$O(4)$ Expansion of Off-Shell Scattering Amplitudes and the Most General Form of Regge Trajectories and Residues for Arbitrary Spins. II*

M. J. KING
Syracuse University, Syracuse, New York 13210

AND

P. K. KUO
Wayne State University, Detroit, Michigan 48202

AND

P. SURANYI†
The Johns Hopkins University, Baltimore, Maryland 21218

(Received 9 February 1969)

We continue our study of the contribution of a Regge-pole family to the scattering amplitude. Total-spin states are constructed, which allow us to expand the off-shell scattering amplitude in a series of representation functions of the Lorentz group. The contribution of the family is transformed into a form which strongly reminds one of a Lorentz-pole contribution, even at nonzero momentum transfers t. The most general dependence of the Regge-pole residues on the total helicity λ and daughter index κ is given in terms of two arbitrary functions of composite variables $tj(j+1)$ and $\Delta\lambda$, where $\Delta=\{[t-(m_1-m_2)^2][t-(m_1+m_2)^2]\}^{1/2}$. The contribution of a family to the scattering amplitude constructed from these residue functions and the trajectories given in a previous paper is the most general one satisfying all kinematic constraint relations at $t=0$, pseudothreshold, and threshold.

I. INTRODUCTION

THIS is the second of a series of papers in which we set out to solve the general problem of constructing a Reggeized scattering amplitude satisfying all requirements of Lorentz invariance and analyticity. The first paper of the series[1] deals with the determination of the most general behavior of a family of Regge trajectories consistent with the above requirements. In this paper we discuss the most general form of the Regge residues. Our aim is to give a form for the residue that can be easily parametrized for phenomenological analysis.

Since unitarity requires that the residue of a pole be factorizable, we shall work with Regge vertex functions, i.e., the vertices which couple Regge trajectories with two-particle states. The problem to be solved here is the determination of the most general structure of the vertex function such that the contribution of a Regge family to the scattering amplitude is analytic at vanishing momentum transfer, $t=0$, and at the same time satisfies the constraint relations at thresholds and pseudothresholds. To achieve this, we study the dependence of the vertex function on the following variables: κ, the daughter number; λ, the total helicity of the two-particle state; and σ and M, the quantum numbers which characterize the family at $t=0$.[1] Although our method differs from that of Cosenza, Sciarrino, and Toller,[2] we essentially accomplish the task outlined in their series of papers.

The underlying group-theoretic part of the problem is described and solved in I for a less general scattering process in which both the initial and final state have a spinless particle. This arrangement avoids the difficulty of having to couple the spins of a two-particle system in a covariant way. A crucial step in our solution to the general problem is the use of "total-spin" amplitudes based on "total-spin" states. This concept of total spin is the covariant generalization of the "north-pole-frame total spin" introduced by Toller.[3] The total-spin states, defined and discussed in Sec. II, transform as if only one of the particles has spin. Hence the amplitudes based on these states can be treated in exactly the same way as in I. Having disposed of the problem of spin, we return to the line of approach of I in Sec. III to give the expansion of a Regge vertex function in a series of representation functions of the Lorentz group. In spite of the fact that the trajectories are in general nonparallel, we are able to sum up the contribution of the whole family into a form which resembles the contribution of a single Lorentz pole. This form is convenient for establishing analyticity at $t=0$ as well as the fulfillment of the constraint relations at $t=0$ (equal-mass case).

The Regge vertex in the form of an infinite sum of representations, however, is not convenient for practical use. Specializing to the unequal-mass case, we study in Sec. IV the behavior of the representation functions near the special values $t=0$, threshold, and pseudothreshold. With the help of the method of composite

* Supported in part by the National Science Foundation.
† On leave of absence from the Central Research Institute for Physics, Budapest, Hungary.
[1] P. K. Kuo and P. Suranyi, preceding paper, Phys. Rev. D 1, 3416 (1970); hereafter referred to as I.
[2] G. Cosenza, A. Sciarrino, and M. Toller, Nuovo Cimento 57, 253 (1968).
[3] M. Toller, Nuovo Cimento 53A, 671 (1968).

variables,[1,4] the infinite sum is reduced to a closed, compact form useful for parametrization. It is also shown in Sec. IV that the unequal-mass constraint relations at threshold and pseudothreshold are satisfied by the contribution of each daughter alone. The procedure of Sec. IV is repeated in Sec. V for the equal-mass case. Here, however, for reasons to be seen later, our results are in a less convenient form and are perhaps less general. The constraint relations at threshold are again satisfied by each daughter alone, but the same is not true at pseudothreshold (which coincides with $t=0$); the constraint relations are satisfied only by the family as a whole. All the important results of this and the previous paper are summarized in Sec. VI.

Two appendices are included. They carry the more technical parts of the discussions of the paper.

II. TOTAL-SPIN AMPLITUDE

In I we have shown that the wave function of a two-particle state has a simple expansion in terms of the d functions of the Lorentz group if one of the particles is spinless. When both particles have spin, the spins have to be combined to form a "total spin" before the simple expansion can be used. At special values of total energy, i.e., threshold and pseudothreshold, the total-spin states are well known.[5] Since at these values both particles are at rest in the c.m. frame, their spins combine just like angular momenta. At threshold,

$$|p_1,p_2,s,\lambda\rangle \sim \sum_{\lambda_1\lambda_2} (s_1\lambda_1,\, s_2-\lambda_2|s\lambda) \times |p_1 s_1\lambda_1\rangle \otimes |p_2 s_2\lambda_2\rangle\,, \quad (2.1)$$

and at pseudothreshold,

$$|p_1,p_2,s,\lambda\rangle \sim \sum_{\lambda_1\lambda_2} (-1)^{s_2-\lambda_2}(s_1\lambda_1,\, s_2-\lambda_2|s\lambda) \times |p_1 s_1\lambda_1\rangle \otimes |p_2 s_2\lambda_2\rangle\,. \quad (2.2)$$

The extra phase factor in the latter arises because the momenta of the two particles are not in the same light cone.

The problem of this section is to define a covariant total-spin state which (i) transforms like a one-particle state with spin s and helicity λ and (ii) reduces to Eqs. (2.1) and (2.2) at threshold and pseudothreshold, respectively. To this end, we generalize the transformation rule of a one-particle helicity state,[6]

$$U(L)|p,s,\lambda\rangle = U(L)U(\Lambda_p)|p_0,s,\lambda\rangle = \sum_\mu |\bar{p},s,\mu\rangle D_{\mu\lambda}{}^s(\Lambda_{\bar{p}}{}^{-1}L\Lambda_p)\,, \quad (2.3)$$

by introducing a new state,

$$|p,s,\lambda(k)\rangle = \sum_\mu |p,s,\mu\rangle D_{\mu\lambda}{}^s(\Lambda_p{}^{-1}\Lambda_k)\,, \quad (2.4)$$

where $D_{\mu\lambda}{}^s$ is the $(2s+1)$-dimensional representation function of the $SU(2)$ group generalized to the $SL(2,C)$ group.[7] It can be verified that this new state transforms according to

$$U(L)|p,s,\lambda(k)\rangle = \sum_\mu |\bar{p},s,\mu(\bar{k})\rangle D_{\mu\lambda}{}^s(\Lambda_{\bar{k}}{}^{-1}L\Lambda_k)\,, \quad (2.5)$$

where $\bar{p}=Lp$ and $\bar{k}=Lk$, i.e., the spin transforms as if the momentum of the particle were k.[8] Having freed the spin from the particle momentum, we can define a two-particle state for which both spins transforms via the same Wigner rotation:

$$|p_1,s_1,\lambda_1(k)\rangle \otimes |p_2,s_2,\lambda_2(k)\rangle\,. \quad (2.6)$$

So, combining these states together by means of Clebsch-Gordan coefficients, we obtain the total-spin states

$$|p_1,p_2,s,\lambda(k)\rangle = \sum_{\lambda_1\lambda_2} (s_1\lambda_1,s_2\lambda_2|s\lambda) \times |p_1 s_1\lambda_1(k)\rangle \otimes |p_2 s_2\lambda_2(k)\rangle\,. \quad (2.7)$$

It follows immediately that this state has the desired transformation property

$$U(L)|p_1p_2s\lambda(k)\rangle = \sum_\mu |\bar{p}_1,\bar{p}_2 s\mu(\bar{k})\rangle D_{\mu\lambda}{}^s(\Lambda_{\bar{k}}{}^{-1}L\Lambda_k)\,, \quad (2.8)$$

where $\bar{p}_i=Lp_i$. For convenience, we choose $k=p_1$ (assuming $m_1\geq m_2$) and denote our total-spin state simply by $|p_1p_2,s\lambda\rangle$:

$$|p_1p_2s\lambda\rangle = \sum_{\lambda_1\lambda_2} (s_1\lambda_1,\, s_2\lambda-\lambda_1|s\lambda)|p_1s_1\lambda_1\rangle \otimes |p_2s_2\lambda_2\rangle D_{\lambda_2\lambda-\lambda_1}{}^{s_2}(\Lambda_2{}^{-1}\Lambda_1)\,, \quad (2.9)$$

whose transformation rule is

$$U(L)|p_1p_2s\lambda\rangle = \sum_\mu |\bar{p}_1\bar{p}_2 s\mu\rangle D_{\mu\lambda}{}^s(\Lambda_{\bar{1}}{}^{-1}L\Lambda_1)\,. \quad (2.10)$$

In the c.m. frame, the D function in Eq. (2.9) becomes

$$D_{\lambda_2\lambda-\lambda_1}{}^{s_2}(\Lambda_2{}^{-1}\Lambda_1) = e^{-\lambda_2\xi_2+(\lambda-\lambda_1)\xi_1}e^{is_2\pi}\delta_{\lambda_2,-\lambda+\lambda_1}\,, \quad (2.11)$$

where

$$\cosh\xi_1 = \frac{t+m_1{}^2-m_2{}^2}{2t^{1/2}m_1}\,, \quad \cosh\xi_2 = \frac{t-m_1{}^2+m_2{}^2}{2t^{1/2}m_2}\,.$$

At threshold, $t=(m_1+m_2)^2$ and $\xi_1=\xi_2=0$, so that Eq. (2.9) reduces to

$$|p_1p_2s\lambda\rangle = (-1)^{s_2} \sum_{\lambda_1\lambda_2} (s_1\lambda_1,\, s_2-\lambda_2|s\lambda) \times |p_1s_1\lambda_1\rangle \otimes |p_2s_2\lambda_2\rangle\,. \quad (2.12)$$

At pseudothreshold, $t=(m_1-m_2)^2$, $\xi_1=0$, $\xi_2=i\pi$ (since

---

[4] P. K. Kuo and P. Suranyi, Phys. Rev. Letters **22**, 1025 (1969).
[5] M. J. King and P. K. Kuo, Phys. Rev. D **1**, 442 (1970).
[6] The notation is the same as in I.
[7] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968).
[8] The transformation rule for spinor states is obtained if we choose k to be the null vector.

$m_1 \geq m_2$); Eq. (2.9) reduces to

$$|p_1p_2s\lambda\rangle = \sum_{\lambda_1\lambda_2} (-1)^{s_2-\lambda_2}(s_1\lambda_1, s_2-\lambda_2|s\lambda) \times |p_1s_1\lambda_1\rangle \otimes |p_2s_2\lambda_2\rangle. \quad (2.13)$$

We have seen that the total-spin states defined by Eq. (2.9) satisfy all requirements. We are now in position to discuss the total-spin amplitude which is the S-matrix element between the total-spin states. It follows directly from Eq. (2.9) that

$$T_{s'\lambda's\lambda} = \sum_{\lambda_i} T_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s_1\lambda_1s_2\lambda-\lambda_1|s\lambda)D_{\lambda_2\lambda-\lambda_1}{}^{s_2}(\Lambda_2{}^{-1}\Lambda_1) \times (s_3\lambda_3s_4\lambda'-\lambda_3|s'\lambda')D_{\lambda_4\lambda'-\lambda_3}{}^{s_4*}(\Lambda_4{}^{-1}\Lambda_3). \quad (2.14)$$

The transform rule for T follows from Eq. (2.10):

$$T_{s'\lambda's\lambda}(p_1p_2p_3p_4) = \sum_{\mu\mu'} T_{s'\mu's\mu}(\bar{p}_1\bar{p}_2\bar{p}_3\bar{p}_4) \times D_{\mu\lambda}{}^{s}(\Lambda_{\bar{1}}{}^{-1}L\Lambda_1)D_{\mu'\lambda'}{}^{s'*}(\Lambda_{\bar{3}}{}^{-1}L\Lambda_3). \quad (2.15)$$

This rule is the same as for the helicity amplitude with $s_2=s_4=0$, $s_1=s$, and $s_3=s'$. It then follows that all the results of I apply equally to the total-spin amplitudes. From this point on, we shall restrict our discussions to the total-spin amplitudes. Once their structure is known, the structure of the helicity amplitudes can be obtained immediately from the inverse of Eq. (2.14):

$$T_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_{s\lambda s'\lambda'} T_{s'\lambda's\lambda}(s_1\lambda_1s_2\lambda-\lambda_1|s\lambda)D_{\lambda-\lambda_1\lambda_2}{}^{s_2}(\Lambda_1{}^{-1}\Lambda_2) \times (s_3\lambda_3s_4\lambda'-\lambda_3|s'\lambda')D_{\lambda'-\lambda_3\lambda_4}{}^{s_4*}(\Lambda_3{}^{-1}\Lambda_4). \quad (2.16)$$

The threshold and pseudothreshold constraint relations for the total-spin amplitudes have been discussed by King and Kuo[5]; they again agree with those for helicity amplitudes with $s_2=s_4=0$, $s_1=s$, and $s_3=s'$.

The problem of total spin has often been mistreated in the literature by building the two-particle state by combining one-particle helicity states simply through a Clebsch-Gordan coefficient $(s\lambda|s_1\lambda_1,s_2\lambda_2)$. The total spin defined in this way is certainly not covariant and the expansion coefficients of the representation functions of the Lorentz group *will be helicity dependent.* Such a helicity dependence was noted in a paper of Bitar,[9] where it was required that at threshold and pseudothreshold the amplitude has extra factors, thus giving rise to the right combinations given by Eqs. (2.1) and (2.2). This requirement is still not enough, however, to satisfy the constraints at the singular points; we have constraints on the expansion coefficients of the amplitude around these points as well. A correct treatment of the problem of total spin was outlined in a paper of Frazer *et al.*,[10] who, however, described in detail only the case of one spinless particle in both initial and final states. Nevertheless, the method they offer, namely, the application of spinor amplitudes, leads to correct results.

[9] K. M. Bitar, Phys. Rev. **180**, 1477 (1969).

[10] W. R. Frazer, F. R. Halpern, H. M. Lipinski, and D. R. Snider, Phys. Rev. **176**, 2047 (1968).

III. EXPANSION OF RESIDUE FUNCTION AND LORENTZ POLE FORM OF CONTRIBUTION OF FAMILY AT $t\neq 0$

In the first part of this section we shall give the Regge vertex functions (the product of which gives the residue function) as a sum over representation functions of the Lorentz group, $d_{s\lambda j}{}^{\sigma j_0}(\xi)$. At first the most general t and κ dependence of the expansion coefficients will be given. This result is a more or less trivial consequence of the introduction of the method of composite variables[1,4] and the treatment of total spin given in Sec. II. We do not distinguish between equal- and unequal-mass cases until Sec. IV and, therefore, do not specify the value of the boost angle ξ.

As one can see from Eqs. (3.17) and (3.18) of I [(I 3.17) and (I 3.18)], the wave function satisfying equation $K\psi=0$ agrees with the Regge vertex function. Of course, a normalization constant remains undetermined by the homogeneous equation, so we shall work at first with an arbitrary normalization and defer this question to Appendix A.

As a first step, we introduce the parity-conserving combinations for the partial-wave projections of the wave function defined by Eq. (I 3.23):

$$\psi_j{}^{\sigma M(\pm)}(q^2,t) = \psi_j{}^{\sigma M}(q^2,t) \pm \psi_j{}^{\sigma -M}(q^2,t). \quad (3.1)$$

It follows from the orthogonality of the functions $d_{s\lambda j}{}^{\sigma M(\pm)}$ that the wave function $\psi_j{}^{\sigma M(\pm)}(q^2,t)$ satisfies Eq. (I 3.24) with $K_j{}^{\sigma j_0\sigma' j_0'}(p^2,q^2,t)$ substituted by $K_j{}^{\sigma M\sigma' M'(\pm)}(p^2,q^2,t)$ and $\psi_j{}^{\sigma j_0}(q^2,t)$ by $\psi_j{}^{\sigma M(\pm)}(q^2,t)$.

Using standard techniques of perturbation theory, one is able to calculate the wave function in terms of those at $t=0$:

$$\psi_j{}^{\sigma M(\pm)}(q^2,t) = \delta_{\sigma\bar{\sigma}}\delta_{M\bar{M}}\psi_0{}^{\bar{\sigma}\bar{M}(\pm)}(q^2) + 4\pi^2 \int dq'^2\, q'^2 \frac{K_j{}^{\bar{\sigma}\bar{M}\sigma M(\pm)}(q^2,q'^2,t)}{K_j{}^{\bar{\sigma}\bar{M}\bar{\sigma}\bar{M}}(q^2,q^2,t) - K_j{}^{\sigma M\sigma M}(q'^2,q'^2,t)} \times \psi_0{}^{\sigma M(\pm)}(q'^2) + \cdots, \quad (3.2)$$

where $\psi_0{}^{\sigma M(\pm)}(q^2)$ is the solution of the eigenvalue equation (I 3.24) at $t=0$. The quantum number $\bar{M}$ is determined at $t=0$, while $\bar{\sigma}$ is the solution of Eq. (I 3.33), $\bar{\sigma}_\kappa{}^{(\pm)} = \alpha_\kappa{}^{(\pm)}(t)+\kappa$. We note that σ and M differ from $\bar{\sigma}$ and $\bar{M}$ by integers.

Using Eqs. (I 3.31) and (3.2), and repeating the considerations of Sec. III of I, we obtain the most general j (or κ) dependence of $\psi_j{}^{\sigma M(\pm)}(t)$ [from now on we omit the q^2 dependence of $\psi_j{}^{\sigma M(\pm)}(q^2,t)$ which is inessential from the point of view of our further considerations]:

$$\psi_j^{\sigma M(\pm)}(t)=t^{n_0/2}\left[\frac{\Gamma(\sigma_>+j+2)}{\Gamma(\sigma_<+j+2)}\frac{\Gamma(\sigma_>-j+1)}{\Gamma(\sigma_<-j+1)}\times\frac{\Gamma(M_>+j+1)}{\Gamma(M_<+j+1)}\frac{\Gamma(j-M_<+1)}{\Gamma(j-M_>+1)}\right]^{1/2}\times[a(t,t_1)\pm t_{M<}b(t,t_1)],\quad(3.3)$$

where $n_0=|M-\bar{M}|+|\sigma-\bar{\sigma}|$ and $\sigma_>$ and $M_>$ are the larger of σ, $\bar{\sigma}$ and M, $\bar{M}$, respectively.

The residue of the daughter trajectories is given essentially by the partial-wave projection of Eq. (I 3.23),

$$\psi_j^{\sigma(\pm)}(t)=\sum_M d_{s\lambda j}{}^{\sigma M(\pm)}(\xi)\psi_j^{\sigma M(\pm)}(t).\quad(3.4)$$

This expression, together with Eq. (3.3), will serve as a basis of our further considerations in the following sections, where we shall give the most general form for Regge vertices, which will be useful for parametrization.

Let us turn now to the question of normalization. The partial-wave scattering amplitude near one of the daughter poles has the form

$$T\sim N_j^2(t)\frac{\psi_{(1)j}{}^{(\pm)}(q_0,|\mathbf{q}|,t)\psi_{(2)j}{}^{(\pm)*}(q_0',|\mathbf{q}'|,t)}{j-\alpha_\kappa(t)},\quad(3.5)$$

where $\psi_{(1)j}{}^{(\pm)}$ and $\psi_{(2)j}{}^{(\pm)}$ are the wave functions for initial and final states, respectively, normalized to unity:

$$\|\psi\|^2=\int|\psi_{(i)j}{}^{(\pm)}(q_0,|\mathbf{q}|,t)|^2dq_0d|\mathbf{q}|\,|\mathbf{q}|^2=1.$$

In Appendix A we show that $N_j^2(t)|_{j=\alpha_\kappa^{(\pm)}(t)}$ can be given as

$$N_j^2|_{j=\alpha_\kappa^{(\pm)}(t)}=G(t,t_1,\pm t_{\bar{M}})\times\left[1-\frac{\partial}{\partial j}F(t,t_1,\pm t_{\bar{M}})\right]^{-1}\Bigg|_{j=\alpha_\kappa^{(\pm)}(t)},\quad(3.6)$$

where $G(t,t_1,\pm t_{\bar{M}})$ is a regular function of its variables near $t=0$, while $F(t,t_1,\pm t_{\bar{M}})=f(t,t_1)\pm t_{\bar{M}}g(t,t_1)$ was defined in Eq. (I 3.33).

Using Eqs. (3.4)–(3.6), we finally obtain, for the normalized Regge vertex function,

$$r_{\kappa,s\lambda}{}^{(\pm)}(t)=\sum_{\sigma,M}d_{s\lambda j}{}^{\sigma M(\pm)}(\xi)\psi_j^{\sigma M(\pm)}(t)\times\left[1-\frac{\partial}{\partial j}F(t,t_1,\pm t_{\bar{M}})\right]^{-1/2}\Bigg|_{j=\alpha^{(\pm)}(t)},\quad(3.7)$$

where we have absorbed function $G(t,t_1,\pm t_{\bar{M}})$ into $\psi_j^{\sigma M(\pm)}(t)$.

The summation over M extends from 0 to s, while in σ it extends from j to infinity.

The coefficients of the expansion (3.7) were studied in several papers. The κ dependence of the first few derivatives of $\psi_j^{\sigma M(\pm)}(t)$ in t at $t=0$ was determined in papers of Domokos and Suranyi[11] and Kuo and Walker.[12] Bronzan[13] was able to give the κ dependence of arbitrary order of derivatives of the functions $r_{\kappa,s\lambda}{}^{(\pm)}(t)$. In a recent paper, Durand, Fishbane, Klein, and Simmons[14] have given a form for the coefficients $\psi_j^{\sigma M(\pm)}(t)$, but their approach is entirely different. So a direct comparison with our results at this stage is very hard.

We regard Eq. (3.7) only as an intermediate step towards the final formulas we shall obtain in Secs. IV and V. The infinite sum in Eq. (3.7) is certainly not useful for practical applications (parametrization, phenomenology). Even from a fundamental point of view it is not satisfying: It does not give the right number of independent constants in a given order of t. We shall show in the following sections, after a study of the structure of functions $d_{s\lambda j}{}^{\sigma M(\pm)}(\xi)$, that the residue is completely determined in terms of one or two arbitrary functions of the composite variables instead of infinitely many.

In the following part of this section, we shall give a useful form of the contribution of a Regge-pole family at arbitrary values of t. This form will be useful for the proof of the following statements.

(i) The contribution of the family to the scattering amplitude is a regular function of t at $t=0$.

(ii) The constraint relations are satisfied at $t=0$ (equal-mass case).

The contribution of a daughter pole with daughter index κ and parity index τ to the scattering amplitude has the following form [see Eq. (3.7)]:

$$T_{\kappa,s\lambda s'\lambda'}{}^{(\tau)}(t)=\sum_{\sigma,M,\sigma',M'}d_{s\lambda j}{}^{\sigma M(\tau)}(\xi)\psi_j^{\sigma M(\tau)}(t)d_{\lambda\lambda'}{}^j(\theta)\times d_{s'\lambda' j}{}^{\sigma' M'(\tau)}(-\xi')\psi_j^{\sigma' M'(\tau)}(t)\times\left[1-\frac{\partial}{\partial j}F(t,t_1,\pm t_{\bar{M}})\right]^{-1}\Bigg|_{j=\alpha_\kappa^{(\pm)}(t)}.\quad(3.8)$$

Equation (3.8) can be recast as

$$T_{\kappa,s\lambda s'\lambda'}{}^{(\tau)}(t)=\frac{1}{2\pi i}\oint dj\sum_{\sigma,M}\sum_{\sigma',M'}d_{s\lambda j}{}^{\sigma M(\tau)}(\xi)\times\psi_j^{\sigma M(\tau)}d_{\lambda\lambda'}{}^{j'}(\theta)d_{s'\lambda' j}{}^{\sigma' M'(\tau)}(-\xi')\times\psi_j^{\sigma' M'(\tau)}(t)[j+\kappa-F(t,t_1,\pm t_{\bar{M}})]^{-1},$$

[11] G. Domokos and P. Suranyi, Nuovo Cimento **56A**, 445 (1968); **57A**, 813 (1968).
[12] P. K. Kuo and J. F. Walker, Phys. Rev. **175**, 1794 (1968).
[13] J. B. Bronzan, Phys. Rev. **180**, 1423 (1969).
[14] L. Durand III, P. M. Fishbane, S. A. Klein, and L. M. Simmons, Phys. Rev. Letters **23**, 201 (1969). We thank Dr. Durand for a correspondence.

where the zero of the denominator is inside the integration contour. Equivalently, by introducing $\bar{\sigma}=j+\kappa$, we may write

$$T_{\kappa,s\lambda s'\lambda'}^{(\tau)}(t)=\frac{1}{2\pi i}\oint d\bar{\sigma}\sum_{\eta,M}\sum_{\eta'M'} d_{s\lambda\bar{\sigma}-\kappa}{}^{\bar{\sigma}+\eta,M(\tau)}(\xi)$$
$$\times\psi_{\bar{\sigma}-\kappa}{}^{\bar{\sigma}+\eta,M(\tau)}(t)d_{\lambda\lambda'}{}^{\bar{\sigma}-\kappa}(\theta)d_{s'\lambda'\bar{\sigma}-\kappa}{}^{\bar{\sigma}+\eta'M'(\tau)}(-\xi')$$
$$\times\psi_{\bar{\sigma}-\kappa}{}^{\bar{\sigma}+\eta'M'(\tau)}(t)[\bar{\sigma}-F(t,t_1,\pm t_{\bar{M}})]^{-1}. \quad (3.9)$$

For any contour of integration, there exists a value of t small enough that $(\bar{\sigma}-F)^{-1}$ can be expanded in powers of $t_1=t(\bar{\sigma}-\kappa)(\bar{\sigma}-\kappa+1)$ and

$$t_{\bar{M}}=[\Gamma(\bar{\sigma}-\kappa+\bar{M}+1)/\Gamma(\bar{\sigma}-\kappa-\bar{M}+1)]t^{\bar{M}/2}.$$

The powers of t_1 and $\pm t_{\bar{M}}$ can be absorbed into $\psi_j{}^{\sigma M(\pm)}(t)$ without altering its structure.

Using the notations of Appendix B and the form of the Clebsch-Gordan coefficients given by Eqs. (I 3.29) and (I 3.30), we may write these modified functions $\psi_j{}^{\sigma M(\pm)}(t)$ as

$$\psi_j{}^{\sigma M(\pm)}(r)=\sum_n t^{n/2}\left[\begin{pmatrix}\sigma & M & | & \bar{\sigma} & \bar{M} & n & 0\\ j & m & | & j & m & 0 & 0\end{pmatrix}\psi_{n,1}{}^{\sigma M}(t)\right.$$
$$\left.\pm\begin{pmatrix}\sigma & -M & | & \bar{\sigma} & \bar{M} & n & 0\\ j & m & | & j & m & 0 & 0\end{pmatrix}t^{M<}\psi_{n,2}{}^{\sigma M}(t)\right], \quad (3.10)$$

where $\psi_{n,1}{}^{\sigma M}(t)$ and $\psi_{n,2}{}^{\sigma M}(t)$ are regular functions of t. In Eq. (3.10) the j dependence is explicitly factorized in the Clebsch-Gordan coefficients. By making use of the Clebsch-Gordan series for the functions $d_{s\lambda j}{}^{\sigma j_0}(\xi)$ [Eq. (B9)] and summing over κ and τ, we obtain the contribution of the whole family of Regge poles. Using the addition theorem for the functions $d_{s\lambda j}{}^{\sigma j_0}(\xi)$[15] and performing the circuital integration, we obtain

$$T_{s\lambda s'\lambda'}\sim\sum_{\kappa=0}^{\infty}\sum_{\tau=\pm}T_{\kappa,s\lambda s'\lambda'}{}^{(\tau)}$$
$$=\sum_{\bar{s}\bar{s}'}\sum_{k=1}^{\infty}\sum_n\sum_{n'}\sum_{r,r'}c_{\bar{s}r,\bar{s}'r';k}{}^{nn',\bar{\sigma},\bar{M}}(t)$$
$$\times t^{n/2+n'/2}d_{r00}{}^{n0}(\xi)d_{r'00}{}^{n'0}(-\xi')$$
$$\times(s\lambda|\bar{s}\lambda;r0)(s'\lambda'|\bar{s}'\lambda';r'0)(d/d\bar{\sigma})^{k-1}$$
$$\times[D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}(\Lambda_3^{-1}\Lambda_1)\pm\eta(-1)^{\bar{s}+\bar{s}'-r-r'}$$
$$\times D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}-\bar{M}}(\Lambda_3^{-1}\Lambda_1)]|_{\bar{\sigma}=F(t,0,0)}, \quad (3.11)$$

where η is the product of the internal parities of the scattered particles.

The functions c are analytic at $t=0$. In addition, it is clear from the construction that if $k>n/2+n'/2$, then

$$c_{\bar{s},r,\bar{s}',r';k}{}^{n,n'\sigma,M}(t)\sim O(t^{k-n/2-n'/2}). \quad (3.12)$$

Equation (3.11) is rather interesting from the point of view that it is essentially a one-Lorentz-pole form for the contribution of the family even at $t\neq0$. The infinitely many derivatives of the function $D_{s\lambda s'\lambda'}{}^{\sigma M}(\Lambda_q{}^{-1}\Lambda_p)$ introduce a "nonlocal" effect, which results in the appearance of nonparallel trajectories. If the trajectories were parallel, no derivatives of the D functions would appear [because the function $F(t,t_1,\pm t_{\bar{M}})$ would be independent of t_1 and $\pm t_{\bar{M}}$]. On the other hand, the "number of derivatives" depends on the external masses as well. For equal masses, as $d_{r00}{}^{n0}(\xi)$ is a bounded function of t near $t=0$, it follows from Eqs. (3.11) and (3.12) that the kth derivative of the amplitude in variable t at $t=0$ would only contain the kth derivative of function $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}$ in the variable $\bar{\sigma}$. For unequal masses the functions $d_{r00}{}^{n0}(\xi)$ are singular at $t=0$, so $t^{n/2}d_{r00}{}^{n0}(\xi)$ is finite and infinitely many derivatives of $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}$ appear at $t=0$.

It is easy to see that the form (3.11) is an analytic function of t (assuming the convergence of the sums) near $t=0$. The multipliers of the $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}$ are analytic functions of t. The functions $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}$, together with their derivatives in the parameter $\bar{\sigma}$ at a place $\bar{\sigma}=F(t,0,0)$, are analytic as well.[16]

We remark that the derivatives of $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}$ in the variable $\bar{\sigma}$ lead to additional logarithms in the asymptotic behavior. If we sum over the derivatives, these logarithms sum up to a fractional leading power of s, giving rise to the right value of the leading trajectory.

If we disregard the effects of nonparallel trajectories (that is to say, the derivatives in $\bar{\sigma}$), the form (3.11) coincides with that of Delbourgo, Salam, and Strathdee,[17] except for the treatment of total spin. Their form, however, is not applicable in the case of unequal masses, when "perturbations" to the total-spin value appear even at $t=0$.

IV. UNEQUAL-MASS CASE

For the case of unequal masses, a convenient variable to use for the d function is $x=e^{-2\xi}$,

$$\cosh\xi=(t+m_1{}^2-m_2{}^2)/2t^{1/2}m_1$$

in the c.m. frame. At $t=0$, $x=0$, and at both threshold and pseudothreshold [$t=(m_1\pm m_2)^2$], $x=1$. From the definition (B1) of Appendix B, the d function has the following power-series expansion:

$$d_{s\lambda j}{}^{\sigma j_0}(\xi)=x^{\frac{1}{2}(|j_0-\lambda|-\sigma)}(a_0+a_1x+a_2x^2+\cdots), \quad (4.1)$$

[15] A. Sciarrino and M. Toller, J. Math. Phys. 8, 1252 (1967).

[16] The analyticity of the $D_{s\lambda s'\lambda'}{}^{\bar{\sigma}\bar{M}}$ functions at $t=0$ has been discussed by D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967).

[17] R. Delbourgo, A. Salam, and J. Strathdee, Phys. Letters 25B, 230 (1967).

where the first coefficient a_0 is given by

$$a_0=N_{s\lambda j}{}^{\sigma j_0}=\frac{\Gamma(\sigma-j_0+1)\Gamma(\sigma+\lambda+1)}{\Gamma(j_0-\lambda+1)}\left[\frac{(2j+1)(2s+1)\Gamma(j_0+j+1)\Gamma(j_0+s+1)}{\Gamma(\sigma-j+1)\Gamma(j-j_0+1)\Gamma(\sigma+j+2)\Gamma(j+\lambda+1)}\right.$$
$$\left.\times\frac{\Gamma(j-\lambda+1)\Gamma(s-\lambda+1)}{\Gamma(\sigma-s+1)\Gamma(s-j_0+1)\Gamma(\sigma+s+2)\Gamma(s+\lambda+1)}\right]^{1/2}, \quad (4.2)$$

if $j_0\geq\lambda$. The case $j_0<\lambda$ is obtained by interchanging j_0 and λ in the above expression. It is convenient to define a new function $f_{s\lambda j}{}^{\sigma j_0}(\xi)$ by

$$d_{s\lambda j}{}^{\sigma j_0}(\xi)=N_{s\lambda j}{}^{\sigma j_0}x^{\frac{1}{2}(|j_0-\lambda|-\sigma)}f_{s\lambda j}{}^{\sigma j_0}(x). \quad (4.3)$$

Then the function f is normalized at $x=0$:

$$f_{s\lambda j}{}^{\sigma j_0}(0)=1. \quad (4.4)$$

Comparing Eq. (4.3) with Eq. (B8), we see that for $j_0=s$,

$$\begin{aligned}f_{s\lambda j}{}^{\sigma s}(X)&=(1-x)^{\sigma-s}\\&\quad\times F[j-\sigma,\ -j-\sigma-1;\ -\sigma-\lambda;\ x/(x-1)]\\&=(1-x)^{\lambda-s-1}\\&\quad\times F[-\lambda-j;\ -\lambda+j+1;\ -\sigma-\lambda;\ x/(x-1)].\end{aligned} \quad (4.5)$$

The second form follows from the first by a transformation of the hypergeometric function. It is useful to show that $f_{s\lambda j}{}^{\sigma s}$ is a regular function of the composite variables x and $x_1=xj(j+1)$ near $x=0$. Indeed, the $(r+1)$th term of the hypergeometric function is equal to

$$\begin{aligned}&\frac{\Gamma(-\lambda-j+r)\Gamma(-\lambda+j+1+r)\Gamma(-\sigma-\lambda)}{\Gamma(-\lambda-j)\Gamma(-\lambda+j+1)\Gamma(-\sigma-\lambda+r)r!}\left(\frac{x}{x-1}\right)^r\\&=\frac{\Gamma(-\sigma-\lambda)}{\Gamma(-\sigma-\lambda+r)r!}\left(\frac{x}{x-1}\right)^r\\&\qquad\times\prod_{i=0}^{r-1}[(\lambda+i)(\lambda+i-1)-j(j+1)].\end{aligned} \quad (4.6)$$

The last product is a polynomial of rth order in $j(j+1)$ and it is multiplied by x^r, thus yielding a polynomial of x and x_1. Our next step is to show that this fact holds for all $f_{s\lambda j}{}^{\sigma j_0}$ by using the recursion formula which can be translated from Eq. (B4) to read

$$\begin{aligned}&\{2j_0[x+(x-1)x(d/dx)]\\&\quad-(x-1)(j_0-\lambda)(\sigma-j_0+1)\}f_{s\lambda j}{}^{\sigma j_0}(x)\\&=(j_0-\lambda)(\sigma-j_0+1)f_{s\lambda j}{}^{\sigma j_0-1}(x)\\&\quad-\frac{[j(j+1)-j_0(j_0+1)][s(s+1)-j_0(j_0+1)]}{(\sigma-j_0)(j_0-\lambda+1)}\\&\qquad\times xf_{s\lambda j}{}^{\sigma j_0+1}(x)\end{aligned} \quad (4.7)$$

for $j_0\geq\lambda$. We start with the maximum value of $j_0=s$, and use Eq. (4.7) to decrease j_0 by unity each time until we reach the value $j_0=\lambda$. By induction, every $f_{s\lambda j}{}^{\sigma j_0}(x)$ is a regular function of x and x_1. The situation $j_0<\lambda$ is handled by the symmetry relation

$$f_{s\lambda j}{}^{\sigma j_0}(x)=f_{sj_0j}{}^{\sigma\lambda}(x),$$

which follows from Eqs. (4.3) and (B3). Since in the c.m. frame

$$x=4tm_1{}^2/(t+m_1{}^2-m_2{}^2+\Delta)^2,$$

where

$$\Delta=\{[t-(m_1-m_2)^2][t-(m_1+m_2)^2]\}^{1/2}, \quad (4.8)$$

if $f_{s\lambda j}{}^{\sigma j_0}(x)$ is a regular function of x and x_1 near $x=0$, then it is also a regular function of t and t_1 near $t=0$. Before we proceed any further, let us define the shorthand notation

$$\begin{aligned}t_\sigma'&\equiv t^\sigma\Gamma(\sigma+j+2)\Gamma(\sigma-j+1),\\t_M&=t^M[\Gamma(j+M+1)/\Gamma(j-M+1)].\end{aligned} \quad (4.9)$$

Then, from the above considerations, we can write

$$\begin{aligned}d_{s\lambda j}{}^{\sigma j_0}&=(2j+1)^{1/2}\left(\frac{t_{|j_0|}}{t_\sigma' t_{|\lambda|}}\right)^{1/2}h_{s\lambda}{}^{\sigma j_0}(t,t_1)\quad\text{if } j_0\geq\lambda\\&=(2j+1)^{1/2}\left(\frac{t_{|\lambda|}}{t_\sigma' t_{|j_0|}}\right)^{1/2}h_{s\lambda}{}^{\sigma j_0}(t,t_1)\quad\text{if } \lambda\geq j_0.\end{aligned} \quad (4.10)$$

Furthermore, we can write Eq. (3.3) as

$$\psi_j{}^{\sigma M(\pm)}(t)=(t_{\sigma>}'t_{M>}/t_{\sigma<}'t_{M<})^{1/2}[a(t,t_1)\pm t_{M<}b(t,t_1)],$$

so that

$$\psi_j{}^{\sigma M(\pm)}d_{s\lambda j}{}^{\sigma M(\pm)}=f_{\sigma M}(\lambda)d_{s\lambda j}{}^{\bar\sigma\bar M}\pm\bar f_{\sigma M}(-\lambda)d_{s\lambda j}{}^{\bar\sigma-\bar M}, \quad (4.11)$$

where

$$\bar f_{\sigma M}(\lambda)=\left(\frac{t_{\sigma>}'t_{M>}}{t_{\sigma<}'t_{M<}}\right)^{1/2}\frac{d_{s\lambda j}{}^{\sigma M}}{d_{s\lambda j}{}^{\bar\sigma\bar M}}a(t,t_1)+\left(\frac{t_{\sigma>}'t_{M>}t_{M<}}{t_{\sigma<}'}\right)^{1/2}\frac{d_{s\lambda j}{}^{\sigma-M}}{d_{s\lambda j}{}^{\bar\sigma\bar M}}b(t,t_1). \quad (4.12)$$

It can be easily verified that $f_{\sigma M}$ is a regular function of t and t_1 near $t=0$. We may now sum over σ and M in

Eq. (3.4):

$$\psi_j^{(\pm)}(t)=\sum_{\sigma M}\psi_j^{\sigma M(\pm)}d_{s\lambda j}^{\sigma M(\pm)}$$

$$=f(t,j,\lambda)d_{s\lambda j}^{\bar\sigma\bar M}\pm f(t,j,-\lambda)d_{s\lambda j}^{\bar\sigma-\bar M},\quad(4.13)$$

where

$$f=\sum_{\sigma M}f_{\sigma M},$$

and it is a regular function of t and t_1 near $t=0$. The $\bar\sigma$ and $\bar M$ are the quantum numbers of the family. We must be cautious about one point, however. When $\bar M>s$,[18] since $d_{s\lambda j}^{\bar\sigma\bar M}\equiv0$, Eq. (4.13) has to be modified. In Eq. (4.11) we use $d_{s\lambda j}^{\bar\sigma\pm s}$ instead, so that in place of Eq. (4.13) we obtain, for $\bar M>s$,

$$\psi_j^{(\pm)}(t)=t^{\frac12(\bar M-s)}\left[\frac{\Gamma(j+\bar M+1)\Gamma(j-s+1)}{\Gamma(j-\bar M+1)\Gamma(j+s+1)}\right]^{1/2}\times[\bar f(t,j,\lambda)d_{s\lambda j}^{\bar\sigma s}\pm f(t,j,-\lambda)d_{s\lambda j}^{\bar\sigma-s}].\quad(4.14)$$

We have only discussed the j dependence of $\bar f$ and $\bar g$. For the purpose of discussing constraint relations at threshold and pseudothreshold, we must also determine the λ dependence as well. This may be done as follows. From Eq. (B8) we see that, near $x=1$, for $j_0=s$,

$$d_{s\lambda j}^{\sigma s}=\left[\frac{\Gamma(j+\lambda+1)\Gamma(j-\lambda+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)}\right]^{1/2}\times(1-x)^{j-s}f(1-x,(1-x)\lambda),\quad(4.15)$$

where f is a regular function of $1-x$ and $(1-x)\lambda$ near $x=1$. Furthermore, this property is obviously not destroyed by the recursion formula (B4), so that $d_{s\lambda j}^{\sigma M}$ has the form of the right-hand side of Eq. (4.15) as well. The function $\bar f$ depends on λ only through the ratio of d functions with the same s, λ, j. The numerical factor in Eq. (4.15) cancels out in such a ratio; we therefore conclude that, near $x=1$,

$$\bar f=\bar f_j(1-x,(1-x)\lambda).$$

Since

$$1-x=2\Delta/(t+m_1^2-m_2^2+\Delta),$$

and at threshold and pseudothreshold $\Delta=0$, the λ dependence of $\bar f$ is such that it is a regular function of composite variables Δ and $\Delta\lambda$. So we can state the final result for the unequal-mass case: The most general form of the Regge residue, for the trajectory $j=\alpha_\kappa^{(\pm)}(t)$, is

$$r_{\kappa,s\lambda}^{(\pm)}(t)=[\bar f(t,j,\lambda)d_{s\lambda j}^{\bar\sigma\bar M}\pm\bar f(t,j,-\lambda)d_{s\lambda j}^{\bar\sigma-\bar M}]\times\left[1-\frac{\partial}{\partial j}F(t,t_1,\pm t_{\bar M})\right]^{-1/2}\Bigg|_{\bar\sigma=j+\kappa;\ j=\alpha_\kappa^{(\pm)}(t)}\quad\text{if } s\geq\bar M,\quad(4.16)$$

$$r_{\kappa,s\lambda}^{(\pm)}(t)=t^{\frac12(\bar M-s)}\left[\frac{\Gamma(j+\bar M+1)\Gamma(j-s+1)}{\Gamma(j-\bar M+1)\Gamma(j+s+1)}\right]^{1/2}\times[\bar f(t,j,\lambda)d_{s\lambda j}^{\bar\sigma s}\pm\bar f(t,j,-\lambda)d_{s\lambda j}^{\bar\sigma-s}]\times\left[1-\frac{\partial}{\partial j}F(t,t_1,\pm t_{\bar M})\right]^{-1/2}\Bigg|_{\bar\sigma=j+\kappa;\ j=\alpha_\kappa^{(\pm)}(t)}\quad\text{if } s\leqslant\bar M,\quad(4.17)$$

where

$$\bar f(t,j,\lambda)=\bar f_\lambda(t,tj(j+1))\quad\text{near } t=0\quad(4.18)$$

and

$$\bar f(t,j,\lambda)=\bar f_j(\Delta,\Delta\lambda)$$

near $\quad(4.19)$

$$\Delta=\{[j-(m_1-m_2)^2][t-(m_1+m_2)^2]\}^{1/2}=0.$$

The above vertex function has been shown to give rise to a Regge-family contribution analytic at $t=0$. Our remaining chore in this section is to show that it guarantees the fulfillment of the kinematic constraint relations at $\Delta=0$ as well.

The constraint relations for the total-spin amplitudes at threshold and pseudothreshold have the same form[5]:

$$\sum_\lambda\Delta_{\lambda\mu}^s e^{-i\frac12\pi\lambda}T_{s'\lambda's\lambda}\sim\Delta^\mu\quad\text{near }\Delta=0,\quad(4.20)$$

where $\Delta_{\lambda\mu}^s=d_{\lambda\mu}^j(\frac12\pi)$. We first show that Eq. (4.20) is satisfied by any term in the expansion (3.4), which gives the λ dependence of $T_{s'\lambda's\lambda}^{(\pm)}$ in the form $d_{s\lambda j}^{\sigma M(\pm)}(\xi_1)d_{\lambda'\lambda}^j(\theta_t)$. When this is substituted into Eq. (4.20) and use is made of the identity[19]

$$d_{\lambda'\lambda}^j(\theta_t)=\sum_m e^{i(\lambda-\lambda')\frac12\pi}\Delta_{\lambda',m}^j\Delta_{\lambda M}^j e^{-im\theta_t},$$

we have

$$\sum_{\lambda m}\Delta_{\lambda\mu}^s d_{s\lambda j}^{\sigma M(\pm)}(\xi_1)\Delta_{\lambda m}^j e^{-im\theta_t}\Delta_{\lambda'm}^j\sim\Delta^\mu\quad\text{near }\Delta=0.$$

Since $\sinh\xi_1\sim\Delta$ and $\cos\theta_t\sim1/\Delta$, the above will be true if we can prove that

$$\sum_\lambda\Delta_{\lambda\mu}^s d_{s\lambda j}^{\sigma M(\pm)}(\xi_1)\Delta_{\lambda m}^j\sim(\sinh\xi_1)^{|\mu-m|}\quad\text{as }\xi_1\to0.$$

The above follows quite readily from the addition theorem of the d functions.[15,20] It remains to be shown that Eqs. (4.16) and (4.17), with any $\bar f$ satisfying Eqs. (4.18) and (4.19), are also consistent with the constraint relations (4.20). It is a necessary step because we have shown that any wave function of the form (3.4) leads to a vertex of the form (4.16) or (4.17), but we do not

[18] This situation certainly can occur; for example, a two-pion state can couple to a trajectory belonging to an $\bar M=1$ family.

[19] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).

[20] We remark that it perhaps takes much less than the addition theorem to prove this. The left-hand side is simply the matrix element $\langle\sigma Ms\lambda\pm|e^{-iK_1\xi_1}|\sigma Mj\lambda\pm\rangle$, whose $\xi_1=0$ behavior can be established by studying the differential equations satisfied by the matrix element. This proof has some advantage over the addition theorem since the latter is only asymptotically true for continued values of j.

know if Eq. (4.16) or (4.17) is too general. To complete this last step, we observe that if $\bar{f}$ is expanded in the powers of λ, because of Eq. (4.19), each power of λ is accompanied by at least one power of Δ. To absorb all the powers of λ, we can use the recursion formula for $\Delta_{\lambda'\lambda}{}^{s}$

$$\lambda\Delta_{\lambda'\lambda}{}^{s}=a\Delta_{\lambda'+1\lambda}{}^{s}+b\Delta_{\lambda'\lambda}{}^{s}+c\Delta_{\lambda'-1\lambda}{}^{s},$$

where the coefficients a, b, and c do not depend on λ. The charged values of λ' are compensated by the powers of Δ. This completes the proof.

The $t\sim0$ or the $\Delta\sim0$ behavior of the vertex function can be further simplified by substituting Eqs. (4.10) or (4.15), respectively, into Eqs. (4.16) and (4.17).

V. EQUAL-MASS CASE

The behavior of the Regge vertex functions near $t=0$ in the equal-mass case ($m_1=m_2$ and/or $m_3=m_4$) is essentially different and much more complicated than in the unequal-mass case. The physical origin of this difference is, on the one hand, the higher symmetry of the equal-mass problem; on the other hand, in the equal-mass case the point $t=0$ is the pseudothreshold as well. Mathematically, one of the consequences of the different circumstances is that the function $d_{s\lambda j}{}^{j_0}(\xi)$ is a regular function of $\cos\xi=t^{1/2}/2m_1$ at $\cos\xi=0$. It is much harder to give a useful form for the function $d_{s\lambda j}{}^{j_0}(\xi)$ near this regular point than around singular points $\xi=0$ or $\xi=\infty$ ($x=0$ or $x=1$).

In the first part of this section we shall show that the kinematic constraints for helicity amplitudes are automatically satisfied, and later on we shall study the possibility of simplification of the form (3.7).

For the sake of simplicity, we discuss only the equal-mass–equal-mass (EE) case; the (EU) case may be obtained using the known structure of unequal-mass vertices (see Sec. IV) and factorization.

The constraints for a (EE) total-spin amplitude can be written in the following form[5]:

$$\sum_{\lambda\lambda'}\Delta_{\mu'\lambda'}{}^{s}\Delta_{\mu\lambda}{}^{s}f_{s\lambda s'\lambda'}\sim O(t^{|\mu-\mu'|}),\qquad(5.1)$$

where $\Delta_{\mu\lambda}{}^{s}=d_{\mu\lambda}{}^{s}(\tfrac{1}{2}\pi)$. We prove that the constraint relation (5.1) is satisfied by the contribution of a Regge-pole family as given by Eq. (3.11). At the same time we have to emphasize that the contribution of a single daughter pole does not satisfy the constraint relation (5.1).

As a first step we can see that a single-Lorentz-pole contribution satisfies Eq. (5.1)[21]:

$$\sum_{\lambda\lambda'}\Delta_{\mu\lambda}{}^{s}\Delta_{\mu'\lambda'}{}^{s}D_{s\lambda s'\lambda'}{}^{\sigma j_0}(\Lambda_3{}^{-1}\Lambda_1)=\sum_{\lambda}d_{\mu\lambda}{}^{s}(\varphi+\tfrac{1}{2}\pi)d_{s\lambda s'}{}^{\sigma j_0}(\alpha)d_{\mu\lambda}{}^{s'}(-\varphi'-\tfrac{1}{2}\pi),\qquad(5.2)$$

where α, φ, and φ' are given by the addition formula of Lorentz transformations[15]; φ and φ' satisfy $\sin(\varphi+\tfrac{1}{2}\pi)\sim\sin(\varphi'+\tfrac{1}{2}\pi)=O(t^{1/2})$ for $t\to0$. Making use of the behavior of the function $d_{\lambda\mu}{}^{s}(\varphi+\tfrac{1}{2}\pi)$ near $\sin(\varphi+\tfrac{1}{2}\pi)=0$, we obtain

$$\sum_{\lambda\lambda'}\Delta_{\mu\lambda}{}^{s}\Delta_{\mu'\lambda'}{}^{s'}D_{s\lambda s'\lambda'}{}^{\sigma j_0}(\Lambda_3{}^{-1}\Lambda_1)\sim\sum_{\lambda}t^{\frac{1}{2}|\mu-\lambda|}d_{s\lambda s'}{}^{\sigma j_0}(\alpha)t^{\frac{1}{2}|\mu'-\lambda'|}=O(t^{\frac{1}{2}|\mu-\mu'|}).\qquad(5.3)$$

The differentiations with respect to σ do not make any difference from the point of view of kinematic constraints. We have to study, on the other hand, the role of the coefficients of the functions $D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\bar{\sigma}\bar{M}}(\Lambda_3{}^{-1}\Lambda_1)$ in Eq. (3.11). The inequalities $n\geq r$ and $n'\geq r'$ are always satisfied and $d_{r00}{}^{n0}(\xi)$ is a bounded function of t at $t=0$, so the important t- and λ-dependent multiplier of the coefficients is $t^{r/2}(s\lambda|\bar{s}\lambda;r0)$. The contribution of a special term of expansion (3.11) to the constraint relation (5.1) has the form (we do not write the differentiations explicitly)

$$\begin{aligned}\sum_{\lambda,\lambda'}\Delta_{\mu\lambda}{}^{s}\Delta_{\mu'\lambda'}{}^{s'}t^{r/2+r'/2}(s\lambda|\bar{s}\lambda;r0)(s'\lambda'|\bar{s}'\lambda';r'0)D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\sigma j_0}(\Lambda_3{}^{-1}\Lambda_1)\\=t^{r/2+r'/2}\sum_{\lambda\lambda'}\sum_{\eta\eta'}\Delta_{\mu-\eta\lambda}{}^{\bar{s}}\Delta_{\mu'-\eta'\lambda'}{}^{\bar{s}'}\Delta_{\eta0}{}^{r}\Delta_{\eta'0}{}^{r'}(s\mu|\bar{s}\mu-\eta r\eta)\\\times(s'\mu'|\bar{s}'\mu'-\eta';r'\eta')D_{\bar{s}\lambda\bar{s}'\lambda'}{}^{\sigma j_0}\sim t^{r/2+r'/2}t^{\frac{1}{2}|\mu-\eta-\mu'+\eta'|}=O(t^{\frac{1}{2}|\mu-\mu'|}).\qquad(5.4)\end{aligned}$$

In Eq. (5.4) we have made use of the Clebsch-Gordan series for the functions $\Delta_{\mu\lambda}{}^{s}$, Eq. (5.3), and the inequalities $r\geq|\eta|$, $r'\geq|\eta'|$.

Equation (5.4) establishes that the constraint relations (5.1) are satisfied by the contribution of a family of Regge poles as given by Eqs. (I 3.33) and (3.7), or by Eq. (3.11).

We can now turn to the question of simplification of Eq. (3.7). For this aim we apply the same device applied in Sec. III for the derivation of the Lorentz-pole form of the contribution of the family, with the only exception that we do not now expand the normalization factor

$$[1-(\partial/\partial j)F(t,t_1,\pm t_{\bar{M}})]^{-1/2}.$$

We get

$$\begin{aligned}r_{\kappa,s\lambda}{}^{(\pm)}(t)=\sum_{\bar{s}=\bar{M}}^{\infty}\sum_{\sigma Mnr}d_{r00}{}^{n0}(\xi)t^{n/2}\\\times\left[\begin{pmatrix}\sigma & M & \bar{\sigma} & \bar{M} & n & 0\\ s & \lambda & s & \lambda & r & 0\end{pmatrix}\psi_{n,1}{}^{\sigma M}(t)d_{\bar{s}\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\right.\\\left.\pm\begin{pmatrix}\sigma & M & \bar{\sigma} & -M & n & 0\\ s & \lambda & \bar{s} & \lambda & r & 0\end{pmatrix}\psi_{n,2}{}^{\sigma M}(t)d_{\bar{s}\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)\right]\\\times[1-(\partial/\partial j)F(t,t_1,\pm t_{\bar{M}})]^{-1/2}|_{j=\alpha_\kappa{}^{(\pm)}(t)}.\end{aligned}$$

[21] K. M. Bitar and G. L. Tindle, Phys. Rev. 175, 1835 (1968).

Using the λ dependence of the Clebsch-Gordan coefficients as obtained from Eqs. (B11) and (I 2.3) and summing over σ, M, n, and r, we obtain

$$r_{\kappa,s\lambda}^{(\pm)}(t)=\sum_{\bar{s}=\bar{M}}^{\infty} t^{\frac{1}{2}|\bar{s}-s|}(s\lambda|\bar{s}\lambda;\,|s-\bar{s}|0)\times[f_{\bar{s}}(t,t^{1/2}\lambda)d_{\bar{s}\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\pm f_{\bar{s}}(t,\,-t^{1/2}\lambda)d_{\bar{s}\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)]\times[1-(\partial/\partial j)F(t,\,t_1,\,\pm t_{\bar{M}})]^{-1/2}|_{j=\alpha_\kappa{}^{(\pm)}(t)}\,, \quad (5.5)$$

where $f_{\bar{s}}(t,t^{1/2}\lambda)$ is a regular function of its variables at $t=0$.

By the successive application of the recursion relation (B5), we can reduce the $\bar{s}$ values in Eq. (5.5) to s and $s-1$ if $\bar{M}<s$:

$$\begin{aligned}r_{\kappa,s\lambda}^{(\pm)}(t)=\{&O_1(td/dt,t,t^{1/2}\lambda)d_{s\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\\ &\pm O_1(td/dt,\,t,\,-t^{1/2}\lambda)d_{s\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)\\ &+O_2(td/dt,t,t^{1/2}\lambda)t^{1/2}[(s+\lambda)(s-\lambda)]^{1/2}\cdot d_{s-1\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\\ &\pm O_2(td/dt,\,t,\,-t^{1/2}\lambda)t^{1/2}[(s+\lambda)(s-\lambda)]^{1/2}\cdot d_{s-1\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)\}\\ &\times[1-(\partial/\partial j)F(t,\,t_1\pm t_{\bar{M}})]^{-1/2}|_{j=\alpha_\kappa{}^{(\pm)}(t)}\,. \end{aligned}\quad (5.6)$$

If $\bar{M}\geq s$, then $\bar{s}\geq s$ and we obtain

$$r_{\kappa,s\lambda}^{(\pm)}(t)=t^{\frac{1}{2}|\bar{M}-s|}\left[\frac{\Gamma(\bar{M}+\lambda+1)}{\Gamma(s-\lambda+1)}\frac{\Gamma(\bar{M}-\lambda+1)}{\Gamma(s+\lambda+1)}\right]^{1/2}\times[O_1(td/dt,t,t^{1/2}\lambda)d_{\bar{M}\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\pm O_1(td/dt,\,t,\,-t^{1/2}\lambda)d_{\bar{M}\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)]\times\left[1-\frac{\partial}{\partial j}F(t,\,t_1,\,\pm t_{\bar{M}})\right]^{-1/2}\Bigg|_{j=\alpha_\kappa{}^{(\pm)}(t)}\,. \quad (5.7)$$

In Eqs. (5.6) and (5.7), the operators O_1 and O_2 are regular at the point (0,0,0).

Expressions (5.6) and (5.7) are less useful than the corresponding ones for the unequal-mass case given by Eqs. (4.16) and (4.17) because of the lack of a simple closed analytic form of the functions $d_{s\lambda j}{}^{\sigma j0}(\xi)$.

For the special case $s=0$, however, we can write the d functions in the following simple form[22]:

$$d_{M0j}{}^{\sigma M}(\xi)=(\sigma+1)(\sinh\xi)^{-\sigma-M-2}[\pi^{1/2}\Gamma(M+1)]^{-1}\times\left[\frac{\Gamma(\sigma-M+1)}{\Gamma(\sigma+M+2)}\frac{\Gamma(j+M+1)}{\Gamma(j-M+1)}\times\frac{\Gamma(\frac{1}{2}(\sigma+j)+1)}{\Gamma(\frac{1}{2}(\sigma-j)+1)}\frac{\Gamma(\frac{1}{2}(\sigma-j+1))}{\Gamma(\frac{1}{2}(\sigma+j+3))}\right]^{1/2}\times F(\tfrac{1}{2}(1+\sigma-j),\,\tfrac{1}{2}(\sigma+j+2);\,\tfrac{1}{2};\,\coth^2\xi)\,, \quad (5.8)$$

if $(\sigma-j)$ is even, and

$$d_{M0j}{}^{\sigma M}(\xi)=-(\sigma+1)(\sinh\xi)^{-\sigma-M-3}\frac{2\cosh\xi}{\pi^{1/2}\Gamma(M+1)}\times\left[\frac{\Gamma(\sigma-M+1)}{\Gamma(\sigma+M+2)}\frac{\Gamma(j+M+1)}{\Gamma(j-M+1)}\times\frac{\Gamma(\frac{1}{2}(\sigma+j+3))}{\Gamma(\frac{1}{2}(\sigma+j)+1)}\frac{\Gamma(\frac{1}{2}(\sigma-j)+1)}{\Gamma(\frac{1}{2}(\sigma-j+1))}\right]^{1/2}\times F(\tfrac{1}{2}(2+\sigma-j),\,\tfrac{1}{2}(\sigma+j+3);\,\tfrac{3}{2};\,\coth^2\xi)\,, \quad (5.9)$$

if $(\sigma-j)$ is odd. The hypergeometric functions appearing in Eqs. (5.8) and (5.9) are regular functions of t and t_1 ($\cosh\,\xi=t^{1/2}/2m$), and the powers of $\sin\xi$ are regular functions of t at $t=0$ as well. Substituting into Eq. (5.7) and taking into account that an operator $O(td/dt,t)$ applied to a regular function of t and t_1 gives again a regular function of t and t_1, we obtain

$$r_\kappa{}^{(t)}=t^{\bar{M}/2}d_{00j}{}^{\bar{\sigma}0}(\xi)f(t,t_1)|_{j=\alpha_\kappa(t)}\,. \quad (5.10)$$

These Regge vertices are different from zero for only one choice of parity, τ, depending on the internal parity of the external particles. For this reason we suppressed the index $\pm$ in Eq. (5.10).

Equation (5.10) shows that in contrast to general belief, the odd-order daughter trajectories are decoupled only at $t=0$.

Finally, we discuss the constraints at the threshold $t=4m_1^2$. At the threshold the behavior of $x=e^{-2\xi}$ is identical with that for the unequal-mass case, so the discussion of constraints in Sec. IV can be applied without any change to the equal-mass case. This discussion would result in the following form for the operators O_i at threshold:

$$O_i=O_i(\Delta,\Delta\lambda)\,,\quad \Delta=[t(t-4m^2)]^{1/2}\,,$$

where O_i are regular functions of their variables at $\Delta=0$. More generally, since the d functions themselves are regular functions of Δ and $\Delta\lambda$, we can write

$$O_i=O_i((t-4m^2)d/dt,\Delta,\Delta\lambda)\,. \quad (5.11)$$

A simple way to combine the requirements on operators O_i at $t=0$ and threshold is

$$O_i=O_i(\Delta^2d/dt,\Delta,\Delta\lambda)\,, \quad (5.12)$$

where the O_i are regular functions of their arguments at the point (0,0,0).

Thus our final result for the Regge vertices in the equal-mass case is

$$\begin{aligned}r_{\kappa,s\lambda}^{(\pm)}(t)=\{&O_1(\lambda)d_{s\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\pm O_1(-\lambda)d_{s\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)\\ &+O_2(\lambda)\Delta[(s+\lambda)(s-\lambda)]^{1/2}d_{s-1\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\pm O_2(-\lambda)\Delta\\ &\times[(s+\lambda)(s-\lambda)]^{1/2}d_{s-1\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)\}|_{j=\alpha_\kappa(t)} \end{aligned}\quad (5.13)$$

[22] In Eqs. (5.8) and (5.9) we make use of the form of the functions $d_{00j}{}^{\sigma 0}$ given by G. Domokos and P. Suranyi [Nucl. Phys. **54**, 529 (1964)], recursion relations given by A. Sebestyen, K. Szegö, and K. Toth [Fortschr. Physik **17**, 167 (1969)], and an expression for Gegenbauer's functions in terms of hypergeometric functions given by W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer, Berlin, 1966).

if $\bar{M}<s$, and

$$r_{\kappa,s\lambda}^{(\pm)}(t)=t^{(\bar{M}-s)/2}\left[\frac{\Gamma(\bar{M}+\lambda+1)\Gamma(\bar{M}-\lambda+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)}\right]^{1/2}$$

$$\times[O_1(\lambda)d_{\bar{M}\lambda j}{}^{\bar{\sigma}\bar{M}}(\xi)\pm O_1(-\lambda)d_{\bar{M}\lambda j}{}^{\bar{\sigma}-\bar{M}}(\xi)]|_{j=\alpha_\kappa(t)} \tag{5.14}$$

if $\bar{M}\geq s$, where the operators O_i are of the form (5.12). For the zero-total-spin case our result is given in Eq. (5.10).

VI. SUMMARY AND DISCUSSION

For ready reference, we collect here all the important results of I and this paper.

(a) The most general behavior of the κth trajectory function of parity ±1, $j=\alpha_\kappa{}^{(\pm)}(t)$, is the solution of the functional equation

$$j+\kappa=f(t,t_1)\pm t_M g(t,t_1)=F(t,\,t_1,\,\pm t_M)\,, \tag{I 3.33}$$

where $t_1=tj(j+1)$, $t_M=t^M[\Gamma(j+M+1)/\Gamma(j-M+1)]$, $\sigma=f(0,0)$, and M are the Lorentz quantum numbers of the family at $t=0$. f and g are regular at $t=0$.

Note that the above trajectory formula requires the equality of trajectories of opposite parity up to order $M-1$ in powers of t.

(b) The total-spin amplitude is defined by

$$T_{s'\lambda';s\lambda}=\sum_{\lambda_i}T_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s_1\lambda_1;\,s_2\lambda-\lambda_1|s\lambda)D_{\lambda_2\lambda-\lambda_1}{}^{s_2}(\Lambda_2{}^{-1}\Lambda_1)$$

$$\times(s_3\lambda_3;\,s_4\lambda'-\lambda_3|s'\lambda')D_{\lambda_4\lambda'-\lambda_3}{}^{s_4*}(\Lambda_4{}^{-1}\Lambda_3)\,, \tag{2.14}$$

where Λ_i is the three-parameter Lorentz transformation which transforms the four-vector $(m_i,\mathbf{0})$ into p_i.

(c) The contribution of the entire family to the scattering amplitude is of the form

$$T_{s'\lambda's\lambda}=\sum_{\bar{s}\bar{s}'}\sum_{k}\sum_{nn'}\sum_{rr'}c_{\bar{s}r\bar{s}'r';k}{}^{nn'\sigma M}(t)t^{\frac{1}{2}(n+n')}d_{r00}{}^{n0}(\xi_1)d_{r'00}{}^{n'0}(-\xi_3)(\bar{s}\lambda;\,r0|s\lambda)(\bar{s}'\lambda';\,r'0|s'\lambda')(d/d\sigma)^k$$

$$\times[D_{\bar{s}'\lambda'\bar{s}\lambda}{}^{\sigma M}(\Lambda_3{}^{-1}\Lambda_1)+\tau(-1)^{\bar{s}+\bar{s}'-r-r'}D_{s'\lambda's\lambda}{}^{\sigma-M}(\Lambda_3{}^{-1}\Lambda_1)]|_{\sigma=F(t,0,0)}\,, \tag{3.11'}$$

where

$$\cosh\xi_1=\frac{t+m_1{}^2-m_2{}^2}{2t^{\frac{1}{2}}m_1},\quad \cosh\xi_3=\frac{t+m_3{}^2-m_4{}^2}{2t^{\frac{1}{2}}m_3}.$$

The function c is regular at $t=0$, and when $k>\frac{1}{2}(n+n')$, $c(t)\sim O(t^{k-\frac{1}{2}(n+n')})$. For parallel daughter trajectories, $k=0$ only. If we do not require any further conditions on the coefficients c in Eq. (3.11), then this form is more general than a contribution of a family of Regge trajectories, and could include cut effects as well.

(d) The Regge vertex of the κth pole (unequal mass) has the general form:

(i) for $M\leq s$,

$$r_{\kappa s\lambda}{}^{(\pm)}(t)=\left[1-\frac{\partial F(t,\,t_1,\,\pm t_M)}{\partial j}\right]^{-1/2}[d_{s\lambda j}{}^{j+\kappa M}(\xi_1)\bar{f}(t,j,\lambda)$$

$$\pm d_{s\lambda j}{}^{j+\kappa-M}(\xi_1)\bar{f}(t,\,j,\,-\lambda)]|_{j=\alpha_\kappa{}^{(\pm)}(t)}; \tag{4.16'}$$

(ii) for $M>s$,

$$r_{\kappa s\lambda}{}^{(\pm)}(t)=t^{\frac{1}{2}(M-s)}\left[\frac{\Gamma(j+M+1)\Gamma(j-s+1)}{\Gamma(j-M+1)\Gamma(j+s+1)}\right]^{1/2}$$

$$\times\left[1-\frac{\partial F(t,\,t_1,\,\pm t_M)}{\partial j}\right]^{-1/2}[d_{s\lambda j}{}^{j+\kappa s}(\xi_1)\bar{f}(t,j,\lambda)$$

$$\pm d_{s\lambda j}{}^{j+\kappa,-s}(\xi_1)\bar{f}(t,\,j,\,-\lambda)]|_{j=\alpha_\kappa{}^{(\pm)}(t)}\,. \tag{4.17'}$$

Near $t=0$, $\bar{f}$ is a regular function of composite variables t and t_1:

$$\bar{f}=\bar{f}_\lambda(t,t_1)\,,$$

and near $\Delta=\{[t-(m_1-m_2)^2][t-(m_1m_2)^2]\}^{1/2}=0$, $\bar{f}$ is a regular function of composite variables Δ and $\Delta\lambda$:

$$\bar{f}=\bar{f}_j(\Delta,\Delta\lambda)\,.$$

It is interesting to note that while the trajectories for parities $+1$ and -1 are equal up to order $M-1$ in powers of t, the residues have a lower degree of conspiracy: They agree up to order μ, where

$$\mu=\min\{\lambda,M\}-1\,.$$

(e) The Regge vertex of the κth pole (equal mass) has the general form:

(i) for $s>M$,

$$r_{\kappa s\lambda}{}^{(\pm)}(t)=\left[1-\frac{\partial F(t,\,t_1,\,\pm t_M)}{\partial j}\right]^{-1/2}\left[O_1\left(\Delta^2\frac{d}{dt},\Delta,\Delta\lambda\right)d_{s\lambda j}{}^{j+\kappa M}(\xi_1)\right.$$

$$\pm O_1\left(\Delta^2\frac{d}{dt},\,\Delta,\,-\Delta\lambda\right)d_{s\lambda j}{}^{j+\kappa,-M}(\xi_1)+O_2\left(\Delta^2\frac{d}{dt},\Delta,\Delta\lambda\right)[(s+\lambda)(s-\lambda)]^{1/2}\Delta d_{s-1\lambda j}{}^{j+\kappa M}(\xi_1)$$

$$\left.\pm O_2\left(\Delta^2\frac{d}{dt},\,\Delta,\,-\Delta\lambda\right)\Delta[(s+\lambda)(s-\lambda)]^{1/2}d_{s-1\lambda j}{}^{j+\kappa-M}(\xi_1)\right]\Bigg|_{j=\alpha_\kappa{}^{(\pm)}(t)}; \tag{5.13'}$$

(ii) for $s\leq M$,

$$r_{\kappa s\lambda}^{(\pm)}(t)=t^{(M-s)/2}\left[\frac{\Gamma(M+\lambda+1)\Gamma(M-\lambda+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)}\right]^{1/2}\left[1-\frac{\partial F(t,t_1,\pm t_M)}{\partial j}\right]^{-1/2}$$
$$\times\left[O_1\left(\Delta^2\frac{d}{dt},\Delta,\Delta\lambda\right)d_{M\lambda j}{}^{j+\kappa M}(\xi_1)\pm O_1\left(\Delta^2\frac{d}{dt},\Delta,-\Delta\lambda\right)d_{M\lambda j}{}^{j+\kappa-M}(\xi_1)\right]\Bigg|_{j=\alpha_\kappa^{(\pm)}(t)},\quad(5.14')$$

where O_1 and O_2 are regular near (0,0,0);

(iii) for $s=0$,

$$r_\kappa(t)=t^{M/2}d_{00j}{}^{j+\kappa 0}(\xi_1)f(t,t_1)[1-\partial/\partial j)F(t,t_1,t_M)]^{-1/2}|_{j=\alpha_\kappa(t)}.\quad(5.10')$$

Note that the odd-order daughters are not decoupled at nonzero values of t.

It is interesting to remark that the following special forms satisfy the constraints at $t=0$ and at threshold as well [they are less general than Eqs. (5.13) and (5.14)]:

$$r_{\kappa s\lambda}^{(\pm)}(t)=\left[1-\frac{\partial F(t,t_1,\pm t_M)}{\partial j}\right]^{-1/2}[f(t_1,\Delta,\Delta\lambda)d_{s\lambda j}{}^{j+\kappa M}(\xi_1)$$
$$\pm f(t_1,\Delta,-\Delta\lambda)d_{s\lambda j}{}^{j+\kappa-M}(\xi_1)]|_{j=\alpha_\kappa^{(\pm)}(t)}\quad\text{for } M\leq s;\quad(6.1)$$

and

$$r_{\kappa s\lambda}^{(\pm)}(t)=t^{\frac{1}{2}(M-s)}\left[\frac{\Gamma(M+\lambda+1)\Gamma(M-\lambda+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)}\right]^{1/2}\left[1-\frac{\partial F(t,\pm t_1,t_M)}{\partial j}\right]^{-1/2}$$
$$\times[f(t_1,\Delta,\Delta\lambda)d_{M\lambda j}{}^{j+\kappa M}(\xi_1)\pm f(t_1,\Delta,-\Delta\lambda)d_{M\lambda j}{}^{j+\kappa-M}(\xi_1)]|_{j=\alpha_\kappa^{(\pm)}(t)}\quad\text{for } M\geq s.\quad(6.2)$$

f is a regular function of t_1, Δ, and $\Delta\lambda$.

(f) The behavior of the d functions near the special points are as follows.

(i) Near $t=0$ (unequal mass),

$$d_{s\lambda j}{}^{\sigma j_0}=\frac{(2j+1)^{1/2}t^{\frac{1}{2}(|\lambda-j_0|-\sigma)}}{[\Gamma(\sigma+j+2)\Gamma(\sigma-j+1)]^{1/2}}f(t,t_1)\times\left[\frac{\Gamma(j+|j_0|+1)\Gamma(j-|\lambda|+1)}{\Gamma(j-|j_0|+1)\Gamma(j+|\lambda|+1)}\right]^{1/2},\quad j_0\geq\lambda$$
$$=\frac{(2j+1)^{1/2}t^{\frac{1}{2}(|\lambda-j_0|-\sigma)}}{[\Gamma(\sigma+j+2)\Gamma(\sigma-j+1)]^{1/2}}f(t,t_1)\times\left[\frac{\Gamma(j-|j_0|+1)\Gamma(j+|\lambda|+1)}{\Gamma(j+|j_0|+1)\Gamma(j-|\lambda|+1)}\right]^{1/2},\quad \lambda\geq j_0.\quad(4.10')$$

(ii) Near $\Delta=\{[t-(m_1-m_2)^2][t-(m_1+m_2)^2]\}^{1/2}=0$, but $t\neq0$,

$$d_{s\lambda j}{}^{\sigma j_0}=\left[\frac{\Gamma(j+\lambda+1)\Gamma(j-\lambda+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)}\right]^{1/2}\Delta^{j-s}f(\Delta,\Delta\lambda).\quad(4.15')$$

The formulas listed above can be used for phenomenological applications, by parametrizing the function f, etc., in a suitable way. In these parametrized expressions, dynamical singularities can be taken into account.

Finally, we should like to remark that Eq. (3.11′) gives the contribution of a family in terms of t-channel helicity amplitudes. To apply this expression to high-energy phenomenology, we have to calculate the crossed-channel helicity amplitudes. The differential cross section can be obtained, however, directly from Eq. (3.11) because of the orthogonality of the crossing matrix.

ACKNOWLEDGMENTS

The authors thank Professor T. Fulton and Professor G. Owen of the Johns Hopkins University for their hospitality. For two of us (MJK and PKK), this work was initiated at The Johns Hopkins University.

APPENDIX A: NORMALIZATION OF REGGE VERTICES

Differentiating the equation $KT=I$ in the variable t, we obtain

$$T\frac{\partial K}{\partial t}T+\frac{\partial T}{\partial t}=0.\quad(A1)$$

Taking the most singular part of the partial-wave projection of the matrix element of Eq. (A1) between states $\langle p_3,E,s,\lambda|$ and $|p_1,E,s',\lambda'\rangle$, we obtain with the help of Eq. (3.5)

$$N_j{}^4(t)\int dq_0dp_0|\mathbf{q}|^2|\mathbf{p}|^2d|\mathbf{p}|d|\mathbf{q}|\psi_{2,j}{}^{(\pm)*}(q_0,|\mathbf{q}|,t)$$
$$\times\frac{\partial K_j{}^{(\pm)}(q_0,|\mathbf{q}|,p_0,|\mathbf{p}|,t)}{\partial t}\psi_{1,j}{}^{(\pm)}(p_0,|\mathbf{p}|,t)$$
$$+N_j{}^2(t)\alpha_\kappa{}^{(\pm)\prime}(t)|_{j=\alpha_\kappa^{(\pm)}(t)}=0.\quad(A2)$$

We shall use a shorthand notation for Eq. (A2):

$$N^4\langle\psi_2|\frac{\partial K_j^{(\pm)}}{\partial t}|\psi_1\rangle\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}+N^2\alpha_\kappa^{(\pm)\prime}(t)=0. \quad (\text{A}2')$$

Using a similar notation, we may write the homogeneous equations for the wave functions as

$$K_{\alpha_\kappa^{(\pm)}(t)}^{(\pm)}|\psi_1\rangle=0 \quad (\text{A}3)$$

and

$$\langle\psi_2|K_{\alpha_\kappa^{(\pm)}(t)}^{(\pm)}=0. \quad (\text{A}4)$$

By making use of Eqs. (A2)–(A4), we obtain

$$0=\frac{d}{dt}\langle\psi_2|K_j^{(\pm)}|\psi_1\rangle\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}$$
$$=\langle\psi_2|\frac{\partial K_j^{(\pm)}}{\partial t}|\psi_1\rangle\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}$$
$$+\alpha_\kappa^{(\pm)\prime}(t)\langle\psi_2|\frac{\partial K_j^{(\pm)}}{\partial j}|\psi_1\rangle\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}. \quad (\text{A}5)$$

Comparing Eqs. (A2′) and (A5), we find

$$N^2=\left[\frac{\partial}{\partial j}\langle\psi_2|K_j^{(\pm)}|\psi_1\rangle\right]^{-1}\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}$$
$$=\bigg[\frac{\partial}{\partial j}\int dq_0 d|\mathbf{q}|\,|\mathbf{q}|^2 dp_0 d|\mathbf{p}|\,|\mathbf{p}|^2$$
$$\times\psi_{2j}^{(\pm)*}(q_0,|\mathbf{q}|,t)\psi_{1j}^{(\pm)}(p_0,|\mathbf{p}|,t)$$
$$\times K_j^{(\pm)}(q_0,|\mathbf{q}|,p_0|\mathbf{p}|,t)\bigg]^{-1}\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}. \quad (\text{A}6)$$

Using the expansions of $\psi_{i,j}^{(\pm)}$ and $K_j^{(\pm)}$ in a series of d functions and diagonalizing, we obtain

$$N^2=\left\{\frac{\partial}{\partial j}[F(j+\kappa,M,t,t_1,\pm t_M)\right\}^{-1}\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}, \quad (\text{A}7)$$

where $F(\sigma,M,t,t_1,\pm t_M)$ is the diagonalized matrix element of operator K defined in Eq. (I 3.32).

As $F(\sigma,M,t,t_1,\pm t_M)$ has a simple zero at $\sigma=f(t,t_1)\pm t_M g(t,t_1)$ [see Eq. (I 3.33)], we may write

$$F(j+\kappa,M,t,t_1,\pm t_M)=\bar{G}(j+\kappa,M,t,t_1,\pm t_M)\times[j+\kappa-f(t,t_1)\mp t_M g(t,t_1)]. \quad (\text{A}8)$$

Substituting Eq. (A8) into Eq. (A7), the normalization constant has the form

$$N^2=G(t,t_1,\pm t_M)[1-(\partial/\partial j)(f\pm t_M g)]|_{j=\alpha_\kappa^{(\pm)}(t)}, \quad (\text{A}9)$$

where $G(t,t_1,\pm t_M)$ is a regular function of its variables at $t=0$.

It is interesting to note that the second multiplier of the right-hand side of Eq. (A9) can be brought to a somewhat different form as well:

$$\left\{1-\frac{\partial}{\partial j}[f(t,t_1)\pm t_M g(t,t_1)]\right\}\bigg|_{j=\alpha_\kappa^{(\pm)}(t)}=\left[\frac{\partial\alpha_\kappa^{(\pm)}(t)}{\partial\sigma}\right]^{-1},$$

where $\sigma\equiv f(0,0)$.

APPENDIX B: PROPERTIES OF REPRESENTATION FUNCTIONS OF HOMOGENEOUS LORENTZ GROUP

We compile here some useful relations for the general d function of the $SL(2,C)$ group. Our sources of information are the works of Sciarrino and Toller,[15] of Freedman and Wang,[23] and of Sebestyen, Szegö, and Toth.[22] We begin with the definition of our notation:

$$d_{s\lambda j}^{\sigma j_0}(\xi)\equiv d_{s\lambda j}^{(a,b)}(\xi)$$
$$\equiv\sum_\mu(a\mu;\,b\lambda-\mu|j\lambda)(a\mu;\,b\lambda-\mu|s\lambda)e^{(2\mu-\lambda)\xi}, \quad (\text{B}1)$$

where $\sigma=a+b$ and $j_0=a-b$. The above definition is continuable in σ and j, keeping $\sigma-j=\kappa$ at fixed integer values; s may be an integer or half-integer, and j_0, $\lambda=s, s-1, \ldots, -s$. All the relations listed here are compatible with these conditions. In the $O(4)$ region ξ is purely imaginary; hence,

$$d_{s\lambda j}^{\sigma j_0 *}(\xi)=d_{s\lambda j}^{\sigma j_0}(-\xi)=d_{j\lambda s}^{\sigma j_0}(-\xi). \quad (\text{B}2)$$

Other symmetry relations are

$$d_{s\lambda j}^{\sigma j_0}(\xi)=d_{s-\lambda j}^{\sigma-j_0}(\xi)=d_{sj_0j}^{\sigma\lambda}(\xi). \quad (\text{B}3)$$

The recursion formula for j_0 is given by

$$[j_0(x+1+2(x-1)xd/dx)+(x-1)(\sigma+1)\lambda]d_{s\lambda j}^{\sigma j_0}(\xi)$$
$$=x^{1/2}\{[(s+j_0)(s-j_0+1)(j+j_0)(j-j_0+1)]^{1/2}$$
$$\times d_{s\lambda j}^{\sigma j_0-1}(\xi)-[(s-j_0)(s+j_0+1)(j-j_0)$$
$$\times(j+j_0+1)]^{1/2}d_{s\lambda j}^{\sigma j_0+1}(\xi)\}, \quad (\text{B}4)$$

where $x=e^{-2\xi}$. The above formula can be interpreted as a recursion formula for λ because of the symmetry relation (B3). There is also a recursion formula for s:

$$(s-1,\lambda;10|s\lambda)\rho_{s-1}{}^s d_{s-1\,\lambda j}^{\sigma j_0}(\xi)+(s+1,\lambda;10|s\lambda)$$
$$\times\rho_{s+1}{}^s d_{s+1\,\lambda j}^{\sigma j_0}(\xi)=i2^{1/2}(d/d\xi)d_{s\lambda j}^{\sigma j_0}(\xi)$$
$$-(s\lambda;10|s\lambda)\rho_s{}^s d_{s\lambda j}^{\sigma j_0}(\xi), \quad (\text{B}5)$$

where

$$\rho_{s-1}{}^s=\left\{\frac{[s^2-(\sigma+1)^2][s^2-j_0{}^2]}{s(s+\frac{1}{2})}\right\}^{1/2},$$

$$\rho_{s+1}{}^s=\left\{\frac{[(s+1)^2-(\sigma+1)^2][(s+1)^2-j_0{}^2]}{(s+1)(s+\frac{1}{2})}\right\}^{1/2}, \quad (\text{B}6)$$

$$\rho_s{}^s=2ij_0(\sigma+1)/s(s+1).$$

For the special case of $j_0=s$, a simple explicit form can

[23] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

be given:

$$d_{s\lambda j}{}^{\sigma s}=\Gamma(\sigma+\lambda+1)\left[\frac{(2j+1)\Gamma(j-\lambda+1)\Gamma(\sigma-s+1)\Gamma(j+s+1)\Gamma(2s+1)}{\Gamma(s+\lambda+1)\Gamma(s-\lambda+1)\Gamma(j+\lambda+1)\Gamma(\sigma+s+2)\Gamma(j-s+1)\Gamma(\sigma+j+2)\Gamma(\sigma-j+1)}\right]^{1/2}$$
$$\times x^{\frac{1}{2}(s-\lambda-\sigma)}(1-x)^{\sigma-s}F(j-\sigma,\ -j-\sigma-1;\ -\sigma-\lambda;\ x/(x-1)),\quad \text{(B7)}$$

or, equivalently, by transforming the hypergeometric function,

$$d_{s\lambda j}{}^{\sigma s}=\frac{(2j+1)^{1/2}}{\Gamma(2j+2)}\left[\frac{\Gamma(j+\lambda+1)\Gamma(\sigma+j+2)\Gamma(j-\lambda+1)\Gamma(j+s+1)\Gamma(2s+2)\Gamma(\sigma-s+1)}{\Gamma(s-\lambda+1)\Gamma(s+\lambda+1)\Gamma(\sigma-j+1)\Gamma(j-s+1)\Gamma(\sigma+s+2)}\right]^{1/2}$$
$$\times x^{\frac{1}{2}(s-\lambda+\sigma)-j}(1-x)^{j-s}F(j-\sigma,\ j-\lambda+1;\ -2j;\ (x-1)/x).\quad \text{(B8)}$$

Finally, we give the generalized Clebsch-Gordan series for the d functions:

$$\begin{pmatrix}\sigma' j_0' & \sigma'' j_0'' & \big| & \sigma j_0\\ s'\lambda' & s''\lambda'' & \big| & s\lambda\end{pmatrix} d_{s\lambda j}{}^{\sigma j_0}(\xi)=\sum_{j'j''}\begin{pmatrix}\sigma' j_0' & \sigma'' j_0'' & \big| & \sigma j_0\\ j'\lambda' & j''\lambda'' & \big| & j\lambda\end{pmatrix} d_{s'\lambda' j'}{}^{\sigma' j_0'}(\xi)\, d_{s''\lambda'' j''}{}^{\sigma'' j_0''}(\xi),\quad \text{(B9)}$$

where the Clebsch-Gordan coefficient couples the two states $|\sigma' j_0' s'\lambda'\rangle$ and $|\sigma'' j_0'' s''\lambda''\rangle$ into $|\sigma j_0 s\lambda\rangle$:

$$|\sigma j_0 s\lambda\rangle=\sum_{s'\lambda' s''\lambda''}\begin{pmatrix}\sigma' j_0' & \sigma'' j_0'' & \big| & \sigma j_0\\ s'\lambda' & s''\lambda'' & \big| & s\lambda\end{pmatrix}|\sigma'' j_0'' s''\lambda''\rangle\otimes|\sigma' j_0' s'\lambda'\rangle.\quad \text{(B10)}$$

It is related to the 9-j symbol through[11]

$$\begin{pmatrix}\sigma' j_0' & \sigma'' j_0'' & \big| & \sigma j_0\\ s'\lambda' & s''\lambda'' & \big| & s\lambda\end{pmatrix}=[(2s'+1)(2s''+1)(2a+1)(2b+1)]^{1/2}(s'\lambda';s''\lambda''|s\lambda)\begin{Bmatrix}a' & a'' & a\\ b' & b'' & b\\ s' & s'' & s\end{Bmatrix},\quad \text{(B11)}$$

where $a+b=\sigma$, $a-b=j_0$, etc.

For the special case of $s''=0$, it simplifies to a 6-j symbol:

$$\begin{pmatrix}\sigma' j_0' & \sigma'' 0 & \big| & \sigma j_0\\ s'\lambda' & 00 & \big| & s\lambda\end{pmatrix}=\left[\frac{(2a+1)(2b+1)}{2a''+1}\right]^{1/2}(-1)^{b+a'+a''+s}\begin{Bmatrix}s & a' & b'\\ a'' & b & a\end{Bmatrix}.\quad \text{(B12)}$$