derived in I, are also valid here. Some care is necessary, however, for obtaining the mass formula for the K meson, as we shall see shortly. Since m_π^2 vanishes at $a=-1$ and m_K^2 at $a=2$, we may write, as in I,

$$m_\pi^2=(\sqrt{2}\epsilon_0+\epsilon_8)F_\pi(\epsilon_0,\epsilon_8)=\sqrt{2}\epsilon_0(1+a)F_\pi(\epsilon_0,\epsilon_8)\,, \tag{23}$$

$$m_K^2=(\sqrt{2}\epsilon_0-\tfrac{1}{2}\epsilon_8)F_K(\epsilon_0,\epsilon_8)=\sqrt{2}\epsilon_0(1-\tfrac{1}{2}a)F_K(\epsilon_0,\epsilon_8)\,. \tag{24}$$

Now, unlike the situation in Paper I, it is possible to obtain some information on the mass of the η. For this purpose, recall that at the boundary $1-a-b+3ab=0$ of the domain (II–III), we have argued that $m_\eta^2\to 0$. The fact that this boundary may not be accessible in practice except in the $SW(3)$ limit is not relevant at the moment. One expects then that m_η^2 would be proportional to $1-a-b+3ab$, so that we may write, in analogy to Eqs. (23) and (24),

$$\begin{aligned}m_\eta^2&=[(\sqrt{2}\epsilon_0-\epsilon_8)-b(\sqrt{2}\epsilon_0-3\epsilon_8)]F_\eta(\epsilon_0,\epsilon_8)\\&=\sqrt{2}\epsilon_0(1-a-b+3ab)F_\eta(\epsilon_0,\epsilon_8)\,.\end{aligned} \tag{25}$$

We would first like to show, from dimensional considerations and the ansatz that the $SW(3)$ limit may be approached uniformly, that to a good approximation

$$F_\pi(\epsilon_0,\epsilon_8)=F_K(\epsilon_0,\epsilon_8)=F_\eta(\epsilon_0,\epsilon_8)\,. \tag{26}$$

Assume for simplicity that ϵ_0 and ϵ_8 have the dimensions of mass, as in the quark model. Then from dimensional considerations, one can write

$$F_i(\epsilon_0,\epsilon_8)=\epsilon_0\phi_i^{(1)}+\epsilon_8\phi_i^{(2)}+M\phi_i^{(3)}\,, \tag{27a}$$

where i stands for π, K, or η and M is a mass which, as mentioned before, must be present in H_0 in Eq. (1), so that the baryons, for example, may have a nonzero mass in the $SW(3)$ limit. The functions ϕ_i depend on the

[9] ϵ_0 and ϵ_8 are related to the pseudoscalar-meson masses and hence will be of the order of m_π, as we may see from the argument of I. If we use asymptotic $SU(6)_W$ symmetry, then we can compute $\epsilon_0\simeq 140$ MeV. See S. Okubo, Phys. Rev. **188**, 2293 (1969); **188**, 2300 (1969).

dimensionless parameters a and ϵ_0/M or ϵ_8/M. We shall now assume that ϵ_0/M and ϵ_8/M are small enough[9] to be neglected, so that Eq. (27) becomes

$$F_i(\epsilon_0,\epsilon_8)\simeq M\phi_i^{(3)}(a)\,. \tag{27b}$$

Note that in $\phi_i^{(3)}$ the dependence on ϵ_0/M or ϵ_8/M is also dropped. Now if a uniform transition to the $SW(3)$ limit is possibl e,one obtains from Eq. (27a) that for any a, $F_i(0,0)=M\phi_i^{(3)}(a)$, so that from Eq. (27b) $F_i(\epsilon_0,\epsilon_8)\simeq F_i(0,0)$. Thus to the extent that Eq. (27b) is valid, $F_i(\epsilon_0,\epsilon_8)$ is given by its $SW(3)$ value. It should be noted now that $F_i(0,0)$ is in fact independent of i. This follows because in the $SW(3)$ limit, the vacuum state is $SU(3)$ invariant for all values of a. Thus, $F_i(\epsilon_0,\epsilon_8)$ is also independent of i to the extent that ϵ_0/M and ϵ_8/M are negligible. This establishes Eq. (26).

In fact it is simple to obtain an explicit expression for $F_i(0,0)$. We have shown in I that using the variational principle and partial conservation of axial-vector current (PCAC), we can obtain

$$\begin{aligned} F_\pi(\epsilon_0,\epsilon_8)&=-\tfrac{2}{3}\sqrt{2}(\xi_0/f_\pi^2)(1+b) \quad (m_\pi^2\to 0)\,,\\ F_K(\epsilon_0,\epsilon_8)&=-\tfrac{2}{3}\sqrt{2}(\xi_0/f_K^2)(1-\tfrac{1}{2}b) \quad (m_K^2\to 0)\,. \end{aligned} \tag{28a}$$

In the $SW(3)$ limit since $m_\pi^2=0$ and $m_K^2=0$ for all a, using Eq. (16) we obtain for $-1\leq a\leq +1$

$$F_i(0,0)\equiv K=-\tfrac{2}{3}\sqrt{2}(\xi_0/f^2)\,, \tag{28b}$$

where f is the common $SU(3)$ value of the decay constants f_π and f_K. Note that, in general, K can depend on a. We might remark here parenthetically that if Eq. (28b) is valid also in the region $1\leq a\leq 2$, where we have taken $\xi_0=0$, we have to accept[7] $f^2=0$ also to get a finite result K. We have not studied this constraint in detail and it may have some special significance. However, for the purpose of our discussion here, it is sufficient to limit oneself to the range $-1\leq a\leq 1$ in the $SW(3)$ limit. The realization of the $SW(3)$ symmetry limit in the region $1\leq a\leq 2$ indeed poses some problems. For instance, in this range of the values of a, the mass formula (25) for η seems to have a double zero in the $SW(3)$ limit, in contrast with the expressions (23) and (24) for m_π^2 and m_K^2 which go to zero as ϵ_0. This seems to be contrary to the perturbative argument, if indeed it can be trusted.

Returning to our main argument, we thus see that provided ϵ_0 and ϵ_8 are negligible compared to M, one obtains the result Eq. (26). Note that if ϵ_0 and ϵ_8 have dimensions different from that of a mass, a little reflection shows that the above type of argument would still go through in obtaining the result Eq. (26).

Although the neglect of the first two terms in Eq. (27) may *a priori* seem appropriate, we would like to show that for the η case this is not strictly true. The evidence for this comes from the use of the variational principle near the $SU(3)$ symmetric point. One can readily show, following the procedure outlined in I, that we must have, at $a=0$,

$$m_\eta^2=m_\pi^2=m_K^2\,, \tag{29a}$$

$$\frac{\partial m_\eta^2}{\partial\epsilon_8}=-\frac{\partial m_\pi^2}{\partial\epsilon_8}=2\frac{\partial m_K^2}{\partial\epsilon_8}\,, \tag{29b}$$

$$\frac{\partial m_\eta^2}{\partial\epsilon_0}=\frac{\partial m_\pi^2}{\partial\epsilon_0}=\frac{\partial m_K^2}{\partial\epsilon_0}\,, \tag{29c}$$

assuming that the η meson becomes a member of the pure pseudoscalar octet at $a=0$. From Eqs. (23)–(26), it is evident that Eq. (29a) is satisfied. But to satisfy Eqs. (29b) and (29c) for m_η^2, one requires $\partial b/\partial\epsilon_8=\partial b/\partial\epsilon_0=0$, which is unacceptable. This feature is, of course, linked with the appearance of an explicit b dependence in the expression for m_η^2. It is simple to see, however, that Eq. (29) can be satisfied if $F_\eta(\epsilon_0,\epsilon_8)$ is given by

$$F_\eta(\epsilon_0,\epsilon_8)=[1+b\alpha(a,b)]K\,, \tag{30}$$

with the condition

$$\alpha(0,0)=1\,. \tag{31}$$

Clearly no such b dependence is needed for F_π and F_K. However, in the $SW(3)$ limit when ϵ_0, $\epsilon_8\to 0$, the third term on the right-hand side of Eq. (27) is the only surviving term, so that the result (26) must be strictly valid in this limit. Realizing $SW(3)$ in the region $-1\leq a\leq 1$ where b must go to zero, one observes that the function K on the right-hand side of Eq. (30) must be the same as in Eq. (28b). Furthermore, if we realize $SW(3)$ in the region $1\leq a\leq 2$, it is evident that since $b\neq 0$ in this case, we must choose $\alpha(a,b)$ to vanish here. This suggests

$$\alpha(a,b)=1-a-b+3ab\,, \tag{32}$$

which also satisfies the constraint (31). Finally, therefore, one obtains

$$\begin{aligned} F_\pi(\epsilon_0,\epsilon_8)&=F_K(\epsilon_0,\epsilon_8)\simeq KM\,,\\ F_\eta(\epsilon_0,\epsilon_8)&\simeq[1+b(1-a-b+3ab)]KM\,. \end{aligned} \tag{33}$$

From Eqs. (23)–(25) and (33), we then obtain the results

$$\frac{m_\pi^2}{m_K^2}=\frac{1+a}{1-\frac{1}{2}a}\,, \tag{34a}$$

$$\frac{m_\eta^2}{m_K^2}=\frac{(1-a-b+3ab)[1+b(1-a-b+3ab)]}{1-\frac{1}{2}a}\,. \tag{34b}$$

The relations for the decay constants f_π, f_K, and f_η can also be obtained. For the relation between f_π and f_K, the argument is the same as in I. From Eqs. (26) and (28), we obtain

$$\frac{f_\pi^2}{f_K^2}=\frac{1+b}{1-\frac{1}{2}b}\,. \tag{35}$$

The η case is not so straightforward, because of the complications arising from η-X mixing. To start with, we do not know the PCAC condition for η, since it would in general involve also the X meson. However, if we neglect the η-X mixing completely, and use the PCAC hypothesis $\partial_\mu A_\mu{}^{(8)}=(1/\sqrt{2})f_\eta m_\eta{}^2\phi_\eta$, we obtain from the variational principle the following results in the limit $m_\eta{}^2\to 0$:

$$\frac{\partial m_\eta{}^2}{\partial\epsilon_0}=-\frac{4}{3}\frac{\xi_0}{f_\eta{}^2}(1-b),\qquad \frac{\partial m_\eta{}^2}{\partial\epsilon_8}=\tfrac{2}{3}\sqrt{2}\frac{\xi_0}{f_\eta{}^2}(1-3b)\quad(m_\eta{}^2\to 0). \tag{36}$$

Now, Eq. (36) in general contradicts the derivatives obtained from Eq. (25), unless we neglect the dependence of b on ϵ_0 and ϵ_8. This neglect is presumably then connected with the neglect of η-X mixing. Now, if this neglect is reasonable, we obtain

$$F_\eta(\epsilon_0,\epsilon_8)=-\tfrac{2}{3}\sqrt{2}(\xi_0/f_\eta{}^2)\quad(m_\eta{}^2\to 0). \tag{37}$$

Using Eq. (26), we now obtain from Eqs. (37) and (28) the result

$$f_\eta{}^2/f_K{}^2=1/(1-\tfrac{1}{2}b). \tag{38}$$

If the η-X mixing is not too large, one might expect Eq. (38) to be approximately correct.

Using the known masses of π, K, and η, we obtain from Eqs. (34), (35), and (38), the following numerical solution:

$$a=-0.89,\quad b=-0.10,\qquad f_K/f_\pi=1.08,\quad f_\eta/f_\pi=1.06,\quad \gamma=5.05m_\pi{}^2f_\pi{}^2. \tag{39}$$

Notice that these values are very close to the results obtained in Papers I and II, as well as in Ref. 9, based upon the asymptotic $SU(6)_W$ theory. The numerical value of γ is obtained by using Eq. (28) with Eq. (23) or (24), if we recall the definition (8). Alternatively, if one desires, one can obtain two sum rules among $m_\pi{}^2$, $m_K{}^2$, $m_\eta{}^2$, f_K/f_π, and f_η/f_π by eliminating a and b from Eqs. (34), (35), and (38).

A few remarks are in order regarding the structure of the mass formulas. Calling $\sqrt{2}\epsilon_0 KM=m_0{}^2>0$, we may rewrite Eqs. (23)–(25) upon using Eq. (33) in the form

$$m_\pi{}^2=(1+a)m_0{}^2,\quad m_K{}^2=(1-\tfrac{1}{2}a)m_0{}^2,\qquad m_\eta{}^2=(1-a-b+3ab)[1+b(1-a-b+3ab)]m_0{}^2. \tag{40}$$

Note that all the squared masses are non-negative in the entire physical region $-1\leq a\leq 2$. The situation is quite different for the mass of the eighth component of the octet given by

$$m_8{}^2=(1-a)m_0{}^2, \tag{41}$$

which satisfies $3m_8{}^2+m_\pi{}^2=4m_K{}^2$. The formula (41) becomes meaningless for $a>1$, since $m_8{}^2$ becomes negative. This suggests that the concept of η_8 is meaningful only for $|a|\leq 1$, where the $SU(3)$ perturbation with respect to a can be made. The formula for the physical η in Eq. (40), by contrast, does not suffer from this defect. There is, however, an extra complication which we would like to mention here. Consider for instance that point $a=2$ where chimeral $SU(3)$ symmetry is exact. At this point K is massless, but π and η are not. Thus decays of the type π, $\eta\to K\bar{K}K\bar{K}$ are possible near $a=2$. Therefore, the mass formulas (40) cannot be strictly correct in the whole physical domain of a, since one would at least expect π and η masses to develop an imaginary part near $a=2$. Note that this situation cannot arise for the kaon mass, because of strangeness conservation. It may, of course, be that these imaginary parts are numerically small. However, as long as one's attention is confined to the region near the physical point $a\simeq-0.89$, no such complication arises, and one would expect the formulas (40) to be reasonably accurate.

As shown in Sec. II, b is presumably quite small compared with unity for the whole physical range of a. We have also noted before in I that the radius of convergence of the $SU(3)$ perturbations is quite large, i.e., $|a|=1$, and that the $SU(3)$ symmetry seems to work quite well even at $a\simeq-0.89$. We now make the following proposal as a possible explanation of this surprising fact: We suggest that the effective parameter for $SU(3)$ perturbations is not a but b, unless some kinematical constraints dictate otherwise, as in the mass formulas. Thus the explicit dependence on a is probably minimal and enters only through kinematical requirements. Since b is small at the physical point, it would be understandable why the $SU(3)$ perturbation theory would work well. Indeed, the relation for $f_\pi{}^2/f_K{}^2$ in Eq. (35) is not directly dependent upon a, but on b and is well represented by an $SU(3)$ perturbation theory, according to which

$$f_\pi{}^2=f^2(1+b),\quad f_K{}^2=f^2(1-\tfrac{1}{2}b), \tag{42}$$

where f is the value of f_π and f_K in the exact $SU(3)$ limit.

In closing this section, we would like to say a few words about the η-K mixing problem. As is evident, our proposal that $SU(3)$ perturbations are really expansions in b would lead to a small mixing between η and X. Using Eq. (42), the first-order $SU(3)$ results indicate

$$f_8{}^2=f^2(1-b). \tag{43}$$

If we now use the pole approximation, and saturate I_{88} in Eq. (9) by the $SU(3)$ octet and singlet states, we obtain

$$I_{88}=\tfrac{1}{2}m_8{}^2f_8{}^2+\tfrac{1}{2}m_1{}^2f_1{}^2=\gamma(1-a-b+3ab), \tag{44}$$

where $f_{1(8)}$ is defined by

$$\langle 0|A_\mu{}^{(8)}(0)|\eta_{1(8)}{}^{(k)}\rangle=(2k_0V)^{-1/2}if_{1(8)}k_\mu(1/\sqrt{2}). \tag{45}$$

Using Eqs. (41), (43), and (44), we obtain

$$m_1^2 f_1^2 = m_K^2 f_K^2 \frac{2ab}{(1-\frac{1}{2}a)(1-\frac{1}{2}b)} = 4\gamma ab. \quad (46)$$

Since we expect $m_1 \approx m_X$, we obtain from our numerical solution (39) the result

$$f_1^2/f_K^2 \approx 0.03, \quad (47)$$

which implies that f_1 is indeed very small, consistent with a small η-X mixing theory. Note that the smallness of f_1 arises mainly because f_1^2 is proportional to b in Eq. (46). Our result here is in sharp contrast to the $W(3)$ theory, investigated in Paper II, which requires a rather large and complicated η-X mixing.

IV. FURTHER GENERALIZATION

So far all of our analysis is based upon the structure of the Hamiltonian density given in Eq. (1). It would be interesting to modify Eq. (1) by adding a term which behaves as $(1,8)\oplus(8,1)$ representation of $SW(3)$ or $W(3)$ group[10]:

$$H(x) = H_0(x) + \epsilon_0 S^{(0)}(x) + \epsilon_8 S^{(8)}(x) + gM^{(8)}(x), \quad (48)$$

where $H_0(x)$ is invariant under $SW(3)$ or $W(3)$ group, and where $M^{(\alpha)}(x)$ and $N^{(\alpha)}(x)$ ($\alpha = 1, \ldots, 8$) are members of the $(1,8)\oplus(8,1)$ representation, satisfying the commutation relations

$$\begin{aligned} [F^{(\alpha)}(t), M^{(\beta)}(x)]_{x_0=t} &= if_{\alpha\beta\gamma} M^{(\gamma)}(x), \\ [F^{(\alpha)}(t), N^{(\beta)}(x)]_{x_0=t} &= if_{\alpha\beta\gamma} N^{(\gamma)}(x), \\ [F_5^{(\alpha)}(t), M^{(\beta)}(x)]_{x_0=t} &= if_{\alpha\beta\gamma} N^{(\gamma)}(x), \\ [F_5^{(\alpha)}(t), N^{(\beta)}(x)]_{x_0=t} &= if_{\alpha\beta\gamma} M^{(\gamma)}(x). \end{aligned} \quad (49)$$

Then, the partial conservation law Eq. (3) is now replaced by

$$\begin{aligned} \partial_\mu V_\mu^{(\alpha)}(x) &= \epsilon_8 f_{\alpha 8\beta} S^{(\beta)}(x) + g f_{\alpha 8\beta} M^{(\beta)}(x), \\ \partial_\mu A_\mu^{(\alpha)}(x) &= (\epsilon_0 d_{\alpha 0\beta} + \epsilon_8 d_{\alpha 8\beta}) P^{(\beta)}(x) \\ &\quad + g f_{\alpha 8\beta} N^{(\beta)}(x), \end{aligned} \quad (50)$$

which will lead to the sum rules

$$\begin{aligned} I_{33} &= \gamma(1+a)(1+b), \\ I_{44} &= \gamma(1-\tfrac{1}{2}a)(1-\tfrac{1}{2}b) - g\tau, \\ I_{88} &= \gamma(1-a-b+3ab), \\ I_{00} &= \gamma(1+2ab), \\ I_{08} &= \sqrt{2}\gamma(a+b-ab), \\ K_{44} &= (9/4)\gamma ab - g\tau, \end{aligned} \quad (51)$$

where τ is defined by

$$\tau = \tfrac{3}{4}\langle 0|M^{(8)}(0)|0\rangle. \quad (52)$$

[10] Some results of such a model in connection with weak interactions have been given by V. S. Mathur and J. Subba Rao, Phys. Rev. Letters **31B**, 383 (1970).

If we are considering the case where the fundamental symmetry is $SW(3)$ rather than the $W(3)$ group, we have only to discard the equations for I_{00} and I_{08}.

First of all, let us consider the $W(3)$ case, and define

$$\begin{aligned} A_\mu^{(-1)}(x) &= (1/\sqrt{3})[A_\mu^{(8)}(x) + \sqrt{2}A_\mu^{(0)}(x)], \\ A_\mu^{(-2)}(x) &= (1/\sqrt{3})[A_\mu^{(0)}(x) - \sqrt{2}A_\mu^{(8)}(x)], \end{aligned} \quad (53)$$

as in Paper I, instead of $A_\mu^{(8)}$ and $A_\mu^{(0)}$. Then, we have

$$\begin{aligned} I_{-1,-1} &= \gamma(1+a)(1+b) = I_{33}, \\ I_{-2,-2} &= \gamma(1-2a)(1-2b), \\ I_{-1,-2} &= 0. \end{aligned} \quad (54)$$

Hence for the $W(3)$ group, the point $a=-1$ corresponds to the exact validity of the $W(2)$ group, while $a=\frac{1}{2}$ leads to the group Z associated with the conservation law $\partial_\mu A_\mu^{(-2)}(x)=0$. Thus, the discussion of these groups at $a=-1$ and $a=\frac{1}{2}$ is essentially unchanged, and the introduction of the new term proportional to $(1,8)\oplus(8,1)$ does not help possible difficulties of the $W(3)$ group mentioned in the previous section. Note that we have to modify our ansatz slightly and require that the uniform $W(3)$ or $SW(3)$ limit should result for $\epsilon_0 \to 0$, $\epsilon_8 \to 0$, and $g \to 0$. The positivity conditions are now replaced by

$$\begin{aligned} \gamma(1+a)(1+b) &\geqslant 0, \\ \gamma(1-2a)(1-2b) &\geqslant 0, \\ \gamma(1-\tfrac{1}{2}a)(1-\tfrac{1}{2}b) &\geqslant g\tau, \\ (9/4)\gamma ab &\geqslant g\tau. \end{aligned} \quad (55)$$

Since we have an extra parameter g, the analog of Figs. 1 and 2 is now replaced by a complicated three-dimensional diagram in the a, b, and τ plane. However, for $g\tau \geq 0$ the positivity conditions Eq. (55) reproduce the previous ones without the new term. But it is now easy to see that points $a=b=2$ and $a=b=0$ do not satisfy the inequality Eq. (55) for $g\tau > 0$. Hence the domain (II) is completely disjoint from the domain (I) while the domain (IV) has no common points of contact with the domain (V). However, on the contrary, for $g\tau < 0$ these points $a=b=2$ and $a=b=0$ are now interior points rather than at the boundaries, so that domains (I) and (II) as well as domains (IV) and (V) form single domains (I-II) and (IV-V), respectively. For the case of the $SW(3)$ theory, the point $a=b=\frac{1}{2}$ is an interior point from the beginning, and, hence, four domains (I), (II), (III), and (IV) become a single connected domain for $g\tau < 0$ and we have only four disjoint regions, (V), (VI), (VII) together with the new single domain (I-II-III-IV).

Actually, it is more convenient to introduce a new variable $\rho \equiv g\tau/\gamma$ rather than $g\tau$, and consider a three-dimensional space with respect to three variables a, b, and ρ. Then, allowed domains specified by inequalities Eq. (55) are now three-dimensional manifolds in this space. We may refer to the cross section $\rho=0$ as being "at sea level," so that the domains $\rho>0$ and $\rho<0$ may be termed above and below sea level, respectively. Then, the previous considerations can be translated as follows. For the sake of definiteness consider the case of the $W(3)$ theory. For the $SW(3)$ case, we have only to replace the two separate domains (II) and (III) by a single domain (II-III). Then, all allowed domains will now be represented by the five disjoint manifolds D_{12}, D_{34}, D_5, D_6, and D_7 in the three-dimensional space. Each of these five manifolds is connected inside itself, while D_{12} and D_{34} share a line boundary $a=b=\frac{1}{2}$, and similarly D_{34} and D_5 have the common boundary line $a=b=-1$. Otherwise, all five manifolds are mutually disjoint. For the case of the $SW(3)$, we replace the two manifolds D_{12} and D_{34} by a new single connected manifold D_{1234}. At sea level $\rho=0$, the cross sections of D_5, D_6, and D_7 are exactly the two-dimensional domains (V), (VI), and (VII), respectively, of Fig. 1, while D_{12} and D_{34} are the two island domains (I) and (II) and (III) and (IV). Below sea level, $\rho<0$, and the two domains (I) and (II), as well as (III) and (IV), become fused together, making the undersea supercontinents D_{12} and D_{34}, respectively. Above sea level, $\rho>0$, D_{12} (D_{34}) will be split into two disjoint infinitely high mountain peaks corresponding to a continuation of domains (I) and (II) (III) and (IV). On the other hand, the other manifolds D_5, D_6, and D_7 do not produce such mountain-like structures at all. We may also remark that the sign of γ remains always the same inside each manifold, so that we have $\gamma\geq 0$ for D_{12}, D_{34}, and D_5, while $\gamma\leq 0$ for D_6 and D_7. Also, the manifolds D_{12}, D_{34}, and D_5 resemble ordinary continents in the sense that their shapes become narrower for the increasing value of ρ. On the contrary, D_6 and D_7 have shapes similar to those of inverted pyramids in the sense that their cross sections for a given value of ρ will be larger for increasing values of ρ. As we remarked already, in the case of the $SW(3)$ theory, we have only four manifolds D_{1234}, D_5, D_6, and D_7 to deal with.

Although one can obtain mass formulas for m_π^2, m_K^2, and m_η^2 in the present case by arguments essentially similar to those used before, these are much more complicated and we shall not attempt to construct them here.

V. CORRECTION TO SOFT-PION THEOREMS

In this section, we revert to the discussion of the original model (1), i.e., we set $g=0$ in the previous section. Also, all results that follow are valid for both $SW(3)$ and $W(3)$ cases.

First let us consider the K_{l3} form factors by setting

$$\langle\pi^0(p')\,|\,V_\mu{}^{(4-i5)}(0)\,|\,K^+(p)\rangle=-(1/\sqrt{2})(4p_0p_0'V^2)^{-1/2}\times[f_+(q^2)(p+p')_\mu+f_-(q^2)(p-p')_\mu],\quad(56)$$

with $q^2=(p-p')^2$. If we consider the soft-pion ($a=-1$) and the soft-kaon ($a=2$) limits, we obtain the well-known soft-boson theorems[11]

$$\begin{aligned}f_+(-m_K^2)+f_-(-m_K^2)&=f_K/f_\pi\quad(a=-1),\\ f_+(-m_\pi^2)-f_-(-m_\pi^2)&=f_\pi/f_K\quad(a=2).\end{aligned}\quad(57)$$

Now, let us set $\Delta=-(m_K^2+m_\pi^2)$ and consider an expression given by

$$I=m_K^2[f_+(\Delta)+f_-(\Delta)-f_K/f_\pi]-m_\pi^2[f_+(\Delta)-f_-(\Delta)-f_\pi/f_K].\quad(58)$$

Regarded as a function of the variable a, I satisfies the condition $I=0$ at the three points $a=2$, 0, and -1. This assertion at $a=2$ and -1 comes from the soft-meson theorems Eq. (57), since $m_\pi^2=0$ at $a=-1$ and $m_K^2=0$ at $a=2$. For $a=0$, we have the validity of the exact $SU(3)$, so that we have $m_K^2=m_\pi^2$, $f_K=f_\pi$ together with $f_+(0)=1$, and $f_-(q^2)\equiv 0$. Hence the quantity I vanishes at the three points $a=0$, -1, and 2. On the basis of the smoothness assumption, one might expect I to be generally small in the whole physical domain, including, in particular, the physical point $a\simeq-0.89$. Therefore, we may approximately set $I=0$ to obtain

$$(m_K^2-m_\pi^2)f_+(\Delta)+(m_K^2+m_\pi^2)f_-(\Delta)-(m_K^2(f_K/f_\pi)-m_\pi^2(f_\pi/f_K))\simeq 0.\quad(59)$$

This may be regarded as an equation incorporating a finite-mass correction into the soft-pion theory. Of course, our method has an ambiguity since we could have obtained the same conclusion if we had used m_π^4 and m_K^4 instead of m_π^2 and m_K^2 in Eq. (59). However, for the sake of simplicity, and minimality of a dependence, we adopted Eq. (59) as a more reasonable choice.

Similarly, let us set

$$J=f_+(\delta)+f_-(\delta)-f_K/f_\pi,\quad(60)$$

where δ is given by $\delta=m_\pi^2-m_K^2$. Then a similar argument gives us $J=0$ at $a=0$ and -1. Since the physical point $a\simeq-0.89$ lies in the range $-1<a<0$, where by the smoothness assumption we expect J to be small, we may set $J\simeq 0$, i.e.,

$$f_+(m_\pi^2-m_K^2)+f_-(m_\pi^2-m_K^2)\simeq f_K/f_\pi.\quad(61)$$

This relation was first obtained by Dashen and Weinstein,[5] who used an $SW(3)$-perturbative argument. Note that, on the contrary, we need not assume the perturbative argument with respect to the $SW(3)$ in our derivation. Our crucial assumption lies in regarding I or J to be as smooth a function of a as possible consistent with the various conditions that I or J satisfy.

We could apply this technique to a variety of problems. As another application, we shall calculate the correction to the Goldberger-Treiman relation. For this

[11] C. G. Callan and S. B. Treiman, Phys. Rev. Letters **16**, 153 (1966); V. S. Mathur, S. Okubo, and L. K. Pandit, *ibid.* **16**, 371 (1966); M. Suzuki, *ibid.* **16**, 212 (1966).

purpose, let us set

$$\langle p(p')|A_\mu{}^{(1+i2)}(0)|n(p)\rangle=\left(\frac{m_N{}^2}{p_0p_0'V^2}\right)^{1/2}\bar{u}(p')[i\gamma_\mu\gamma_5G_A{}^{(N)}(q^2)+\gamma_5q_\mu G_p{}^{(N)}(q^2)]u(p)\,,$$

$$\langle p(p')|A_\mu{}^{(4+i5)}(0)|\Lambda(p)\rangle=\left(\frac{m_Nm_\Lambda}{p_0p_0'V^2}\right)^{1/2}\bar{u}(p')[i\gamma_\mu\gamma_5G_A{}^{(\Lambda)}(q^2)+\gamma_5q_\mu G_p{}^{(\Lambda)}(q^2)+\gamma_5(p+p')_\mu H^{(\Lambda)}(q^2)]u(p)\,,$$

$$\langle n(p')|A_\mu{}^{(4+i5)}(0)|\Sigma^-(p)\rangle=\left(\frac{m_Nm_\Sigma}{p_0p_0'V^2}\right)^{1/2}\bar{u}(p')[i\gamma_\mu\gamma_5G_A{}^{(\Sigma)}(q^2)+\gamma_5q_\mu G_p{}^{(\Sigma)}(q^2)+\gamma_5(p+p')_\mu H^{(\Sigma)}(q^2)]u(p)\,, \tag{62}$$

$$\langle \Lambda(p')|A_\mu{}^{(1+i2)}(0)|\Sigma^-(p)\rangle=\left(\frac{m_\Lambda m_\Sigma}{p_0p_0'V^2}\right)^{1/2}\bar{u}(p')[i\gamma_\mu\gamma_5G_A{}^{(\Sigma-\Lambda)}(q^2)+\gamma_5q_\mu G_p{}^{(\Sigma-\Lambda)}(q^2)+\gamma_5(p+p')_\mu H^{(\Sigma-\Lambda)}(q^2)]u(p)\,.$$

Now, the ordinary Goldberger-Treiman relation would be exact at the soft-pion limit $a=-1$. This gives us

$$\begin{aligned}&2m_NG_A{}^N(0)=-\sqrt{2}f_\pi G_{NN\pi}\,,\\&(m_\Lambda+m_\Sigma)G_A{}^{(\Sigma-\Lambda)}(0)-(m_\Sigma{}^2-m_\Lambda{}^2)H^{(\Sigma-\Lambda)}(0)\\&\qquad=-f_\pi G_{\Sigma\Lambda\pi}\quad(a=-1)\,.\end{aligned} \tag{63}$$

Similarly, in the soft-kaon limit ($a=2$), we would obtain

$$\begin{aligned}&(m_\Lambda+m_N)G_A{}^{(\Lambda)}(0)-(m_\Lambda{}^2-m_N{}^2)H^{(\Lambda)}(0)=-f_KG_{\Lambda NK}\,,\\&(m_\Sigma+m_N)G_A{}^{(\Sigma)}(0)-(m_\Sigma{}^2-m_N{}^2)H^{(\Sigma)}(0)\\&\qquad=-\sqrt{2}f_KG_{\Sigma NK}\quad(a=2)\,.\end{aligned} \tag{64}$$

Hence, if we set

$$\begin{aligned}I_1\equiv{}&m_K{}^2[f_\pi G_{NN\pi}+\sqrt{2}m_NG_A{}^{(N)}(0)]\\&-\tfrac{1}{2}m_\pi{}^2[f_KG_{\Sigma NK}+(1/\sqrt{2})(m_\Sigma+m_N)G_A{}^{(\Sigma)}(0)\\&-(1/\sqrt{2})(m_\Sigma{}^2-m_N{}^2)H^{(\Sigma)}(0)]+\tfrac{1}{4}\sqrt{3}m_\pi{}^2[f_KG_{\Lambda NK}\\&+(m_\Lambda+m_N)G_A{}^{(\Lambda)}(0)\\&\qquad-(m_\Lambda{}^2-m_N{}^2)H^{(\Lambda)}(0)]\,,\\I_2\equiv{}&m_K{}^2[f_\pi G_{\Lambda\Sigma\pi}+(m_\Lambda+m_\Sigma)G_A{}^{(\Sigma-\Lambda)}(0)\\&-(m_\Sigma{}^2-m_\Lambda{}^2)H^{(\Sigma-\Lambda)}(0)]-\tfrac{1}{2}\sqrt{3}m_\pi{}^2[f_KG_{\Sigma NK}\\&+(1/\sqrt{2})(m_\Sigma+m_N)G_A{}^{(\Sigma)}(0)\\&-(1/\sqrt{2})(m_\Sigma{}^2-m_N{}^2)H^{(\Sigma)}(0)]+\tfrac{1}{2}m_\pi{}^2[f_KG_{\Lambda NK}\\&+(m_\Lambda+m_N)G_A{}^{(\Lambda)}(0)-(m_\Lambda{}^2-m_N{}^2)H^{(\Lambda)}(0)]\,,\end{aligned} \tag{65}$$

then we have $I_1=I_2=0$ for $a=2$, 0, and -1. Again, the identity at $a=2$ and -1 is due to the validity of the Goldberger-Treiman relations Eqs. (63) and (64), while $I_1=I_2=0$ at $a=0$ follows from the validity of the exact $SU(3)$ invariance at this point. Invoking the smoothness assumption again, we would expect that in the whole physical domain $I_1\simeq0$, $I_2\simeq0$. These relations in principle provide a way to calculate the finite pion-mass correction of order $(m_\pi/m_K)^2$ to the Goldberger-Treiman relations. Many of the coupling constants that appear in Eq. (65) are not known, so that one has to await future experiments in order to test these sum rules. We would also like to point out that the relations in Eq. (65) are very similar to the ones derived by Dashen and Weinstein using the $SW(3)$ perturbative method.

We would like to point out that the sum rules of the type (59) or (65) may also be interpreted as providing a relation between the departures from experiments of the soft-pion and the soft-kaon results. Thus only if the original soft-pion sum rule is known to be rather accurately satisfied, would one expect the soft-kaon sum rule to be an approximate result in any reasonable sense. On the basis of these arguments, we would expect the soft-kaon Goldberger-Treiman relations (64) to be rather bad, since the soft-pion relations (63) themselves are satisfied with an accuracy of only about 10%.

In conclusion, we would like to remark that an interesting comparison between the $W(3)$ and $SW(3)$ theories seems to arise in the recent suggestions regarding the origin of the Cabibbo angle. Most of these approaches[12] suggest that $\tan^2\theta=0$ at $a=-1$ and $\tan^2\theta=1$ at $a=0$. However, in the formulation of Gatto *et al.*[12] $\tan^2\theta=\infty$ at $a=\frac{1}{2}$, whereas the numerically conjectured formula[13] $\tan^2\theta=m_\pi{}^2f_\pi{}^2/m_K{}^2f_K{}^2$ blows up at $a=2$, suggesting that the former approach may in some way be connected to the $W(3)$ theory, whereas the latter form may arise more naturally in an $SW(3)$ theory.

After this paper was written, we discovered a paper by Kuo,[14] who has shown that the chimeral and ordinary $SU(3)$ symmetries are unitarily related. In our formalism this would impose constraints if the $SW(3)$ symmetry is realized in the usual manner where vacuum is also an $SW(3)$ scalar. However, starting with the $SW(3)$-symmetry group, realized through Goldstone bosons, it is a matter of convention whether vacuum is symmetric under the usual $SU(3)$ or chimeral $SU(3)$ (since it cannot be symmetric under both), and these two descriptions are indeed equivalent. The arbitrariness arises because of the fact that there is no way to fix the relative parity of strangeness-carrying mesons with respect to the nonstrange mesons. If one adopts the usual description of $SW(3)$ symmetry where

[12] R. Gatto, G. Sartori, and M. Tonin, Phys. Letters **28B**, 128 (1968); N. Cabibbo and L. Maiani, *ibid.* **28B**, 131 (1968); R. J. Oakes, *ibid.* **29B**, 683 (1969).

[13] Some basis for this formula has recently been given by R. J. Oakes (Ref. 12).

[14] T. K. Kuo, Phys. Rev. (to be published).

the vacuum state is a scalar under the ordinary $SU(3)$, the chimeral $SU(3)$ subgroup of $SW(3)$ will be realized as a Goldstone symmetry, in contrast to the ordinary $SU(3)$ subgroup, and our theory presented here would go through unchanged. These points will be discussed in greater detail elsewhere.

Lastly, we may make the following remark: We showed that b as a function of a is discontinuous at $a=-1$ and $a=2$. From this, we concluded that we may have essential singularities at these points, provided that b is an analytic function of a except for a few isolated points in the complex plane of the variable a. However, there is another possibility that b may have branch cuts, instead of the essential singularities, passing through points $a=-1$ and 2, since these will also give the desired discontinuity. An interesting possibility is the conjecture that the Kuo transformation $a\to(2-a)(1+4a)^{-1}$, $\epsilon_0\to-\frac{1}{3}(1+4a)\epsilon_0$ may transform physical quantities on the first Riemann sheet in this cut plane into those on the unphysical second sheet.

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Inclusion of Toller-Angle Dependence in the Multi-Regge Integral Equation*

Dennis Silverman and Chung-I Tan
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
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The nonforward multiperipheral integral equation for the Reggeon-particle absorptive amplitude is generalized to include complete dependence on the Toller-angle variable.

I. INTRODUCTION

Much progress has been made in formulating the multiperipheral bootstrap equation using a multi-Regge production model.[1,2] In a recent publication,[3] Goldberger, Tan, and Wang have constructed a simplified integral equation for the Reggeon-particle absorptive amplitude $\mathcal{A}(p,p_0;Q)$ in a formulation of the multi-Regge model. Their construction seemed to depend on the assumption that the double Regge coupling is independent of the Toller angle ω, and an approximate justification for this assumption was suggested by Tan and Wang.[4] It is our purpose to show that an integral equation which includes the complete dependence on the Toller angle can be written for the absorptive amplitude $\mathcal{A}$. This establishes the full generality of the integral-equation approach through the $\mathcal{A}$ amplitude.

In the process of formulating this equation, we elaborate the relation between the ω angle and the other invariants. We then express the integration of the loop momentum in terms of a particular set of invariants which manifestly cover the entire phase space. These variables also allow us to explicitly continue the integral equation to the forward case $t=0$.

II. CHEW-GOLDBERGER-LOW EQUATION

Our starting point will be the Chew-Goldberger-Low (CGL) equation[1,5] for Regge multiperipheral dynamics where now arbitrary Toller angle dependence is assumed for the double Regge coupling $\beta(t',\omega',t'')$. The B amplitude introduced by CGL is related to the elastic two-body absorptive part $A(p,p_0;Q)$ for

$$(p+\tfrac{1}{2}Q)+(p_0-\tfrac{1}{2}Q)\to(p-\tfrac{1}{2}Q)+(p_0+\tfrac{1}{2}Q)$$

by

$$A(p,p_0;Q)=\int\frac{d^4p'}{(2\pi)^3}\delta^+((p-p')^2-\mu^2)\times g(t_+,t_+';t_-,t_-')B(p,p',p_0;Q).\quad(1)$$

We use the CGL equation for B with a double Regge coupling and propagator function $G(t_\pm',\omega_\pm',t_\pm'')=\beta^*(t_-',\omega_-',t_-'')\beta(t_+',\omega_+',t_+'')$, which is now assumed to depend on the Toller angles $\omega_\pm'$ [in contrast to Eq. (2) of Ref. 3]:

$$B(p,p',p_0;Q)=B_0(p,p',p_0;Q)+\int\frac{d^4p''}{(2\pi)^3}\delta^+((p-p')^2-\mu^2)G(t_\pm',\omega_\pm',t_\pm'')\times(\Sigma/\mu^2)^{\alpha(t_+')+\alpha(t_-')}B(p',p'',p_0;Q)\quad(2)$$

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[3] M. L. Goldberger, C.-I Tan, and J. M. Wang, Phys. Rev. **184**, 1920 (1969). We use a slightly different notation, $\mathcal{A}(p,p_0;Q)$, for the absorptive amplitude of the reaction Reggeon$(p+\frac{1}{2}Q)$+particle$(p_0-\frac{1}{2}Q)\to$ Reggeon$(p-\frac{1}{2}Q)$+particle$(p_0+\frac{1}{2}Q)$, while reserving $A(p,p_0;Q)$ for the physical on-shell absorptive amplitude.

[4] C.-I Tan and J. M. Wang, Phys. Rev. **185**, 1899 (1969).

[5] M. L. Goldberger, Erice Summer School, 1969 (unpublished), is a thorough and stimulating presentation of the integral-equation approach to multiperipheral dynamics.