$O(4)$ Expansion of Off-Shell Scattering Amplitudes and the Most General Form of Regge Trajectories and Residues for Arbitrary Spins. I*

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We work out an analytic method for the discussion of broken symmetries. After the demonstration of the power of the method on a quantum-mechanical example, we apply it to the question of broken $SL(2,C)$ symmetry and families of Regge trajectories. As a result, we obtain all the constraints on daughter trajectory functions in a closed functional form for arbitrary Lorentz quantum numbers σ and M.

I. INTRODUCTION

THE importance of being able to give a group-theoretical classification of the singularities of a relativistic scattering amplitude has long been demonstrated by the works of Domokos and Suranyi,[1,2] of Toller and his collaborators,[3] and of Freedman and Wang.[4] One of the goals of such analysis is to give a general form for the Reggeized scattering amplitudes which automatically satisfy all kinematic requirements. These include (i) factorization of the residues of poles (unitarity), (ii) singularities at the boundaries of the physical regions, (iii) singularities and constraints at the physical thresholds, (iv) singularities and constraints at the pseudothresholds, and (v) absence of singularities at vanishing momentum transfer $t=0$ (when $t=0$ does not coincide with the pseudothreshold).

The first requirement can be satisfied if one regards the pole contribution as a product of two vertex functions and a propagator. The second is automatically satisfied by the usual partial-wave expansion of the helicity amplitudes. The remaining requirements, however, are not as simple. A great deal of effort has been devoted to the point $t=0$ (v) alone.[1,3–5] The outcome is the recognition that, in general, poles exist as members of families. A family usually consists of parent poles and daughter poles of both parities. The singularities and constraints for helicity amplitudes have been studied extensively in recent years,[6] and can be regarded as completely understood. However, the problem of satisfying them simultaneously by a Regge-behaved amplitude has been discussed by few authors[7] and remains largely unsolved. A unique feature of the Regge model is its ability to relate resonance data with scattering data. Therefore, it is important to be able to extrapolate from the physical region of the direct channel to that of the crossed channel through the singularity points. Such understanding is also important in the construction of a more complete future theory, if it is to contain Regge behavior as a part of its features.

One can mention two difficulties in the solution of this kinematic problem. (i) Very little is known about the behavior and structure of the representation functions and the Clebsch-Gordan (C-G) coefficients of the four-dimensional group. (In the analyticity approach,[8,9] which seemingly does not use group theory, the same group-theoretical identities have to be used.) (ii) There is a lack of a set of amplitudes suitable for the four-dimensional harmonic analysis. The usual helicity formalism is useful only if the total spin introduced by Toller[3] and Freedman and Wang[4] at $t=0$ can be generalized to all t values.

In this series of articles we attempt to give a general solution to the problem of a Regge-behaved amplitude satisfying all kinematical requirements. The present article (Paper I) deals with the most general behavior of a family of poles near $t=0$. This problem can be studied separately from the vertex functions, because the positions of the poles of the S matrix do not depend on the physical states used to represent the matrix. Therefore, in this part we may, without loss of generality, restrict ourselves to processes in which the initial (and final) state contains a spinless particle. This way the spin of the remaining particle *is* the total spin, thus bypassing the difficulty (ii) mentioned above. A covariant total spin will be introduced when we consider the structure of vertex functions in Paper II. The other difficulty is

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overcome by a method which we call the method of composite variables.[10]

The key word of our approach is group theory. In general terms, our problem can be stated as follows. Suppose that a symmetry of a dynamical system generated by a group G is broken by some external field V_k transforming like a given representation of G. Let H be the subgroup of G generated by those generators of G which leave V_k invariant; then the symmetry of H remains unbroken by V_k. The Casimir operators M_i of H can be diagonalized together with other good quantum numbers. The dynamical parameters (eigenvalues, eigenfunctions, etc.) will then depend on m_i, the eigenvalues of M_i, in an entirely general way. However, if we require an analytic dependence on V_k (physically this means that the system responds smoothly to a small disturbance V_k), then we obtain severe constraints on the possible m_i dependences of the dynamical quantities. A simple example of this situation is given in Sec. II; the example also serves to illustrate the method of composite variables. The method of composite variables is then applied to the problem of broken $O(4)$ symmetry of relativistic scattering amplitudes in Sec. III, obtaining the main results of this paper. In Sec. IV, we discuss our results and compare them with results derived by using different methods.

II. METHOD OF COMPOSITE VARIABLES

The method can be best illustrated through a simple example of a linear eigenvalue problem in quantum mechanics. Consider a system with rotational symmetry being placed in a homogeneous external vector field $\mathbf{V}$. The Hamiltonian $\mathcal{H}(\mathbf{V})$ is not an invariant under rotations of the system except when $\mathbf{V}=0$. However, $\mathcal{H}$ still transforms like a scalar if both the system and $\mathbf{V}$ are rotated together. This means that $(\partial/\partial V_i)\mathcal{H}(\mathbf{V})|_{\mathbf{V}=0}$ transforms like a vector and $(\partial^2/\partial V_i\partial V_j)\mathcal{H}(\mathbf{V})|_{\mathbf{V}=0}$ like a tensor, etc. In other words, $\mathcal{H}$ has the expansion in terms of harmonic polynomials of $\mathbf{V}$:

$$\mathcal{H}(\mathbf{V})=\sum_{l,m} H_m{}^l(V^2)V^lY_m{}^l(\hat{V})\,, \qquad (2.1)$$

where $H_m{}^l(V^2)$ is an irreducible spherical tensor operator. $H_m{}^l(V^2)$ depends analytically on V^2 near zero if $\mathcal{H}$ does on $\mathbf{V}$. We shall assume this to be true. The eigenfunctions of $\mathcal{H}(0)$ are chosen as a basis set, with angular momentum quantized along $\mathbf{V}$. Our method begins to depart from the conventional perturbation approach by considering the general dependence on m of the matrix element of $\mathcal{H}$:

$$\langle jm\alpha|\mathcal{H}|j'm'\alpha'\rangle =\delta_{mm'}\sum_l V^l\langle jml0|j'm\rangle\langle j\alpha||H^l(V^2)||j'\alpha'\rangle\,, \qquad (2.2)$$

where α stands for quantum numbers other than jm. Since m remains a good quantum number when $V\neq 0$, $\mathcal{H}$ is diagonal in m and the dependence on m is specified by the C-G coefficient, which has the following form:

$$\langle jml0|j'm\rangle=\left(\frac{(j_>+m)!(j_>-m)!}{(j_<+m)!(j_<-m)!}\right)^{1/2} \times[\alpha m^{l-|j-j'|}+\beta m^{l-|j-j'|-2}+\cdots]\,, \qquad (2.3)$$

where $j_>$ ($j_<$) is the greater (lesser) one of j, j', and α, β, ... are independent of m. This form of the C-G coefficient allows one to absorb one power of V into each power of m in (2.2) and to write the matrix element in the form

$$\langle jm\alpha|\mathcal{H}|j'm'\alpha'\rangle=\delta_{mm'}V^{|j-j'|}\left(\frac{(j_>+m)!(j_>-m)!}{(j_<+m)!(j_<-m)!}\right)^{1/2} \times H_{j\alpha,j'\alpha'}(V,mV)\,. \qquad (2.4)$$

$H_{j\alpha,j'\alpha'}$ depends analytically on V and the composite variable mV near $V=mV=0$. We see that if it were not for the square-root factor in (2.4), we could have concluded immediately that the eigenvalues of $\mathcal{H}$ must be analytic functions of V and mV. In a simple case like this, the square-root factors can be eliminated by multiplying the columns and rows of $\mathcal{H}$ by suitable factors which do not affect the eigenvalues. For the purpose of a more general treatment, we prove that this is true to any order in a Rayleigh-Schrödinger perturbation theory. Let us denote by $H_{jj'}$ the matrix element of $\mathcal{H}$ (suppressing the index α) and by E_j the eigenvalue which approaches H_{jj} as $V\to 0$. Then to any finite order the equation which determines E_j will have coefficients composed of quantities like

$$H_{jj},\ H_{jj_1}H_{j_1j},\ H_{jj_1}H_{j_1j_2}H_{j_2j},\ \text{etc.},\quad j_1,j_2,\ldots\neq j\,.$$

It is easy to see that none of these quantities will contain a square-root factor. Thus it follows that the eigenvalues of $\mathcal{H}$ will be analytic functions of V and mV:

$$E_{jm\alpha}(V)=F_{j\alpha}(V,mV)\,. \qquad (2.5)$$

By expanding $F_{j\alpha}$ in a Taylor series of two variables, all the results of the conventional perturbation method follow immediately.

It should be noted that the above result is not true for a degenerate level[11] because a secular equation would have been solved and nonanalytic dependence may be introduced. However, if the degeneracy can be removed by a discrete symmetry, the method of composite variables can still be applied, as shown in Sec. III.

III. RELATIVISTIC TWO-BODY PROBLEM

A. Off-Shell Scattering Amplitude as Function on Lorentz Group

In this section we turn to the main application of our method of composite variables; namely, we discuss

[10] A brief account of this method and the main results of this paper have been reported earlier by us [Phys. Rev. Letters 22, 1025 (1969)].

[11] We wish to thank Gabor Domokos for a discussion on this point.

some important properties of the relativistic two-particle–two-particle transition amplitude.

To simplify the description of a scattering process, one usually diagonalizes the amplitude in quantum numbers which correspond to operators commuting with the scattering operator. So, if the total four-momentum of the two particles in the initial state is a timelike four-vector, one can expand in eigenstates of the total angular momentum operator. For zero total four-momentum (or four-momentum transfer) $E_\mu=0$, the little group is larger; it coincides with the homogeneous Lorentz group. In this case the natural generalization of the angular momentum expansion is that in terms of the eigenfunctions of the Casimir operators of the homogeneous Lorentz group. The larger symmetry at the point $E_\mu=0$ is reflected by the particle spectrum as well. Regge poles are grouped together into infinite families.[1–4]

Our aim is to study the behavior of Regge trajectories belonging to a family at $E_\mu=0$ (unequal external masses and/or $E_\mu^2=t=0$) by group-theoretical methods. However, if we want to go beyond the information obtained from a simple little-group expansion, we are forced to work with an off-shell scattering amplitude. In what follows, we assume the existence of the inverse of this amplitude satisfying the equation

$$\sum_b \int d^4q_1 d^4q_2 \delta^4(p_1+p_2-q_1-q_2) K_{ab}(p_1,p_2,q_1,q_2) \times T_{bc}(q_1,q_2,l_1,l_2)=\delta_{ac}\delta^4(p_1-l_1)\delta^4(p_2-l_2), \quad (3.1)$$

where K_{ab} is the inverse of the off-shell scattering amplitude T_{bc}; p_1, p_2, q_1, q_2, and l_1, l_2 are the momenta of scattered particles in the final, intermediate, and initial states, respectively. Similarly, a, b, and c stand for the discrete variables. In what follows, we assume the possibility of Wick's rotation and work in Euclidean metrics. We merely remark that our general considerations will not be affected even by the presence of complex singularities. We use the terms "Lorentz transformation" and "Lorentz group" for the corresponding compact transformations and group $[SU(2)\otimes SU(2) \sim O(4)]$ throughout this paper.

Equation (3.1) can be regarded as an operator equation $KT=I$, where operators K and T are represented in the space of off-shell two-particle states $|p_1,p_2;a\rangle$, satisfying the orthogonality relation

$$\langle p_1,p_2;a|q_1,q_2;b\rangle=\delta(p_1-q_1)\delta(p_2-q_2)\delta_{ab} \quad (3.2)$$

and the completeness relation

$$I=\sum_a \int d^4p_1 d^4p_2 |p_1,p_2;a\rangle\langle p_1,p_2;a| \quad (3.3)$$

in the subspace of two-particle states of given type. We often refer to K_{ab} as the kernel of Eq. (3.1).

The two-particle state can be regarded as a function on the group $O(4)\otimes O(4)$,[3]

$$|L_1,L_2;a\rangle=U_1(L_1)U_2(L_2)|p_{10},p_{20};a\rangle, \quad (3.4)$$

where only the time components of the vectors, p_{10} and p_{20}, differ from zero: $p_1^2=p_{10}^2$ and $p_2^2=p_{20}^2$. The Lorentz transformations L_1 and L_2 commute with each other. The generators of these groups can be given in terms of differential operators in the momentum spaces. These operators are certainly well defined for matrix elements which have a regular dependence on momenta. Presumably the kernel K_{ab} is such a regular function of momenta (every Feynman diagram is a regular function of momenta in the Euclidean region). Writing the two-particle states in form (3.4), we regard $K_{ab}(p_1,p_2;q_1,q_2)$ as a function on the $O(4)\otimes O(4)\otimes O(4)\otimes O(4)$ group if we write

$$\langle L_1,L_2;a|K|\tilde{L}_1,\tilde{L}_2;b\rangle =\delta(p_1+p_2-q_1-q_2)K_{ab}(L_1,L_2;\tilde{L}_1,\tilde{L}_2). \quad (3.5)$$

The Lorentz transformations L_1, L_2, $\tilde{L}_1$, and $\tilde{L}_2$ are not entirely independent, however, because of momentum conservation.

Lorentz invariance implies

$$K_{ab}(LL_1,LL_2;L\tilde{L}_1,L\tilde{L}_2)=K_{ab}(L_1,L_2;\tilde{L}_1,\tilde{L}_2). \quad (3.6)$$

The covariance conditions for $K_{ab}(L_1,L_2;\tilde{L}_1,\tilde{L}_2)$ have the following form (here we write out explicitly the spin and helicity indices of particle 1 in the final state):

$$K_{s_1\lambda_1,\ldots}(L_1R,L_2;\tilde{L}_1,\tilde{L}_2) =\sum_{\mu_1} D_{\mu_1\lambda_1}{}^{s_1}(R)K_{s_1\mu_1,\ldots}(L_1,L_2;\tilde{L}_1,\tilde{L}_2). \quad (3.7)$$

Similar equations hold if we modify the other arguments of $K_{ab}(L_1,L_2;\tilde{L}_1,\tilde{L}_2)$ by a rotation. One can greatly simplify Eq. (3.1) by introducing the total momentum $E=p_1+p_2=q_1+q_2$ and some different combinations of momenta p_1 and p_2 and of momenta q_1 and q_2 as integration variables:

$$\sum_b \int d^4q\, K_{ab}(p,q;E)T_{bc}(q,l;E)=\delta_{ac}\delta^4(p-l). \quad (3.8)$$

To find the covariance properties of the kernel $K_{ab}(p,q;E)$, we have to study the two-particle states $|E;p;a\rangle\equiv|p_1,p_2;a\rangle$. Similarly, for the Lorentz groups acting on momenta p_1 and p_2, we can define the Lorentz groups acting on momenta p and E. We have to emphasize, however, that the generators of this new $O(4)\otimes O(4)$ group *cannot* be expressed by the generators of the $O(4)\otimes O(4)$ group transforming p_1 and p_2. This is trivially shown by the fact that the invariants of this group are different (p^2,E^2 and p_1^2, p_2^2). There is an $O(4)$ subgroup, however, which is identical in both groups, namely, the one which corresponds to the simultaneous Lorentz transformation of p and E or p_1 and p_2.

We remark that the definition of the two-particle state vectors as functions on the group $O(4)\otimes O(4)$, acting on p and E, is not a trivial question and the equality $|E;p;a\rangle\equiv|p_1,p_2;a\rangle$ is, generally speaking, not enough for the definition of this function. To simplify matters, in the first part of our paper we study the simple problems in which particles with momentum vectors p_2 and q_2 have zero spin. Such a special choice of processes certainly enables us to discuss any family of Regge poles. In addition, we use the freedom in the definition of p and q, choosing $p=p_1$ and $q=q_1$, For this simple case, it is comparatively easy to define the state vector $U_E(L_E)U_1(L_1)|E_0;p_0;s,\lambda\rangle$ as a function on the $O(4)\otimes O(4)$ group consistently with the group properties, the equation $|E;p;s,\lambda\rangle=|p_1,p_2;s,\lambda\rangle$, and the invariance and covariance properties of the vector $U_1(L_1)U_2(L_2)|p_{10},p_{20};s,\lambda\rangle$.

We represent Lorentz transformations in the form

$$L=R_z(\varphi)R_y(\vartheta)B(\xi)R_z(\alpha)R_y(\beta)R_z(\gamma),$$

where $R_i(\varphi)$ is a rotation around the i axis by an angle φ, while $B(\xi)$ is a boost along the z axis by a hyperbolic angle ξ. We denote by Λ the following type of special Lorentz transformation:

$$\Lambda=R_z(\varphi)R_y(\vartheta)B(\xi).$$

A four-momentum vector p uniquely determines a Lorentz transformation Λ, which "reconstructs" p from a "normal" vector $p_0=(\sqrt{(p^2)},0,0,0)$. With the help of this fact, one can define uniquely $U_E(L_E)U_1(L_p)\times|E_0;p_{10};s,\lambda\rangle$ on the subset of the group $O(4)\otimes O(4)$ in which $L_E=\Lambda_E$, $L_p=\Lambda_p$ by

$$U_E(\Lambda_E)U_p(\Lambda_1)|E_0,p_{10};s,\lambda\rangle =U_1(\Lambda_1)U_2(\Lambda_2)|p_{10},p_{20};s,\lambda\rangle, \quad (3.9)$$

where the Lorentz transformations Λ_E, Λ_1, and Λ_2 are not pure rotations. To extend this definition, we apply an arbitrary simultaneous Lorentz transformation to Eq. (3.9). As we mentioned, simultaneous Lorentz transformations are well defined on both sides. Using this fact and the covariance property of the state vector $U_1(L_1)U_2(L_2)|p_{10},p_{20};s,\lambda\rangle$, we obtain, after the repeated use of Eq. (3.9),

$$\begin{aligned}U_E(\tilde{\Lambda}_E)U_p(\tilde{\Lambda}_1)|E_0,p_{10};s,\lambda'\rangle D_{\lambda'\lambda}{}^s(\tilde{\Lambda}_1{}^{-1}L\Lambda_1)\\ =U_E(\tilde{\Lambda}_E)U_p(\tilde{\Lambda}_1)U_E(\tilde{\Lambda}_E{}^{-1}L\Lambda_E)\\ \times U_p(\tilde{\Lambda}_1{}^{-1}L\Lambda_1)|E_0;p_{10};s,\lambda\rangle, \quad (3.10)\end{aligned}$$

where L is an arbitrary Lorentz transformation, $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_E$ are the Lorentz transformations belonging to the transformed p_1 and E vectors, while $\tilde{\Lambda}_1{}^{-1}L\Lambda_1$ and $\tilde{\Lambda}_E{}^{-1}L\Lambda_E$ are Wigner rotations. Our basic requirement on the function $U_E(L_E)U_p(L_1)|E_0;p_{10};s,\lambda\rangle$ is the relation

$$U_E(L_3)U_p(L_4)U_E(L_E)U_p(L_1)|E_0,p_{10};s,\lambda\rangle =U_E(L_3L_E)U_p(L_4L_1)|E_0;p_{10};s,\lambda\rangle. \quad (3.11)$$

Applying the transformation $U_E(\tilde{\Lambda}_E)^{-1}U_p(\tilde{\Lambda}_1)^{-1}$ to both sides of Eq. (3.10), we obtain

$$U_E(R_1)U_p(R_2)|E_0,p_{10};s,\lambda\rangle =\sum_\mu |E_0,p_{10};s,\mu\rangle D_{\mu\lambda}{}^s(R_2). \quad (3.12)$$

The state

$$|L_E,L_p,s,\lambda\rangle=|\Lambda_E,L_p,s,\lambda\rangle=U_E(L_E)U_p(L_p)|E_0,p_{10};s\lambda\rangle$$

is completely determined by Eqs. (3.9) and (3.12). As the operator K is diagonal in E, the reduced matrix element $K_{s\lambda s'\lambda'}(p,q,E)$ is defined as a function on the group $O(4)\otimes O(4)\otimes O(4)$ with the invariance property

$$K_{s\lambda s'\lambda'}(LL_p,LL_q,LL_E)=K_{s\lambda s'\lambda'}(L_p,L_q,L_E) \quad (3.13)$$

and the covariance property

$$K_{s\lambda s'\lambda'}(L_pR_p,L_qR_q,L_ER_E) =\sum_{\mu\mu'} D_{\mu\lambda}{}^s(R_p)D_{\mu'\lambda'}{}^{s'}(R_q)K_{s\mu s'\mu'}(L_p,L_q,L_E). \quad (3.14)$$

Using Eq. (3.12), we can write the completeness relation of two-particle states in the following way:

$$\begin{aligned}I&=\sum_\lambda\int d^4Ed^4p\ |E,p;s\lambda\rangle\langle E,p;s,\lambda|\\ &=\sum_\lambda\int d\Lambda_E dL_p\frac{E^2dE_p{}^2dp^2}{32\pi^2}U_E(\Lambda_E)U_p(L_p)|E_0,p_0;s\lambda\rangle\\ &\qquad\times\langle E_0,p_0;s\lambda|U_E{}^\dagger(\Lambda_E)U_p{}^\dagger(L_p). \quad (3.15)\end{aligned}$$

We close this part of the paper with the description of the method applied below for the derivation of several properties of Regge trajectories and residues. Let us write Eq. (3.8) in the angular momentum representation. Using the conservation of angular momentum in the c.m. system $[E_\mu=(E,0,0,0)]$, we obtain

$$\sum_b\int K_{ab}{}^j(p_0,p;q_0,q;E)T_{bc}{}^j(q_0,q;l_0,l;E)dq_0q^2dq =\delta_{ac}\delta(p_0-l_0)(1/p^2)\delta(p-l), \quad (3.16)$$

where we have introduced the notation p, q, and l for the absolute value of the three-momenta **p**, **q**, and **l**. A pole of $T_{bc}{}^j$ in the variable j at $j=\alpha$ has a contribution, using the factorization theorem,

$$T_{bc}{}^j(q_0,q;l_0,l;E)\approx\frac{\psi_b{}^{(1)}(q_0,q;E)\psi_c{}^{(2)}(l_0,l;E)}{j-\alpha}. \quad (3.17)$$

It is easy to conclude that the Regge vertex $\psi^{(1)}$, or, as we shall often call it, the wave function, satisfies the homogeneous equation

$$\sum_b\int K_{ab}{}^\alpha(p_0,p;q_0,q;E)\psi_b{}^{(1)}(q_0,q;E)dq_0q^2dq=0. \quad (3.18)$$

Equation (3.18) can be regarded as an eigenvalue equation for α, where the eigenfunction ψ is the Regge vertex. If we diagonalize $K_{ab}{}^{j}$ in its indices and in its continuous variables, the zeros of the diagonalized matrix elements will give us the eigenvalues α_i. The eigenvalue problem defined by Eq. (3.18) is certainly much more involved than the one discussed in Sec. II. The main trouble is that the zeros of the kernel K [the solutions of Eq. (3.18)] do not lie at physical values of the angular momentum, j, for arbitrary values of $E_\mu{}^2=t$. So Eq. (3.18) has to be "solved" in two steps. In the first step we define an auxiliary parameter λ and we solve the equation $(K-\lambda)\psi=0$, by expanding K and ψ in representation functions of the homogeneous Lorentz group and diagonalizing the resulting matrix (that is to say, we solve the characteristic equation for λ). The eigenvalues λ will depend on t and the diagonalized "angular momentum" variables σ, j_0, j, and m (actually λ does not depend on m because of the Wigner-Eckart theorem):

$$\lambda=F(\sigma,j_0,j,t)\,. \tag{3.19}$$

In Eq. (3.19), σ, j, and j_0 have the values corresponding to unitary representations of the group $O(4)$: $\sigma-j=\kappa\geqslant 0$; $j-|j_0|=\eta\geqslant 0$; both κ and η are integers; j_0 is an integer or half-integer.

However, as we are interested in the solution of the equation $K\psi=0$ instead of $(K-\lambda)\psi=0$, we have to find such values of parameters for which λ vanishes. This can be accomplished by an analytic continuation in j, keeping κ fixed. The solution of the eigenvalue problem will be

$$F(j+\kappa,j_0,j,t)=0\,,$$

where j is some complex number, depending on j_0, κ, and t.

B. Expansion of Kernel and Wave Function

The above-defined function on the $O(4)\otimes O(4)\otimes O(4)$ group can be expanded in representations of the group $O(4)$ owing to the analyticity of K and ψ in the Euclidean region:

$$K_{s\lambda s'\lambda'}(L_p,L_q,L_E)=\sum_{\sigma,j_0,jm}\sum_{\sigma',j_0',j',m'}\sum_{n,l,\mu} t^{n/2}K_{jmj'm'l\mu}{}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)\times D_{jms\lambda}{}^{\sigma j_0}(L_p)D_{s'\lambda' j'm'}{}^{\sigma' j_0'}(L_q^{-1})D_{l\mu 00}{}^{n0}(L_E)\,. \tag{3.20}$$

The special values for the lower indices of the D functions, s, λ, s', λ', and 0, 0, follow from Eq. (3.14). It is easy to prove that $K_{jmj'm'l\mu}{}^{\sigma j_0\sigma' j_0' n}$ does not depend on the helicities λ and λ'. The Lorentz invariance of the kernel expressed by Eq. (3.13) requires

$$K_{jmj'm'l\mu}{}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)=\begin{pmatrix}\sigma j_0 & \sigma' j_0' & \big| & n0\\ jm & j'm' & \big| & l\mu\end{pmatrix}\bar{K}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)\,, \tag{3.21}$$

where

$$\begin{pmatrix}\sigma j_0 & \sigma' j_0' & \big| & n0\\ jm & j'm' & \big| & l\mu\end{pmatrix}$$

is a C-G coefficient of the $O(4)$ group. $\bar{K}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)$ is a regular function of t.

In the rest of our discussion we select a special frame by choosing $L_E=I$, the identity transformation. Then Eq. (3.20) simplifies to

$$K_{s\lambda s'\lambda'}(L_p,L_q,I)=\sum_{\sigma j_0 jm\sigma' j_0' n} t^{n/2}D_{jms\lambda}{}^{\sigma j_0}(L_p)D_{s'\lambda' jm}{}^{\sigma' j_0'}(L_q^{-1})\times\bar{K}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)\times\begin{Bmatrix} j & \frac{1}{2}(\sigma+j_0) & \frac{1}{2}(\sigma-j_0)\\ \frac{1}{2}n & \frac{1}{2}(\sigma'-j_0') & \frac{1}{2}(\sigma'+j_0')\end{Bmatrix}, \tag{3.22}$$

where we have used the explicit expression of the C-G coefficient in Eq. (3.20) in terms of the $6j$ symbol for the case $l=\mu=0^2$.

Similarly to K, we can expand the wave function ψ as well:

$$\psi_{s\lambda}(q,E)\equiv\psi_{s\lambda}(L_q,L_E)=\sum_{\sigma j_0 jm} D_{jms\lambda}{}^{\sigma j_0}(L_q)\psi_j{}^{\sigma j_0}(q^2,L_E)\,. \tag{3.23}$$

Using the completeness relation (3.15) and expansions (3.22) and (3.23) and the orthogonality properties of the functions $D_{s\lambda jm}{}^{\sigma j_0}(L)$, we obtain from the equation $(K-\lambda)\psi=0$

$$\int dq^2q^2\sum_{\sigma' j_0'}\left[\tilde{K}_j{}^{\sigma j_0\sigma' j_0'}(p^2,q^2,t)-\frac{\lambda\delta(p^2-q^2)}{p^2}\delta_{\sigma\sigma'}\delta_{j_0j_0'}\right]\psi_j{}^{\sigma' j_0'}(q^2,t)=0\,, \tag{3.24}$$

where

$$\tilde{K}_j{}^{\sigma j_0\sigma' j_0'}(p^2,q^2,t)=\sum_n t^{n/2}\times\begin{Bmatrix} j & \frac{1}{2}(\sigma+j_0) & \frac{1}{2}(\sigma-j_0)\\ \frac{1}{2}n & \frac{1}{2}(\sigma'-j_0') & \frac{1}{2}(\sigma'+j_0')\end{Bmatrix}\bar{K}^{\sigma j_0\sigma' j_0' n}(p^2,q^2,t)\,.$$

In what follows, we suppress the variables p^2 and q^2, because they are irrelevant from the point of view of our subsequent considerations.

Before we start with the discussion of the properties of the solution of the characteristic equation

$$\|K_j{}^{\sigma j_0\sigma' j_0'}(t)-\lambda\|=0\,,$$

we have to discuss the question of discrete symmetries. Parity will play an important role, because parity conservation requires

$$\tilde{K}_j^{\sigma j_0 \sigma' j_0'}(t) = \tilde{K}_j^{\sigma -j_0, \sigma' -j_0'}(t).$$

At $t=0$, $K_j^{\sigma j_0 \sigma' j_0'}(0) \equiv K^{\sigma j_0}\delta_{\sigma\sigma'}\delta_{j_0 j_0'}$ is diagonal and the equality of the "unperturbed eigenvalues" $K^{\sigma j_0} = K^{\sigma - j_0}$ introduces an accidental degeneracy. To resolve this degeneracy, we introduce states and wave functions with definite parity. This can be done most easily in the angular momentum representaion. The partial-wave projection of $K_{s\lambda s'\lambda'}(L_p, L_q, I)$ can be given if we choose $L_p = \Lambda_p$ and $L_q = \Lambda_q$ (then $K_{s\lambda s'\lambda'}$ is the helicity amplitude). Using the addition theorem for the functions $D_{mm'}{}^j(R)$, we obtain

$$\sum_m D_{jms\lambda}{}^{\sigma j_0}(\Lambda_p) D_{s'\lambda' jm}{}^{\sigma' j_0'}(\Lambda_q^{-1}) = d_{j\lambda s}{}^{\sigma j_0}(B_p) D_{\lambda\lambda'}{}^j(R) d_{j\lambda' s'}{}^{\sigma' j_0'}(B_q),$$

where B_p and B_q are the boost parts of the Lorentz transformations Λ_p and Λ_q, respectively, while R is a rotation transforming the three-vector $\mathbf{p}$ into the three-vector $\mathbf{q}$. The partial-wave amplitude is given by

$$K_{s\lambda s'\lambda'}{}^j(B_p, B_q, I) = \sum_{\sigma j_0 \sigma' j_0' n} t^{n/2} d_{j\lambda s}{}^{\sigma j_0}(B_p) K^{\sigma j_0 \sigma' j_0' n}(t) \times d_{j\lambda' s'}{}^{\sigma' j_0'}(B_q) \begin{Bmatrix} j & \frac{1}{2}(\sigma + j_0) & \frac{1}{2}(\sigma - j_0) \\ \frac{1}{2}n & \frac{1}{2}(\sigma' - j_0') & \frac{1}{2}(\sigma' + j_0') \end{Bmatrix}. \quad (3.25)$$

The parity-conserving partial-wave amplitudes are defined by

$$K_{s\mu s'\mu'}{}^{j(\pm)} = K_{s\lambda s'\lambda'}{}^j \pm K_{s\lambda s' -\lambda'}{}^j,$$

where $\mu = |\lambda|$ and $\mu' = |\lambda'|$. Using the symmetry relation $d_{s\lambda j}{}^{\sigma j_0}(\alpha) = d_{s-\lambda j}{}^{\sigma - j_0}(\alpha)$ and the expansion (3.25), we arrive at the expansion

$$K_{s\mu s'\mu'}{}^{j(\pm)}(B_p, B_q, I) = \sum_{\sigma M \sigma' M'} d_{j\mu s}{}^{\sigma M(\pm)}(B_p) \times K_j{}^{\sigma M \sigma' M'(\pm)}(t) d_{j\mu' s'}{}^{\sigma' M'(\pm)}(B_q), \quad (3.26)$$

where $M \geqslant 0$, $M' \geqslant 0$,

$$d_{j\lambda s}{}^{\sigma M(\pm)}(\alpha) = d_{j\lambda s}{}^{\sigma M}(\alpha) \pm d_{j\lambda s}{}^{\sigma - M}(\alpha), \quad (3.27)$$

and

$$K_j{}^{\sigma M \sigma' M'(\pm)}(t) = \sum_n t^{n/2} \left[\begin{Bmatrix} j & \frac{1}{2}(\sigma + M) & \frac{1}{2}(\sigma - M) \\ \frac{1}{2}n & \frac{1}{2}(\sigma' - M') & \frac{1}{2}(\sigma' + M') \end{Bmatrix} K_{ss'}{}^{\sigma M \sigma' M' n}(t) \pm \begin{Bmatrix} j & \frac{1}{2}(\sigma + M) & \frac{1}{2}(\sigma - M) \\ \frac{1}{2}n & \frac{1}{2}(\sigma' + M') & \frac{1}{2}(\sigma' - M') \end{Bmatrix} K^{\sigma M \sigma' - M' n}(t) \right]. \quad (3.28)$$

We can diagonalize $K_j{}^{\sigma M \sigma' M'(+)}(t)$ and $K_j{}^{\sigma M \sigma' M'(-)}(t)$ independently [the functions $d_{js\lambda}{}^{\sigma j_0(+)}(\alpha)$ and $d_{js\lambda}{}^{\sigma j_0(-)}(\alpha)$ are orthogonal], so that the accidental degeneracy is resolved. In the following part of the paper we shall discuss the determination of the eigenvalues of matrices $K_j{}^{\sigma M \sigma' M'(\pm)}(t)$ in detail.

C. Diagonalization of Kernel and Equations for Trajectories

As in the example of Sec. II, we start with the investigation of the structure of the C-G coefficients. By the recursion formula[12] for the $6j$ symbol, one can establish their j dependence:

$$\begin{Bmatrix} j & \frac{1}{2}(\sigma + M) & \frac{1}{2}(\sigma - M) \\ \frac{1}{2}n & \frac{1}{2}(\sigma' - M') & \frac{1}{2}(\sigma' + M') \end{Bmatrix} = \left(\frac{\Gamma(\sigma_> + j + 2)\Gamma(\sigma_> - j + 1)\Gamma(j - M_< + 1)\Gamma(j + M_> + 1)}{\Gamma(\sigma_< + j + 2)\Gamma(\sigma_< - j + 1)\Gamma(j - M_> + 1)\Gamma(j + M_< + 1)} \right)^{1/2} P_1{}^{(n-n_0)/2}(j(j+1)) \quad (3.29)$$

and

$$\begin{Bmatrix} j & \frac{1}{2}(\sigma + M) & \frac{1}{2}(\sigma - M) \\ \frac{1}{2}n & \frac{1}{2}(\sigma' + M') & \frac{1}{2}(\sigma' - M') \end{Bmatrix} = \left(\frac{\Gamma(\sigma_> + j + 2)\Gamma(\sigma_> - j + 1)\Gamma(j - M_< + 1)\Gamma(j + M_> + 1)}{\Gamma(\sigma_< + j + 2)\Gamma(\sigma_< - j + 1)\Gamma(j - M_> + 1)\Gamma(j + M_< + 1)} \right)^{1/2} \left(\frac{\Gamma(j + M_< + 1)}{\Gamma(j - M_< + 1)} \right) P_2{}^{(n-n_0')/2}(j(j+1)), \quad (3.30)$$

where $n_0 = |\sigma - \sigma'| + |M - M'|$, $n_0' = |\sigma - \sigma'| + M + M'$, and $P^m(x)$ stands for an mth-order polynomial in x. With these relations, we see that the two composite variables are

$$t_1 = tj(j+1) \quad \text{and} \quad t_{M_<} = t^{M_<}[\Gamma(j + M_< + 1)/\Gamma(j - M_< + 1)].$$

[12] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957), p. 98.

Just as in the case in Sec. II, the j dependence of (3.28) can be summarized as

$$K_j^{\sigma M\sigma' M'}(\pm)=t^{n_0/2}\left(\frac{\Gamma(\sigma_>+j+2)\Gamma(\sigma_>-j+1)\Gamma(j-M_<+1)\Gamma(j+M_>+1)}{\Gamma(\sigma_<+j+2)\Gamma(\sigma_<-j+1)\Gamma(j-M_>+1)\Gamma(j+M_<+1)}\right)^{1/2}[A(t,t_1)\pm t_{M<}B(t,t_1)], \quad (3.31)$$

where A and B are arbitrary analytic functions of t and t_1. It is tedious but straightforward to check that the product $K_j^{\sigma M\sigma'' M''}(\pm)K_j^{\sigma'' M''\sigma' M'}(\pm)$ has just the same j dependence as (3.31), It follows that the quantities

$$K_j^{\sigma M\sigma' M'}(\pm)K_j^{\sigma' M'\sigma M}(\pm),$$
$$K_j^{\sigma M\sigma' M'}(\pm)K_j^{\sigma' M'\sigma'' M''}(\pm)K_j^{\sigma'' M''\sigma M}(\pm), \text{ etc.,}$$

will all be free from the square-root factor in (3.28) and depend analytically on t, t_1, and $\pm t_M$. Barring the chance of accidental degeneracy, we may conclude that after diagonalization the elements of K will be of the form

$$F^{\sigma M}(\pm)=F(\sigma, M, t, t_1, \pm t_M). \quad (3.32)$$

We may now continue j in the complex plane, keeping $\sigma-j=\kappa$ fixed, to seek the solution of $F=0$. Let the solution be $j=\alpha_\kappa^{(\pm)}$; we can write

$$\alpha_\kappa^{(\pm)}+\kappa=\mathfrak{F}(t, t_1, \pm t_M),$$

or, equivalently, since t_M^2 can be expressed as a polynomial in t and t_1,

$$\alpha_\kappa^{(\pm)}+\kappa=f(t,t_1)\pm t_M g(t,t_1), \quad (3.33)$$

where f and g are regular in t and t_1 near zero.

IV. DISCUSSION OF RESULTS

Equation (3.33) gives all the results implied by analyticity on the behavior of energy levels. The functional forms of f and g are, of course, dynamical questions. Once they are known, the trajectory functions $\alpha_\kappa^{(\pm)}$ can simply be solved as functions of t (or E). However, even without any knowledge of f and g, the fact that they are analytic still gives all the constraints among $\alpha_\kappa^{(\pm)}$.

Let us first examine (3.33) for some special values of M. If $M=0$, we have, since $t_0=1$,

$$\alpha_\kappa^{(\pm)}+\kappa=f(t,t_1)\pm g(t,t_1)=g^{(\pm)}(t,t_1). \quad (4.1)$$

We see that the two trajectories of opposite parities are completely unrelated (no conspiracy); we shall see that this is not so for all other values of M (conspiracy). If $g^{(\pm)}$ is expanded in a Taylor series (suppressing the superscript),

$$g(t,t_1)=\sigma+g_{10}t+g_{01}t_1+g_{20}t^2+g_{11}tt_1+g_{02}t_1^2+\cdots,$$

then the Taylor series for α_κ can be obtained by iteration:

$$\begin{aligned}\alpha_\kappa(t)=\sigma-\kappa+[g_{10}+g_{01}(\sigma-\kappa)(\sigma-\kappa+1)]t\\ +\{g_{20}+g_{11}(\sigma-\kappa)(\sigma-\kappa+1)+g_{02}(\sigma-\kappa)^2(\sigma-\kappa+1)^2\\ +g_{01}[g_{10}+g_{01}(\sigma-\kappa)(\sigma-\kappa+1)]\\ \times(2\sigma-2\kappa+1)\}t^2+\cdots. \quad (4.2)\end{aligned}$$

This agrees with results obtained by more elaborate calculations.[2,9] For $M=\frac{1}{2}$, (3.33) becomes

$$\alpha_\kappa^{(\pm)}+\kappa=f(t,t_1)\pm t_{1/2}g(t,t_1). \quad (4.3)$$

In terms of $E=\sqrt{t}$, it takes a simpler form because $t_1=t\alpha_\kappa^{(\pm)}(\alpha_\kappa^{(\pm)}+1)$ can be expressed in terms of $t_{1/2}\equiv E(\alpha_\kappa^{(\pm)}+\frac{1}{2})$ and t:

$$\alpha_\kappa^{(\pm)}+\kappa=h(E^2, \pm E(\alpha_\kappa^{(\pm)}+\tfrac{1}{2})). \quad (4.4)$$

By expanding h, we obtain the previously reported result[2,13]:

$$\begin{aligned}\alpha_\kappa^{(\pm)}(E)=\sigma-\kappa\pm h_{01}(\sigma-\kappa+\tfrac{1}{2})E\\ +[h_{10}+h_{01}^2(\sigma-\kappa+\tfrac{1}{2})+h_{02}(\sigma-\kappa+\tfrac{1}{2})^2]E^2+\cdots. \quad (4.5)\end{aligned}$$

A similar result for $M=1$ is also given below[14]:

$$\begin{aligned}\alpha_\kappa^{(\pm)}(t)=\sigma-\kappa+[f_{10}+(f_{01}\pm g_{00})X]t\\ +\{f_{20}+(f_{11}\pm g_{10})X+(f_{02}\pm g_{01})X^2+(f_{01}\pm g_{00})\\ \times[f_{10}-(f_{01}\pm g_{00})X](2\sigma-2\kappa+1)\}t^2+\cdots, \quad (4.6)\end{aligned}$$

with $X\equiv(\sigma-\kappa)(\sigma-\kappa+1)$. In general, opposite-parity trajectories are degenerate to order t^{M-1} for boson trajectories (integer values of M) and to order E^{2M-1} for fermion trajectories (half-integer values of M), as is implied directly by (3.33). The MacDowell symmetry,

$$\alpha_\kappa^{(+)}(E)=\alpha_\kappa^{(-)}(-E), \quad (4.7)$$

for fermion trajectories also follows immediately from (3.33).

Recently, some general formulas for α_κ have become available: Durand, Fishbane, and Simmons[15] have considered the Lorentz expansion of scattering amplitudes of spinless particles and suggest that the $M=0$ trajectory should behave like[16]

$$\alpha_\kappa+\kappa=\sum_{j=0}^{\kappa}\frac{\Gamma(\kappa+1)\Gamma(2\alpha_\kappa+\kappa+2)}{\Gamma(\kappa-j+1)\Gamma(2\alpha_\kappa+\kappa-j+2)}t^j a_j(t), \quad (4.8)$$

where $a_j(t)$ are arbitrary analytic functions of t. Bronzan,[17] on the other hand, has approached the problem by the method of enforcing Mandelstam analyticity term by term and has derived[18] trajectory

[13] P. K. Kuo and J. F. Walker, Phys. Rev. **175**, 1794 (1968); G. Konisi and T. Saito, Progr. Theoret. Phys. (Kyoto) **41**, 108 (1969).
[14] Second paper in Ref. 9.
[15] L. Durand III, P. Fishbane, and M. Simmons, Jr., Phys. Rev. Letters **22**, 261 (1969). *Note added in manuscript.* See also Phys. Rev. Letters **23**, 201 (1969).
[16] The same expression has also been suggested by J. C. Taylor (unpublished).
[17] J. B. Bronzan, Phys. Rev. **180**, 1423 (1969); **181**, 2111 (1969).

functions for integer values of M:

$$\alpha_\kappa^{(\pm)}+\kappa=\sigma+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial\sigma}\right)^n \times\left[A(\sigma,t,\kappa)\pm t^M\frac{\Gamma(\sigma-\kappa+M+1)}{\Gamma(\sigma-\kappa-M+1)}B(\sigma,t,\kappa)\right]^{n+1}, \quad (4.9)$$

where

$$A(\sigma,t,\kappa)=\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_{iq}\frac{\kappa!\Gamma(2\sigma-\kappa+2)}{(\kappa-i)!\Gamma(2\sigma-\kappa-i+2)},$$
$$B(\sigma,t,\kappa)=\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}B_{iq}\frac{\kappa!\Gamma(2\sigma-\kappa+2)}{(\kappa-i)!\Gamma(2\sigma-\kappa-i+2)}, \quad (4.10)$$

and the constants A_{iq}, B_{iq} may be dependent on σ but not on t or κ. It is then an interesting question whether these results agree with our general formula (3.33), because quite different basic assumptions are involved in deriving them. Bronzan based his calculation on the analytic S-matrix theory, while one of our basic assumptions is that the scattering amplitude may be continued off-shell in the mass variables.

In what follows, we show that Bronzan's formula may be summed to a form, of which the formula of Durand *et al.* is a special case, which in turn may be recast in the form of our general expression. We further show that to a given order in t (or E) the same number of independent parameters is required from both Bronzan's and our expressions, thus establishing the equivalence between them.

Let $F(\sigma)$ stand for the square bracket in (4.9) and let $z=\sigma_0$ be the solution of

$$z-\sigma-F(z)=0, \quad (4.11)$$

which tends to σ as $t\to 0$; we can rewrite (4.9) as

$$\alpha_\kappa^{(\pm)}+\kappa=\sigma+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{\partial}{\partial\sigma}\right)^n[F(\sigma)]^{n+1}$$
$$=\sigma+\sum_{n=0}^{\infty}\frac{1}{2\pi i}\oint\frac{dz}{n+1}\left(\frac{F(z)}{z-\sigma}\right)^{n+1}$$
$$=\sigma-\frac{1}{2\pi i}\oint dz\ln\frac{z-\sigma-F(z)}{z-\sigma}=\sigma_0, \quad (4.12)$$

where, to maintain analyticity at $t=0$, the contour must be chosen to enclose both σ and σ_0 but to exclude any other zeros of (4.11). The cut of the logarithmic function extends from σ to σ_0; hence the integral can be evaluated by partial integration. Substituting this back into (4.11), we have

$$\alpha_\kappa^{(\pm)}+\kappa=\sigma+A(\alpha_\kappa^{(\pm)}+\kappa,t,\kappa)\pm t_M B(\alpha_\kappa^{(\pm)}+\kappa,t,\kappa)\quad\text{for } M\neq 0, \quad (4.13)$$

$$\alpha_\kappa^{(\pm)}+\kappa=\sigma^{(\pm)}+A^{(\pm)}(\alpha_\kappa^{(\pm)}+\kappa,t,\kappa)\quad\text{for } M=0. \quad (4.14)$$

We see that (4.8) is equivalent to the above if we identify

$$a_j(t)=\sum_{q=j}^{\infty}t^{q-j}A_{jq}, \quad (4.15)$$

and if the A's are independent of σ. (More about the σ dependence of the A's and B's later.) We now introduce a new variable $\sigma_\kappa=\alpha_\kappa^{(\pm)}+\kappa$ and use it to eliminate κ from (4.13) and (4.14). We further recognize that

$$\frac{\kappa!\Gamma(2\alpha^{(\pm)}+\kappa+2)}{(\kappa-i)!\Gamma(2\alpha^{(\pm)}+\kappa-i+2)}=\prod_{j=0}^{i}[t(\sigma_\kappa-j)(\sigma_\kappa-j+1)-t\alpha_\kappa^{(\pm)}(\alpha_\kappa^{(\pm)}+1)],$$

so that $A(\sigma_\kappa,t,\kappa)$ and $B(\sigma_\kappa,t,\kappa)$ are really analytic functions of σ_κ, t, and $t_1=t\alpha_\kappa^{(\pm)}(\alpha_\kappa^{(\pm)}+1)$. Assuming we can solve (4.13) and (4.14) for σ_κ in terms of t, t_1, and t_M, the solutions would be

$$\sigma_\kappa=\alpha_\kappa^{(\pm)}+\kappa=f(t,t_1)\pm t_M g(t,t_1)\quad\text{for } M\neq 0;$$
$$\sigma_\kappa=\alpha_\kappa^{(\pm)}+\kappa=f^{(\pm)}(t,t_1)\quad\text{for } M=0.$$

It would be difficult to start from these forms in the derivation of (4.13) and (4.14), because there is some arbitrariness in the latter, in that it is possible for different implicit functions to possess the same explicit forms. The σ dependence of the functions A and B of Bronzan is the consequence of such an arbitrariness. In fact, no generality is lost by assuming that they are independent of σ.[17] With this assumption, it is relatively simple to count the number of independent parameters required to determine to a given order in t (or E) from both expressions and find that they indeed agree.

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[18] This is only true for $M\neq 0$. For $M=0$ one should replace σ by $\sigma^{(\pm)}$, A by $A^{(\pm)}$, and leave out the B term.